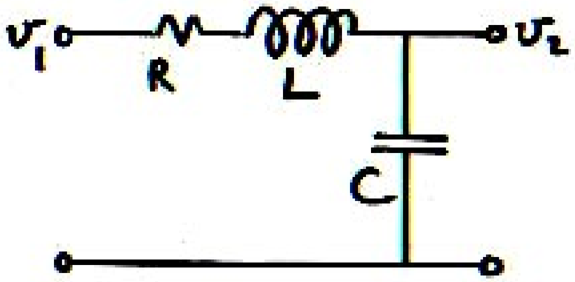
5

BUILDING

LOW ENTROPY EXPRESSIONS

WITH MINIMUM WORK

Double-pole low-pass LC filter



$$\frac{v_2}{v_1} = \frac{1}{1 + sRC + s^2LC}$$

$$= \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$= \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

roots

in which

$$\omega_0 \equiv \frac{1}{\sqrt{LC}}$$ corner (resonant) frequency

$$Q \equiv \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{R_0}{R} \quad \text{where } R_0 \equiv \sqrt{\frac{L}{C}}$$

characteristic resistance

$Q < 0.5$: roots ω_1 and ω_2 are real

$Q > 0.5$: roots ω_1 and ω_2 are complex

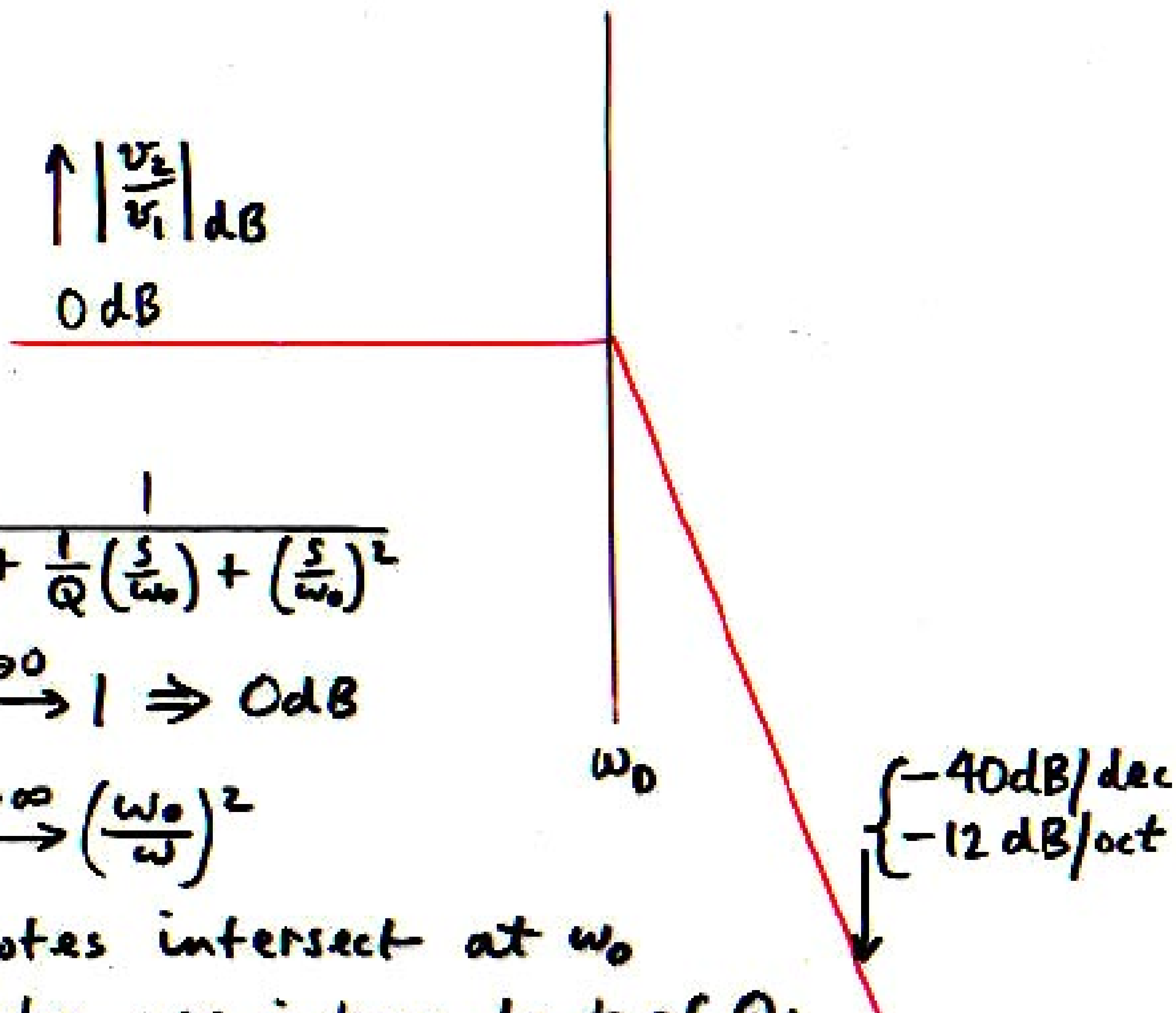
$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| \xrightarrow{\omega \to 0} 1 \Rightarrow 0\,dB$$

$$\xrightarrow{\omega \to \infty} \left(\frac{\omega_0}{\omega}\right)^2$$

Asymptotes intersect at ω_0

Asymptotes are independent of Q;
Q affects shape only in
neighborhood of ω_0



$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| \xrightarrow{\omega \to 0} 1 \Rightarrow 0\,dB$$

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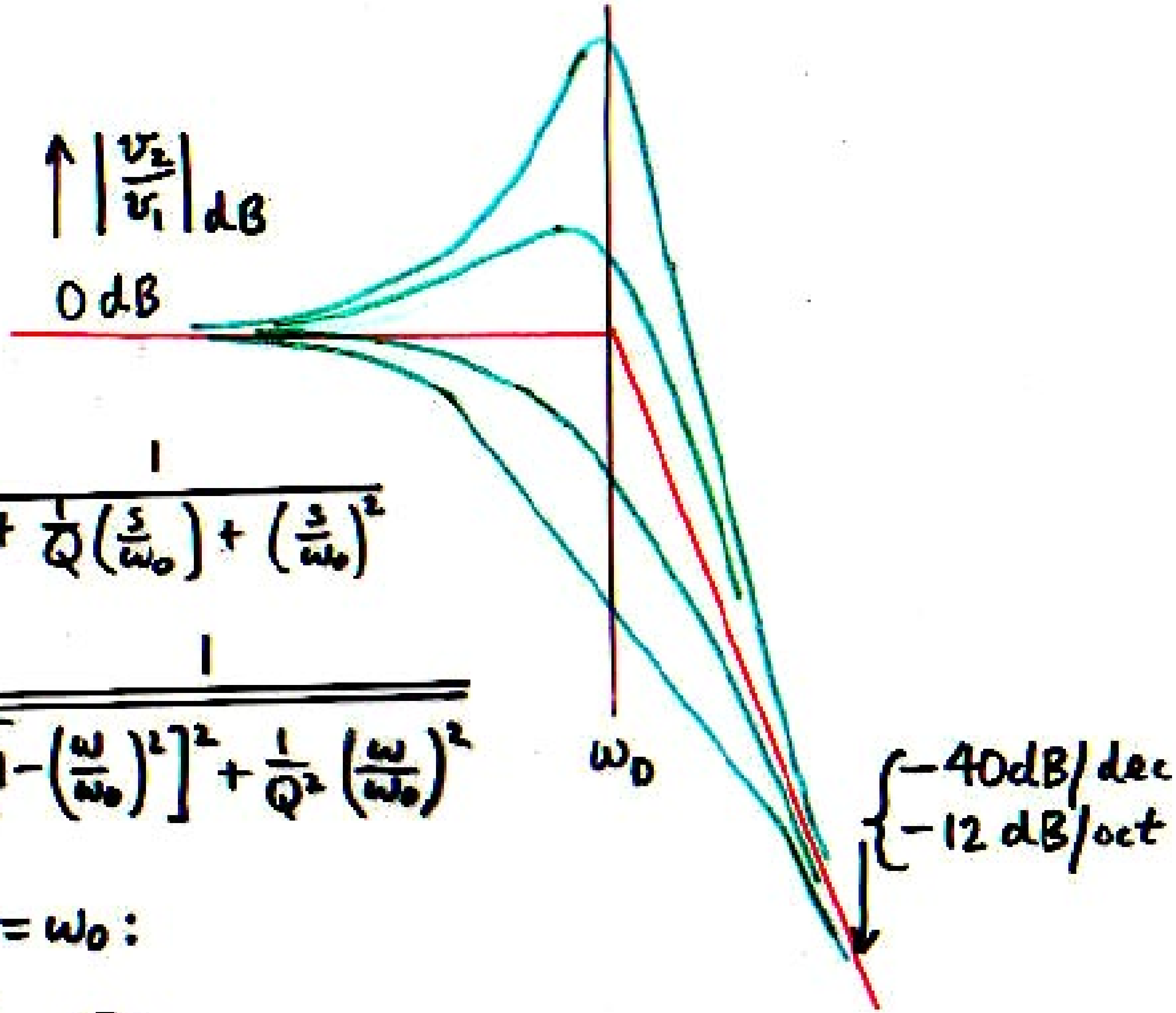
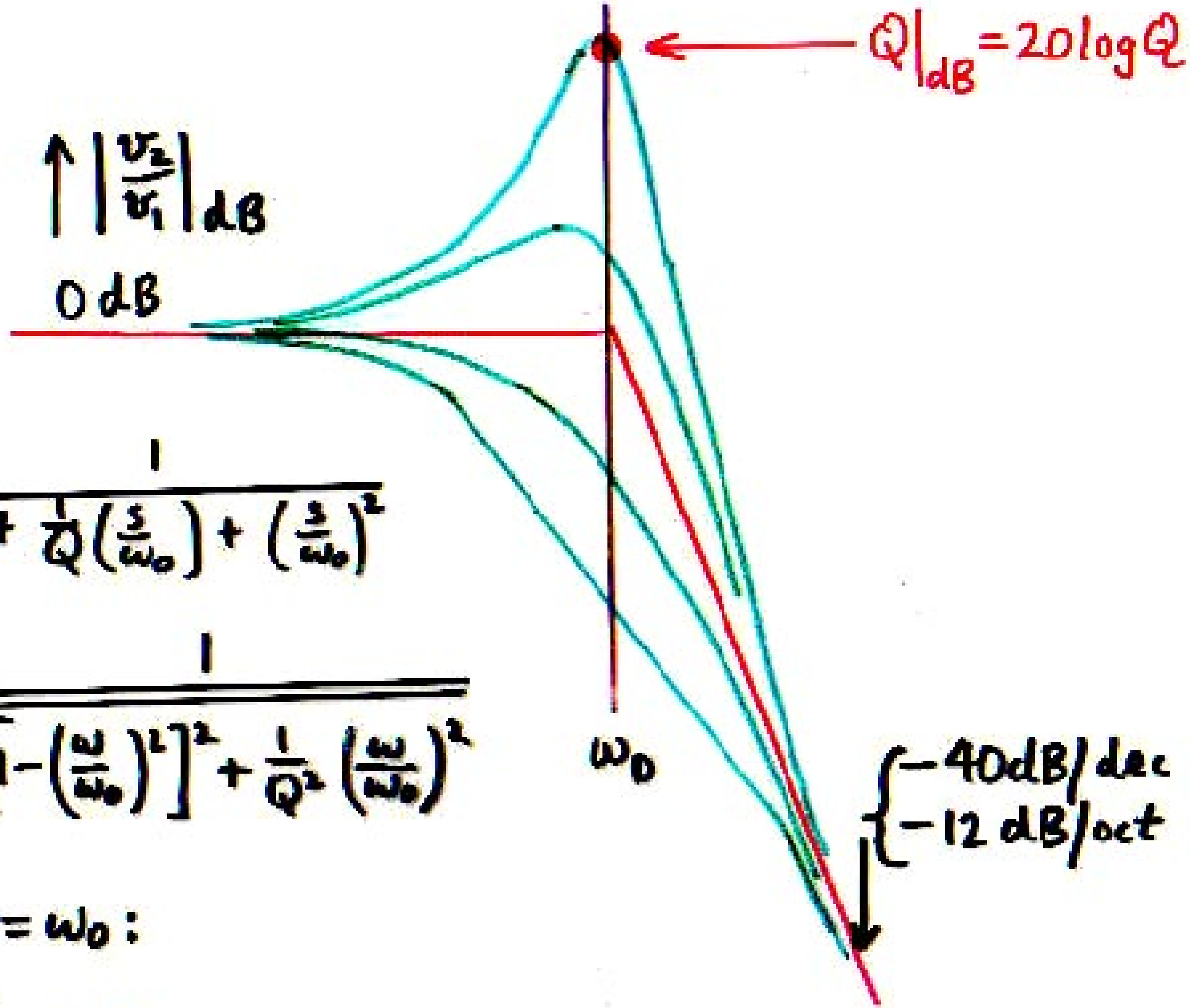


$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$



$$\frac{v_2}{v_1} = \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

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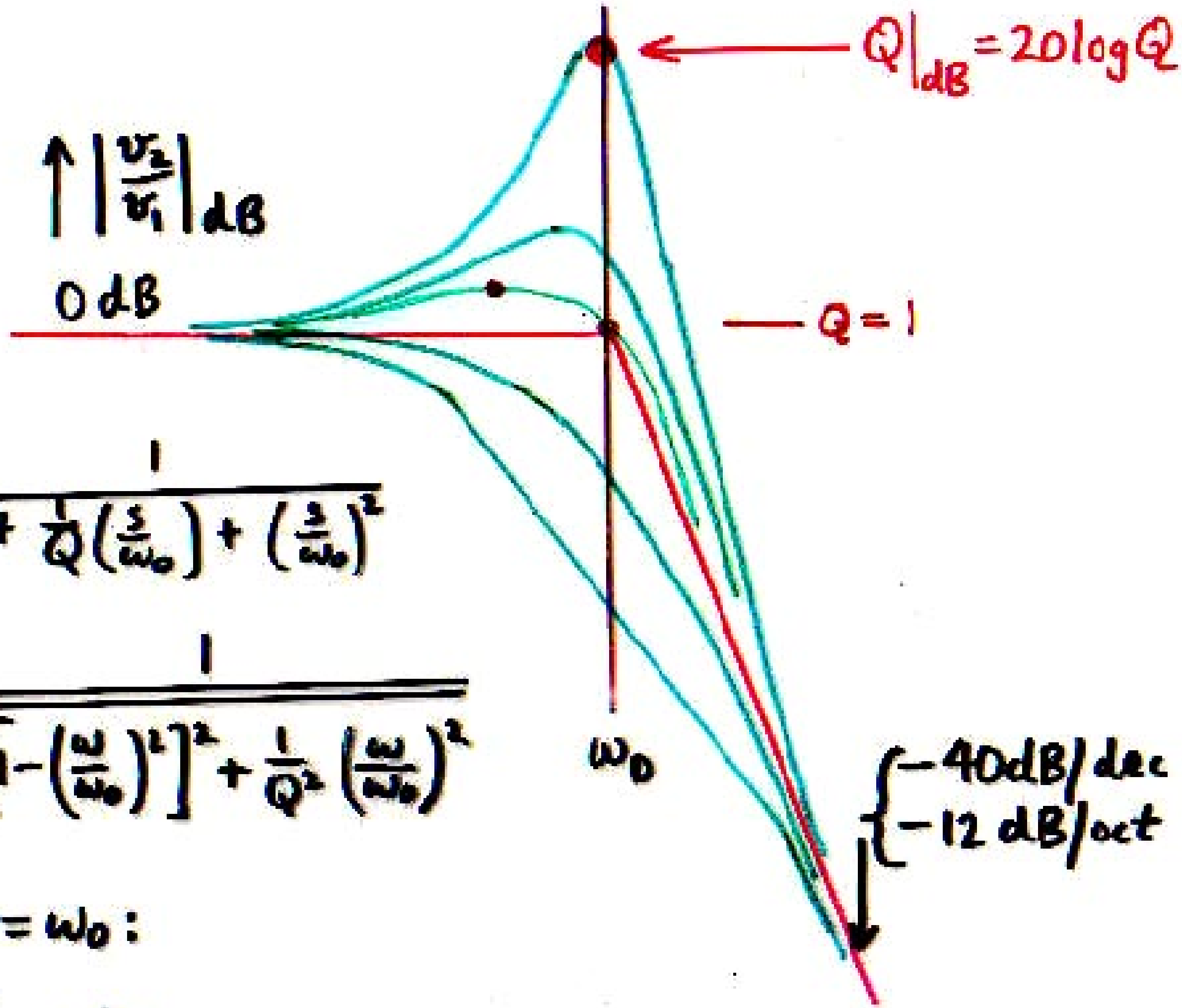

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At $\omega = \omega_0$:

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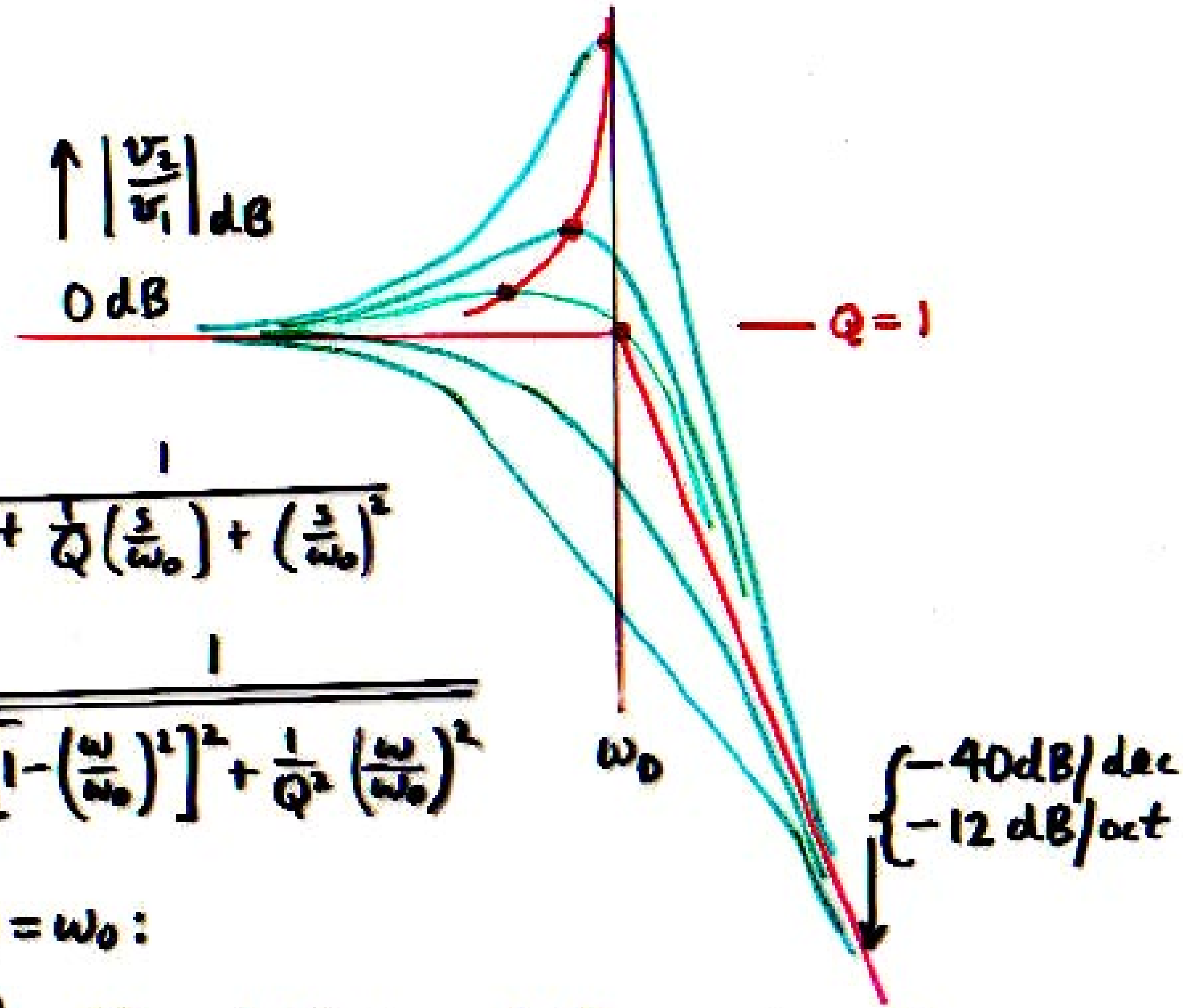
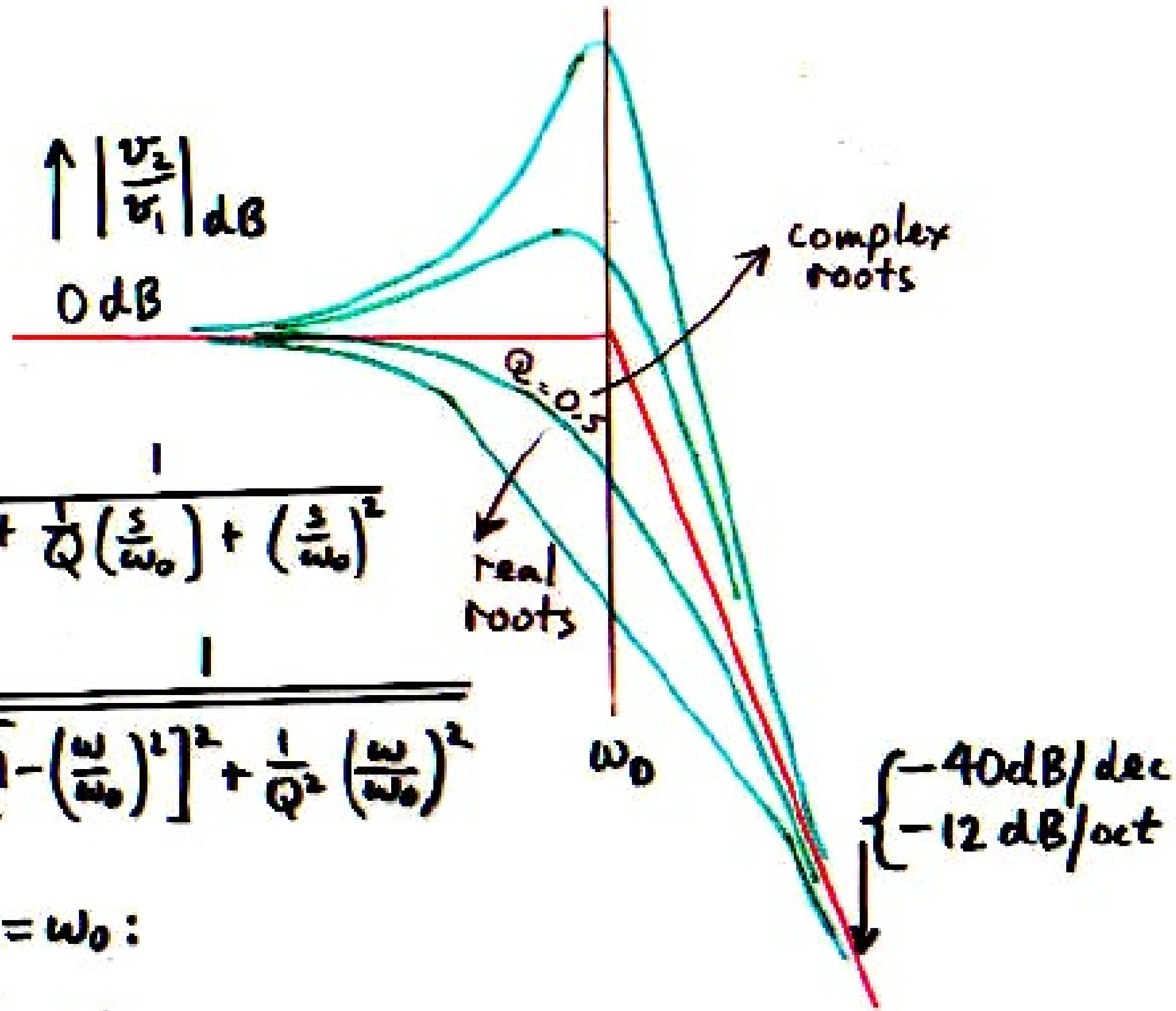


$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$\left|\frac{v_2}{v_1}\right| = Q$ which is not the maximum; the maximum moves to the left for lower Q.



$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

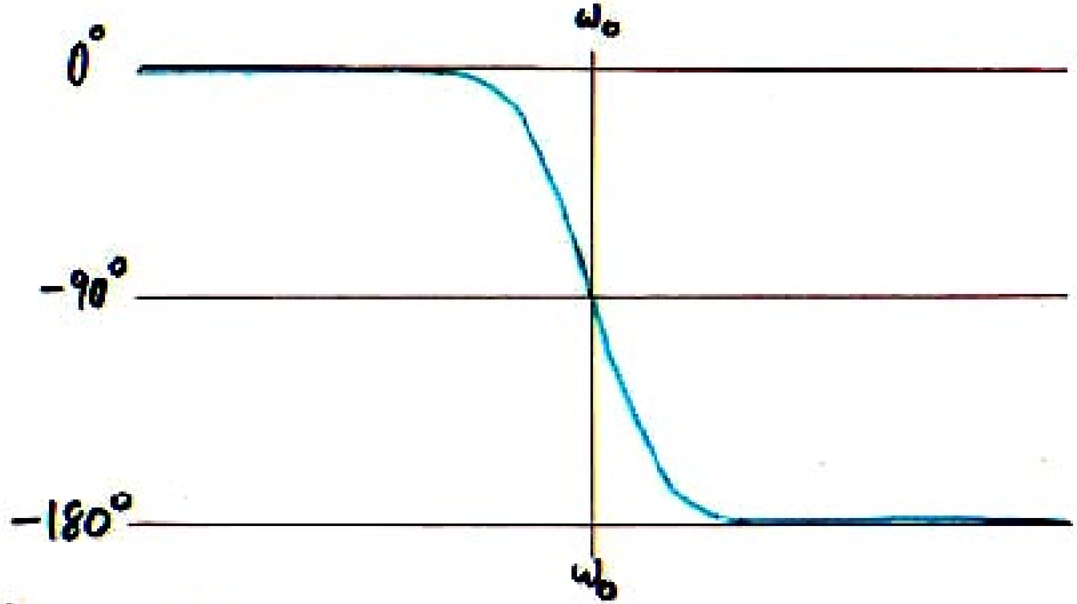
At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$

Phase shape:

$$\angle \frac{v_2}{v_1} = -\tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

$$\left.\begin{array}{l} \xrightarrow{\omega\to 0} 0^\circ \\ \xrightarrow{\omega=\omega_0} -90^\circ \\ \xrightarrow{\omega=\infty} -180^\circ \end{array}\right\} \text{independent of } Q$$



Phase shape:

$$\angle \frac{v_2}{v_1} = -\tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

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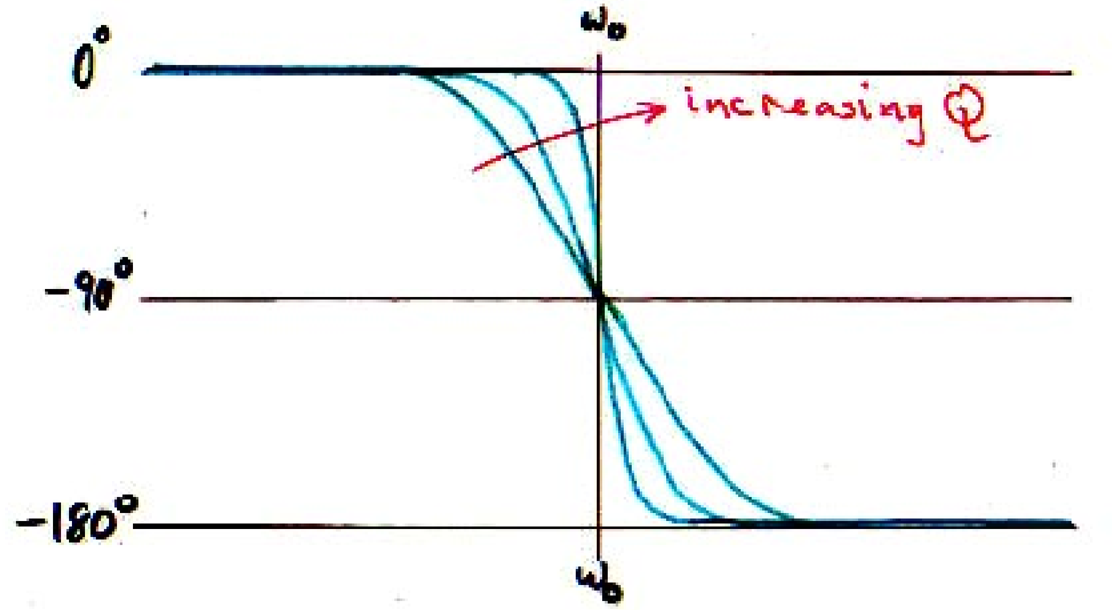
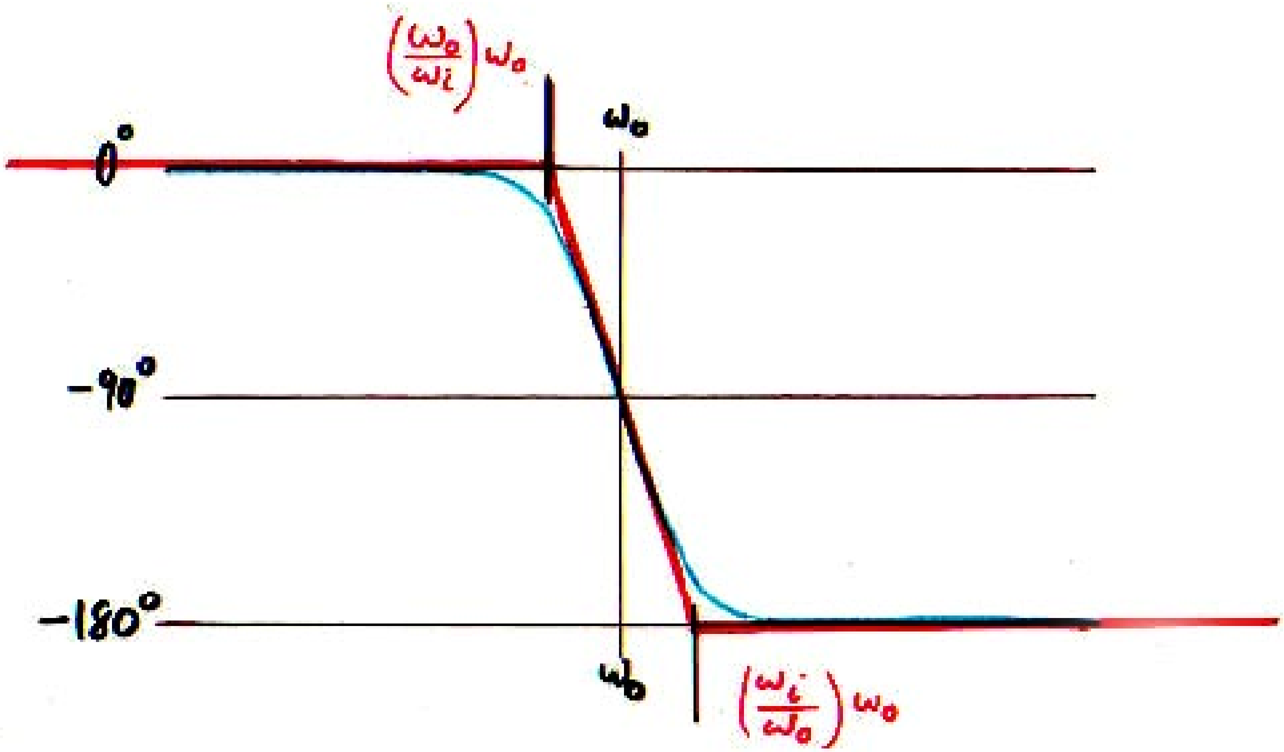


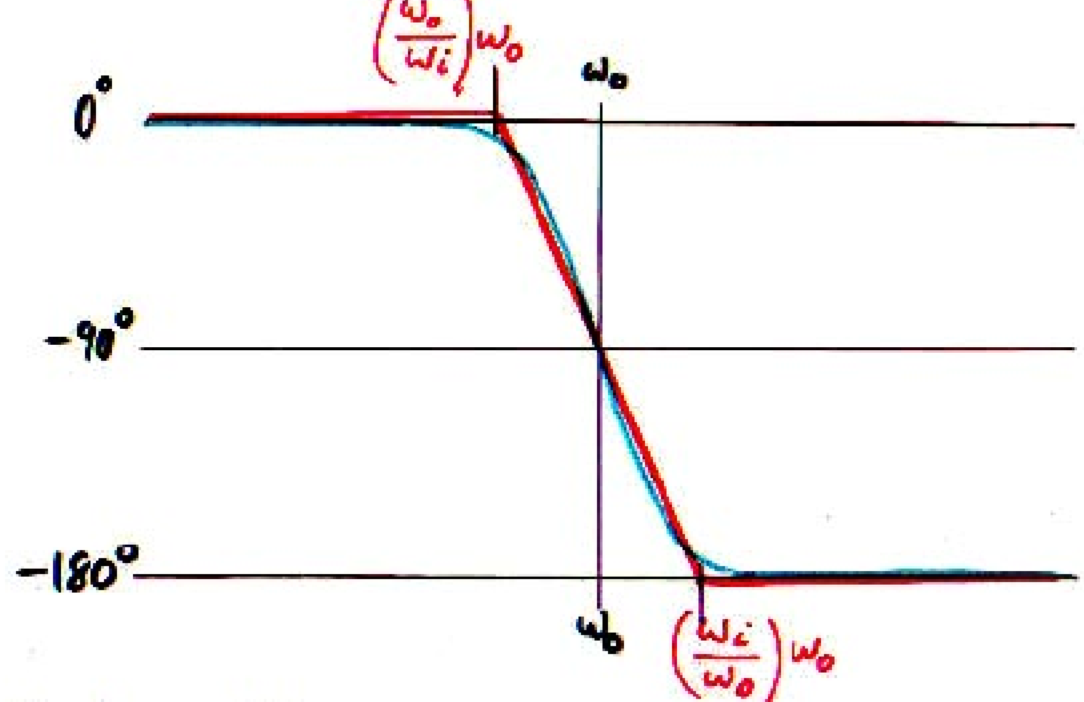
Increased Q causes sharper phase change between the 0° and −180° asymptotes.

Need: a straight-line approximation.



Choose same slope at $\omega = \omega_o$:

$$\frac{\omega_i}{\omega_o} = \left(e^{\frac{\pi}{2}}\right)^{\frac{1}{2Q}} = (4.81)^{\frac{1}{2Q}}$$



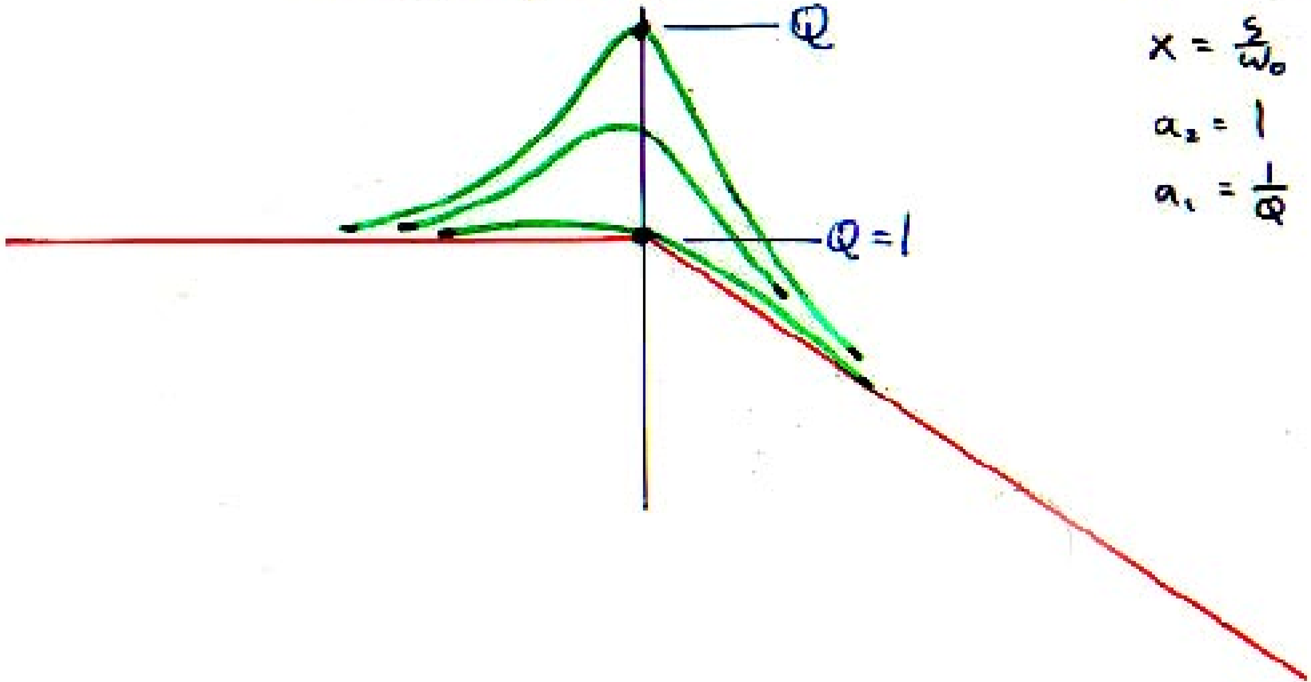
Better choice:

$$\frac{\omega_i}{\omega_o} \approx 5^{\frac{1}{2Q}}$$

Second-order response:

$$A = A_1 \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}$$

$$A = A_1 \frac{1}{1+a_1(x)+a_2(x)^2} = A_1 \frac{1}{1+\frac{a_1}{\sqrt{a_2}}(\sqrt{a_2}x)+(\sqrt{a_2}x)^2}$$



$x = \frac{s}{\omega_0}$

$a_2 = 1$

$a_1 = \frac{1}{Q}$

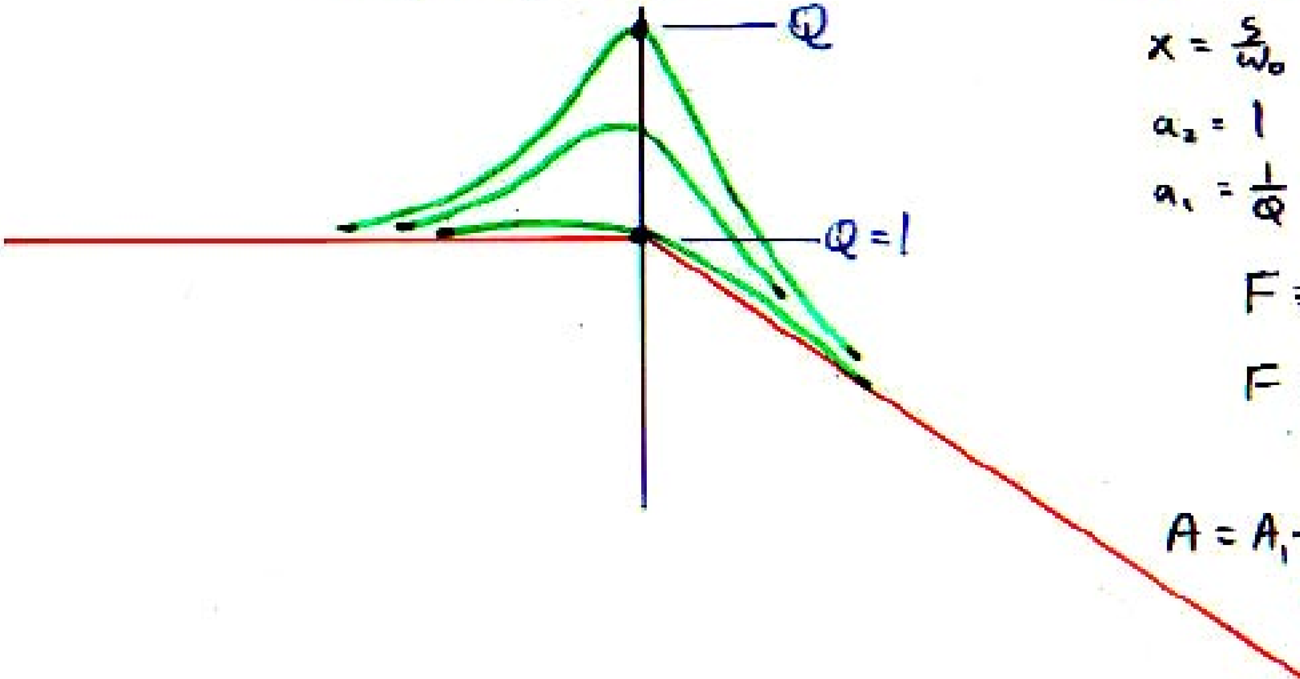
$x = s$

$a_2 = \frac{1}{\omega_0^2}$

$a_1 = \frac{1}{\omega_0 Q}$

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$



$$A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} = A_1 \frac{1}{1 + \frac{a_1}{\sqrt{a_2}}(\sqrt{a_2}x) + (\sqrt{a_2}x)^2}$$

$x = \frac{s}{\omega_0}$

$a_2 = 1$

$a_1 = \frac{1}{Q}$

$x = s$

$a_2 = \frac{1}{\omega_0^2}$

$a_1 = \frac{1}{\omega_0 Q}$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4a_2/a_1^2}$$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F} x\right)}$$

Second-order response:

$$A = A_1 \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}$$

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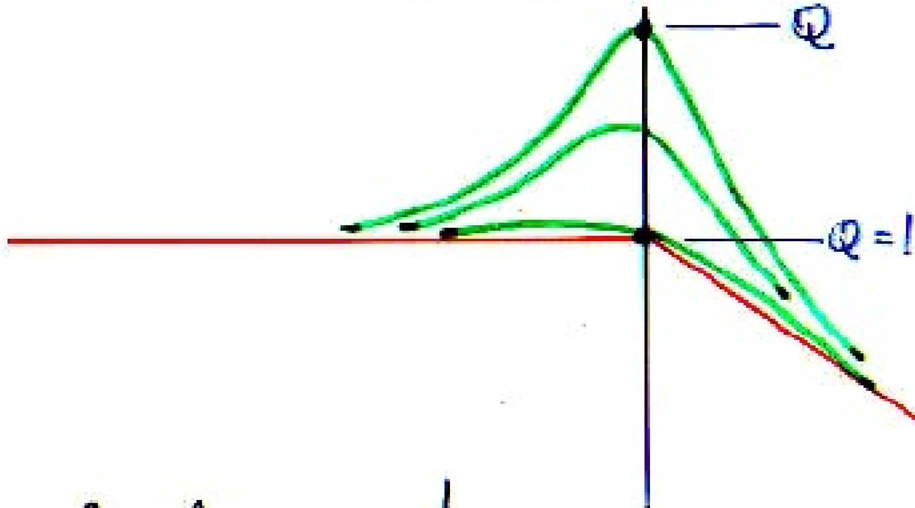
$x = \frac{s}{\omega_0}$	$x = s$
$a_2 = 1$	$a_2 = \frac{1}{\omega_0^2}$
$a_1 = \frac{1}{Q}$	$a_1 = \frac{1}{\omega_0 Q}$



$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1-4a_2/a_1^2}$$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2}$$

$$A = A_1 \frac{1}{\left(1+\frac{s}{Q\omega_0/F}\right)\left(1+\frac{s}{F\omega_0/Q}\right)}$$

$$A = A_1 \frac{1}{(1+a_1 F x)\left(1+\frac{a_2}{a_1 F}x\right)}$$

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

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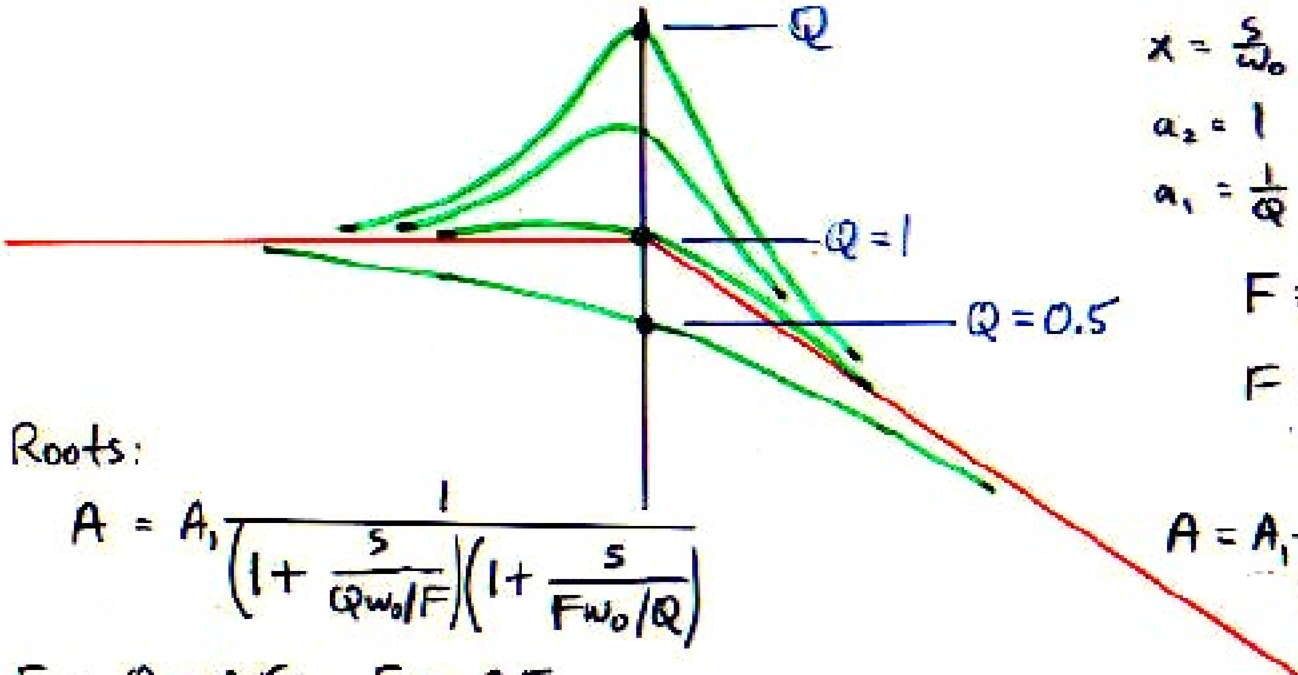
$x = \frac{s}{\omega_0}$	$x = s$
$a_2 = 1$	$a_2 = \frac{1}{\omega_0^2}$
$a_1 = \frac{1}{Q}$	$a_1 = \frac{1}{\omega_0 Q}$



$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4a_2/a_1^2}$$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}$$

Roots:

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0/F}\right)\left(1 + \frac{s}{F\omega_0/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F} x\right)}$$

For $Q = 0.5$: $F = 0.5$

$$A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_0}\right)^2}$$

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

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$x = \frac{s}{\omega_0}$	$x = s$
$a_2 = 1$	$a_2 = \frac{1}{\omega_0^2}$
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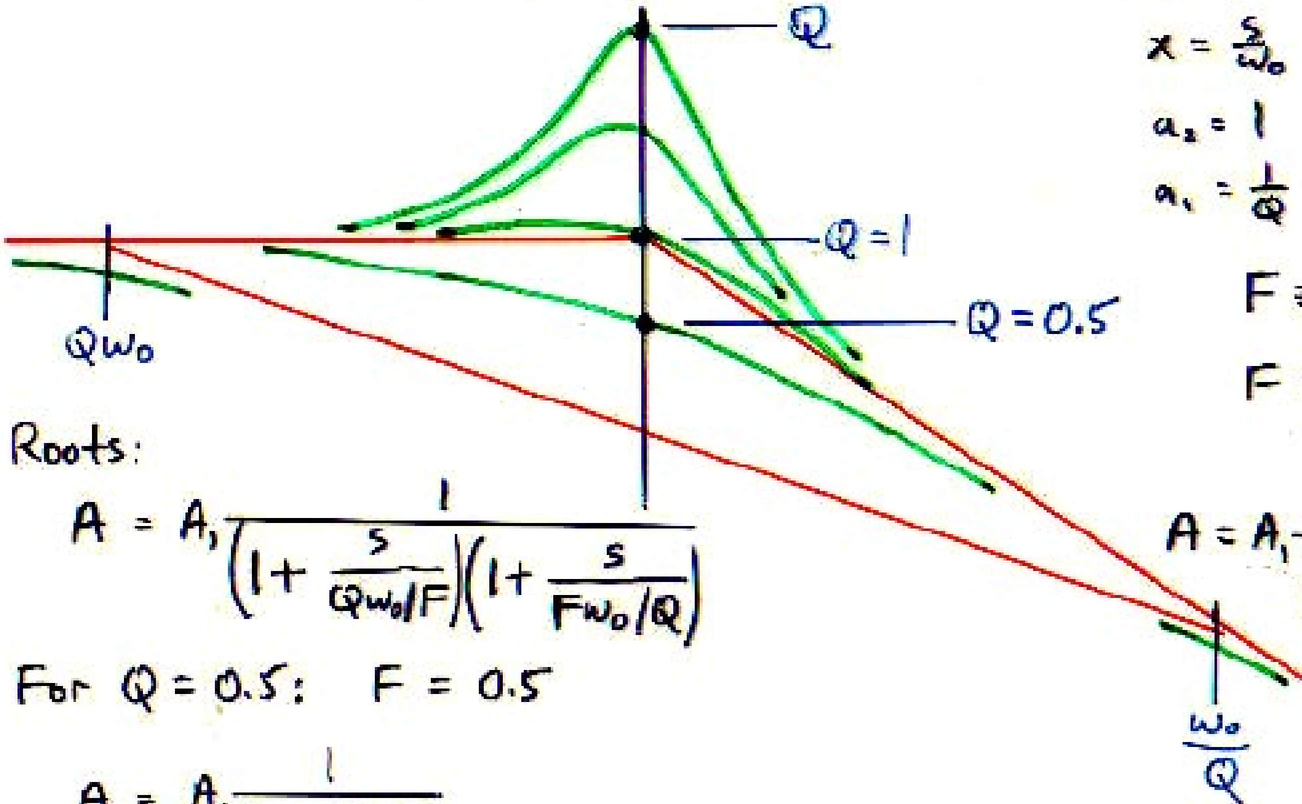


Roots:

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0/F}\right)\left(1 + \frac{s}{F\omega_0/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F} x\right)}$$

For $Q = 0.5$: $F = 0.5$

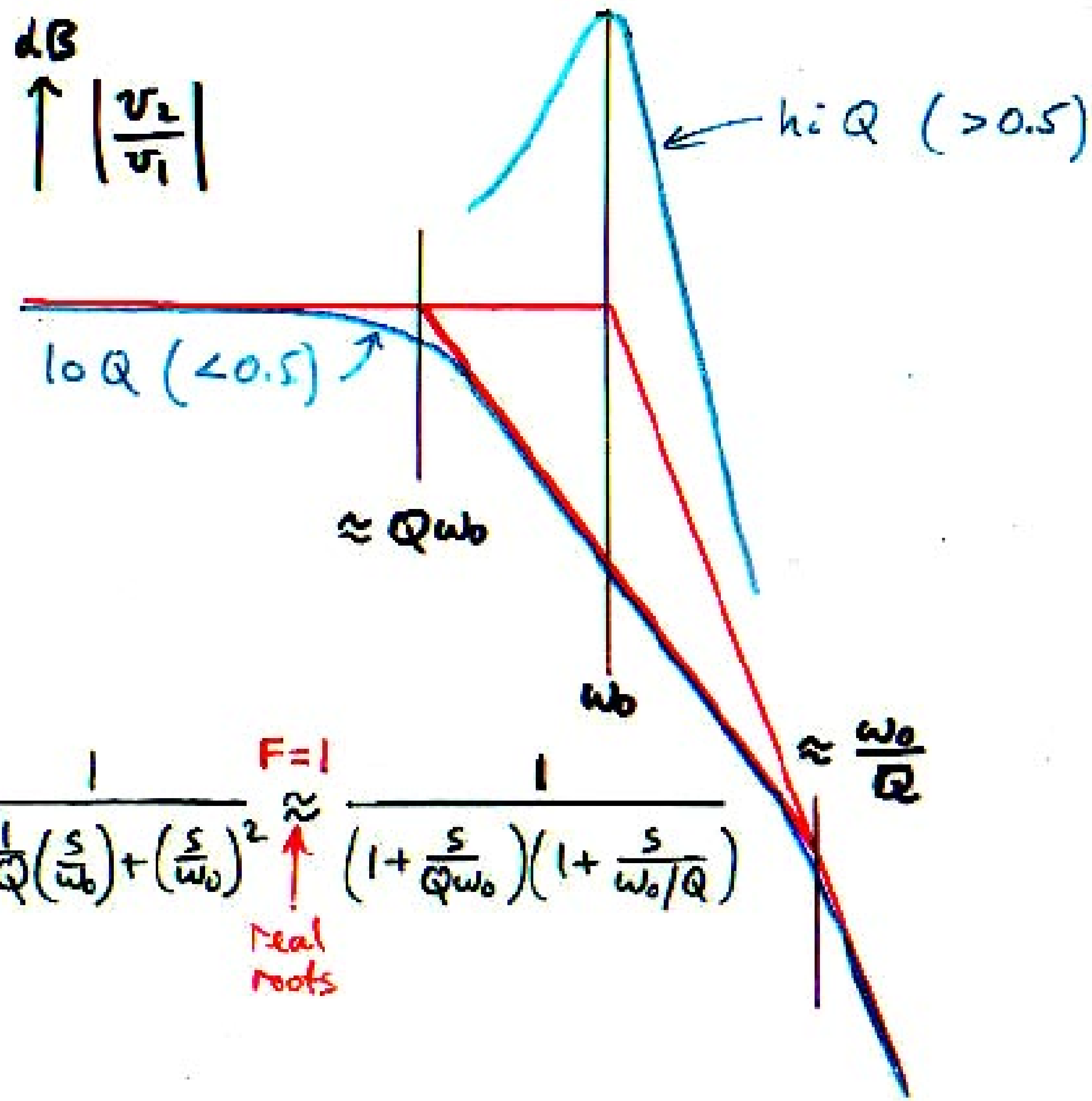
$$A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_0}\right)^2}$$

For $Q \ll 0.5$: $F \approx 1$

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0}\right)\left(1 + \frac{s}{\omega_0/Q}\right)}$$

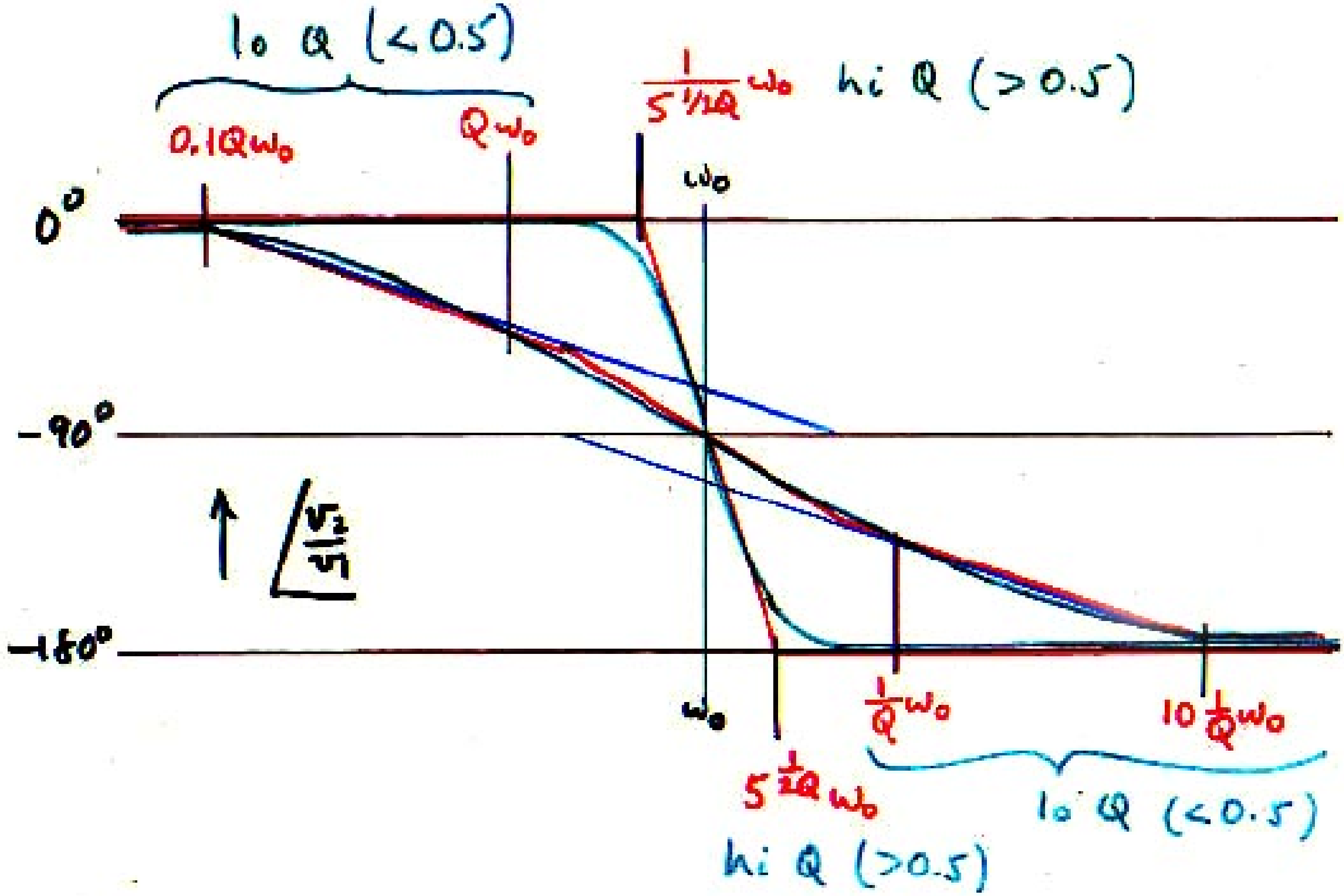
$$A = A_1 \frac{1}{(1 + a_1 x)\left(1 + \frac{a_2}{a_1} x\right)}$$

Low-pass 2-pole characteristic:



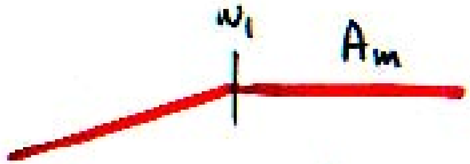
$$\frac{v_2}{v_1} = \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2} \overset{F=1}{\approx} \frac{1}{\left(1+\frac{s}{Q\omega_0}\right)\left(1+\frac{s}{\omega_0/Q}\right)}$$

real roots

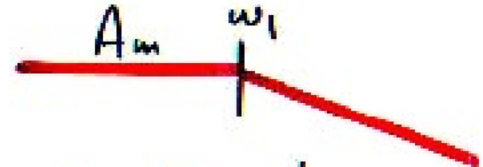


lo Q (<0.5)
0.1Q ω_0
Q ω_0
$\frac{1}{5^{1/2Q}}\omega_0$
hi Q (>0.5)
ω_0
0°
−90°
−180°
$\angle \frac{V_2}{V_1}$
ω_0
$5^{\frac{1}{2Q}}\omega_0$
hi Q (>0.5)
$\frac{1}{Q}\omega_0$
$10\frac{1}{Q}\omega_0$
lo Q (<0.5)

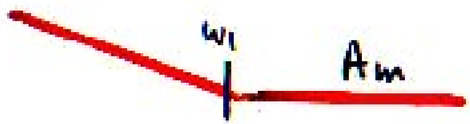
Normal and inverted poles and zeros:



$$A = A_m \frac{1}{\left(1+\frac{\omega_1}{s}\right)}$$ inverted pole

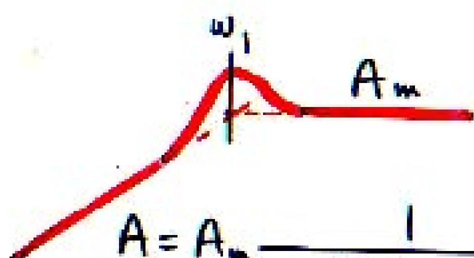


$$A = A_m \frac{1}{\left(1+\frac{s}{\omega_1}\right)}$$ normal pole

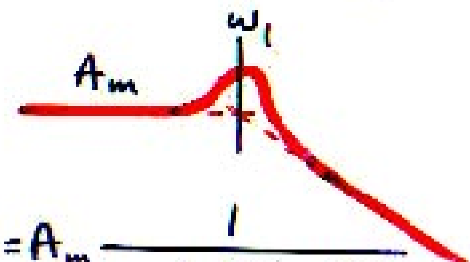


$$A = A_m\left(1+\frac{\omega_1}{s}\right)$$ inverted zero

$$A = A_m\left(1+\frac{s}{\omega_1}\right)$$ normal zero

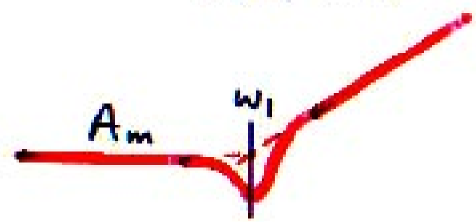


$$A = A_m \frac{1}{1+\frac{1}{Q}\left(\frac{\omega_1}{s}\right)+\left(\frac{\omega_1}{s}\right)^2}$$



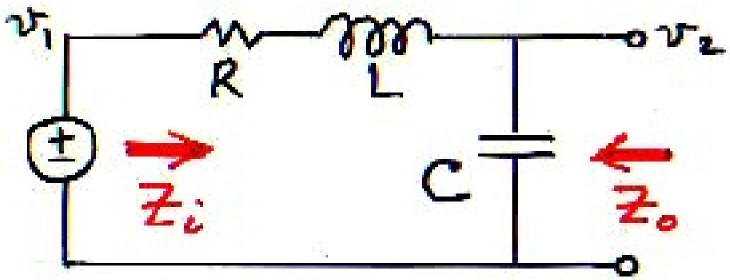
$$A = A_m \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_1}\right)+\left(\frac{s}{\omega_1}\right)^2}$$

$$A = A_m\left[1+\frac{1}{Q}\left(\frac{\omega_1}{s}\right)+\left(\frac{\omega_1}{s}\right)^2\right]$$



$$A = A_m\left[1+\frac{1}{Q}\left(\frac{s}{\omega_1}\right)+\left(\frac{s}{\omega_1}\right)^2\right]$$

Input and Output Impedances of low-pass filter



$$\omega_o \equiv \frac{1}{\sqrt{LC}}$$

$$R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q = \frac{R_o}{R} = \frac{1}{\omega_o C R} = \frac{\omega_o L}{R}$$

$$Z_i = \frac{1}{sC} + R + sL$$

$$\frac{1 + sCR + s^2LC}{sC}$$

$$Z_o = \frac{\frac{1}{sC}\left(R + sL\right)}{\frac{1}{sC} + R + sL}$$

$$= \frac{R + sL}{1 + sCR + s^2LC}$$

Express in terms of ω_o, Q, R_o:

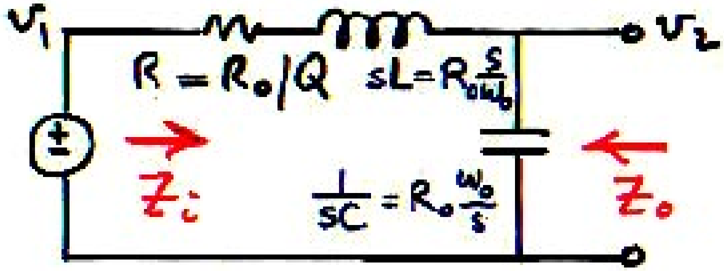
$$Z_i = \frac{1}{\omega_o C}\,\frac{1 + \omega_o CR\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$= R_o\,\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$Z_o = \omega_o L\,\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{R}{sL}\right)}{1 + \omega_o CR\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

$$= R_o\,\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Input and Output Impedances of low-pass filter



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad Q = \frac{R_0}{R}$$

$$R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s} \qquad Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

$$= R_0\,\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)} \qquad = R_0\,\frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

Note how the algebra is shortened when the analysis starts with the normalized element values.

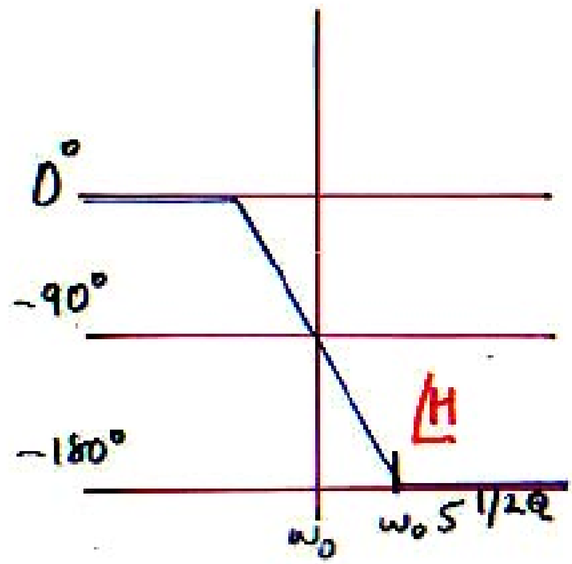
$$Z_i = R_0 \times \left[1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_0}}\right]$$

ref. value

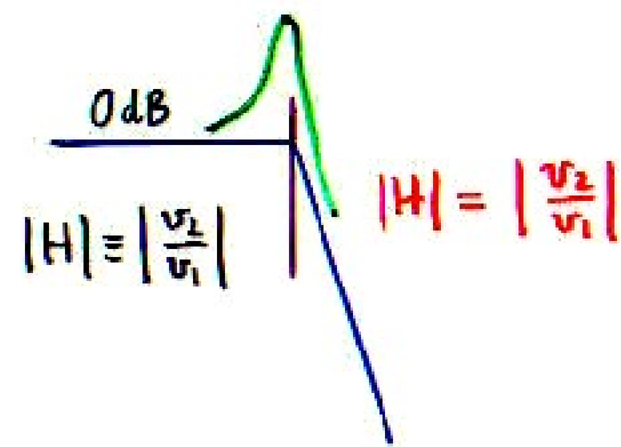
$\frac{1}{H}$

single slope

R_0



0°
-90°
-180°
∠H
ω_0
$\omega_0 5^{1/2Q}$



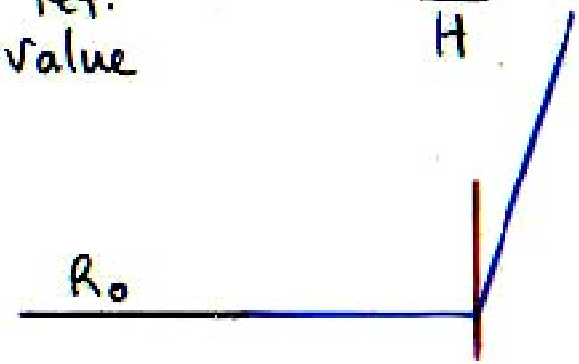
0dB
$|H| \equiv \left|\frac{v_2}{v_1}\right|$
$|H| = \left|\frac{v_2}{v_1}\right|$

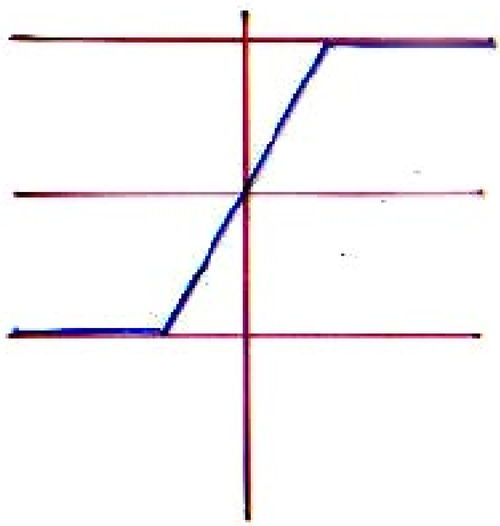
$$Z_i = R_o \times \left[1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

ref. value

$\frac{1}{H}$

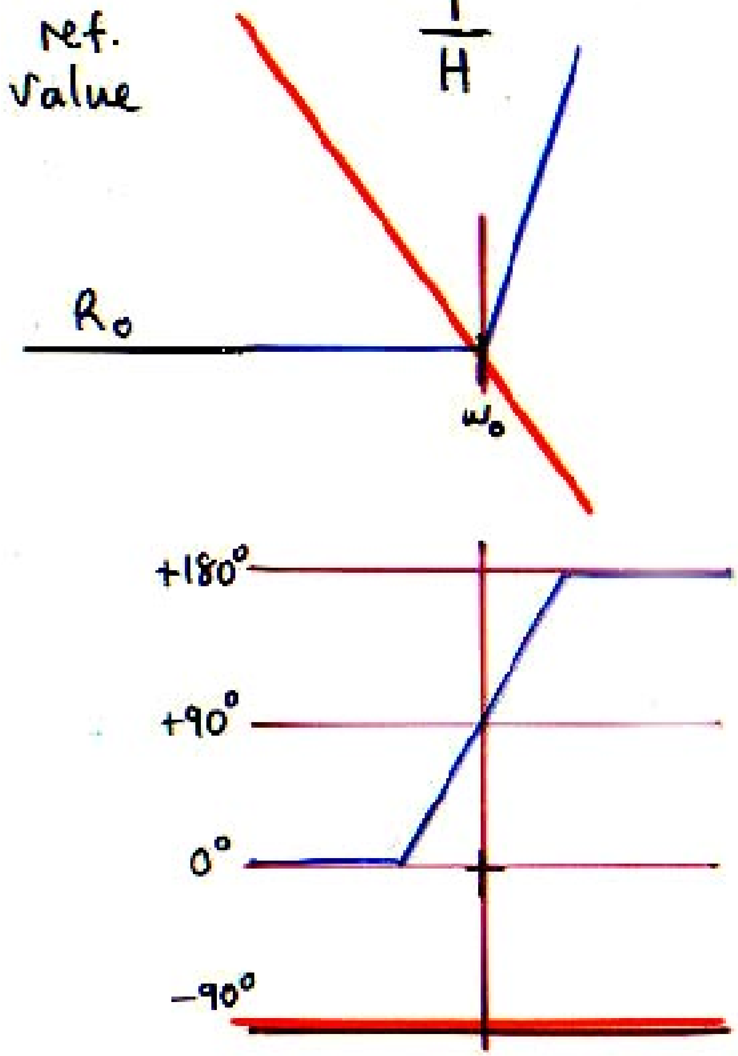
single slope





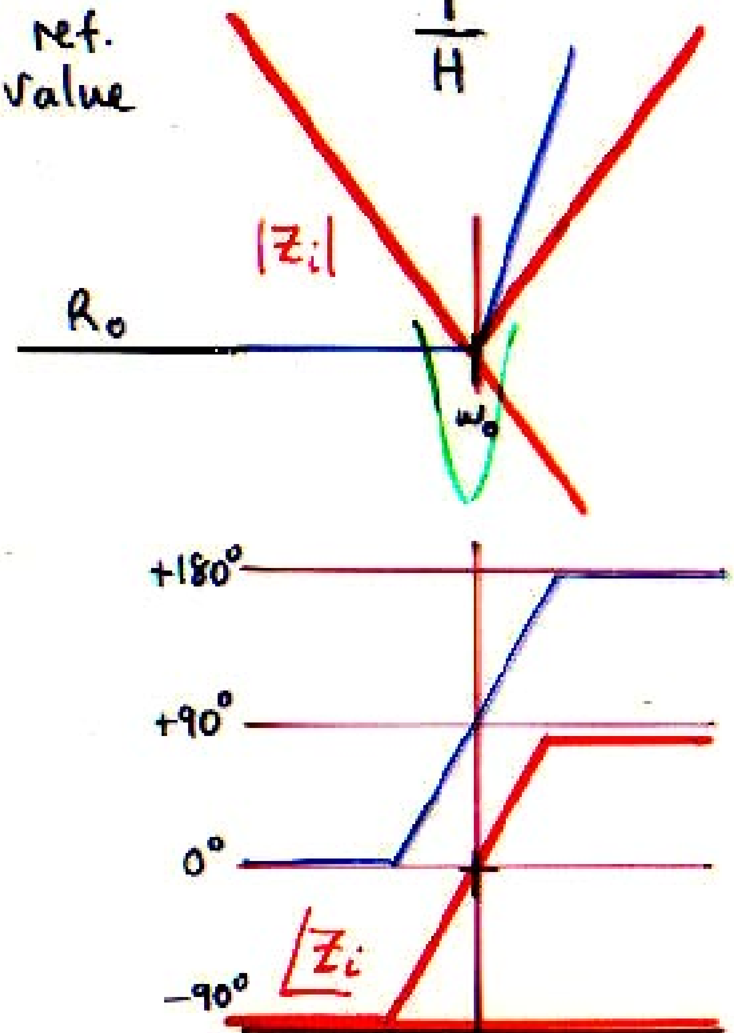
$$Z_i = R_o \times \left[1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

single slope

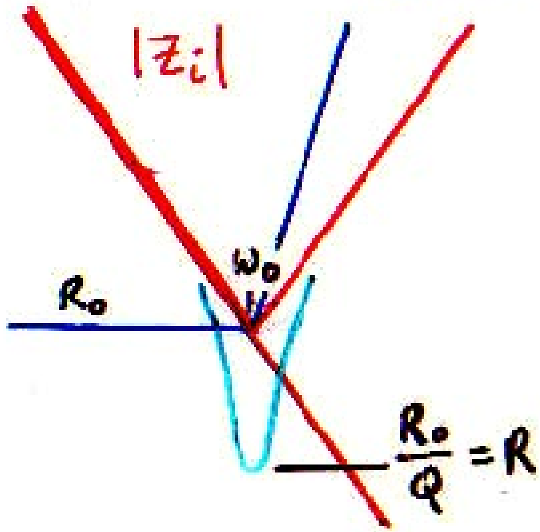


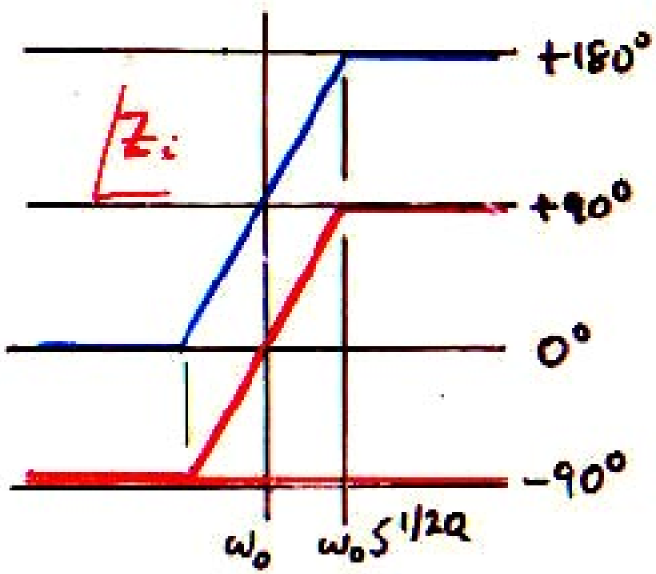
$$Z_i = R_o \times \left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right] \quad \times \quad \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

ref. value $\qquad \frac{1}{H} \qquad\qquad$ single slope

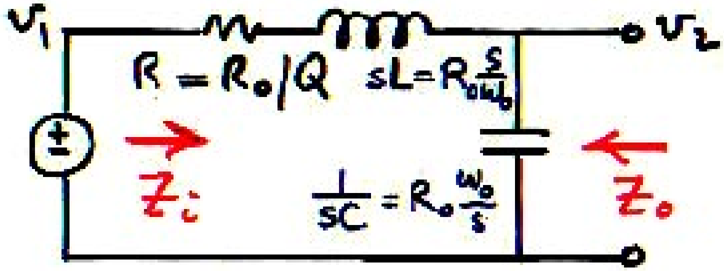


Asymptote sketches for high Q (>>0.5)





Input and Output Impedances of low-pass filter



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_0}{R}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}$$

$$= R_0 \frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)}$$

$$Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

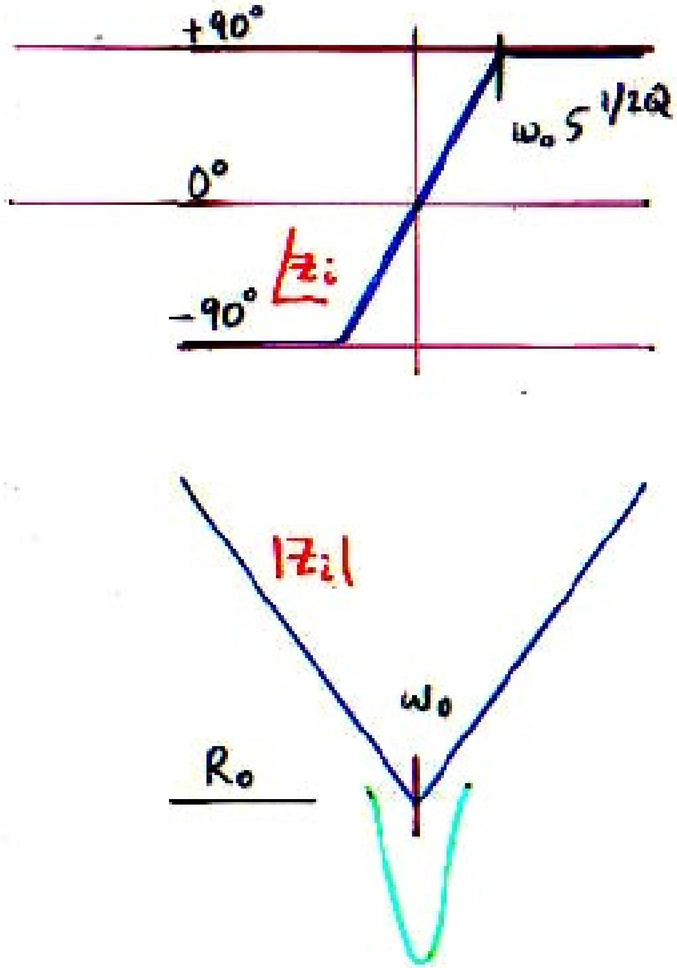
$$= R_0 \frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

Note how the algebra is shortened when the analysis starts with the normalized element values.

$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2} \right] \times \left(1 + \frac{\omega_o/Q}{s} \right)$$

ref. value — $\frac{1}{Z_i}$ — inverted zero

$$\frac{R_o}{}$$



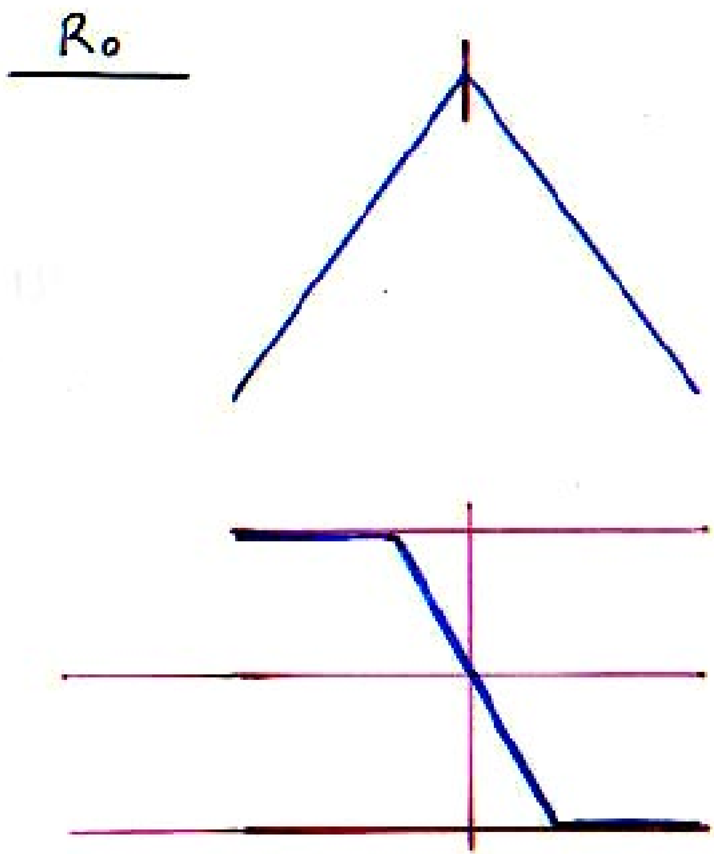
+90°
$\omega_o \, 5^{1/2Q}$
0°
$\angle z_i$
−90°
$|z_i|$
ω_0
R_0

$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_0}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \right] \times \left(1 + \frac{\omega_0 / Q}{s} \right)$$

ref. value

$\frac{1}{Z_i}$

inverted zero

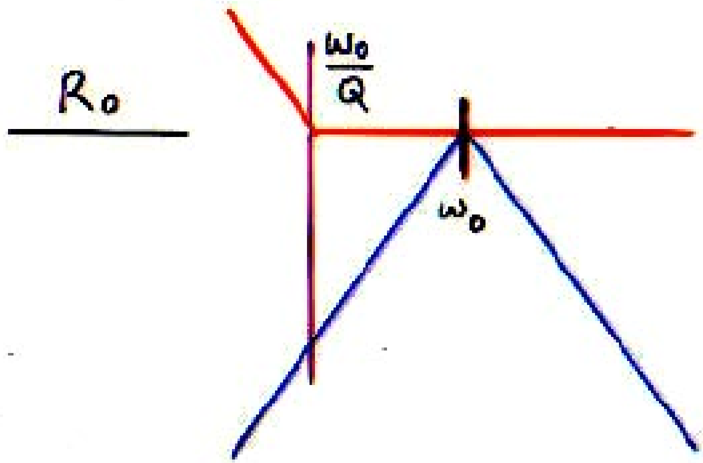


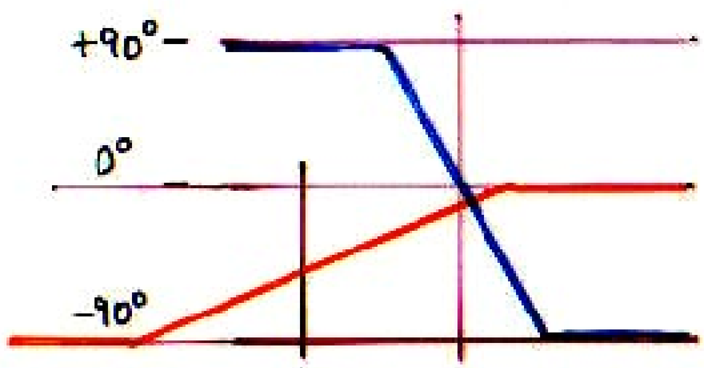
$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1 + \frac{\omega_o/Q}{s}\right)$$

ref. value

$\frac{1}{Z_i}$

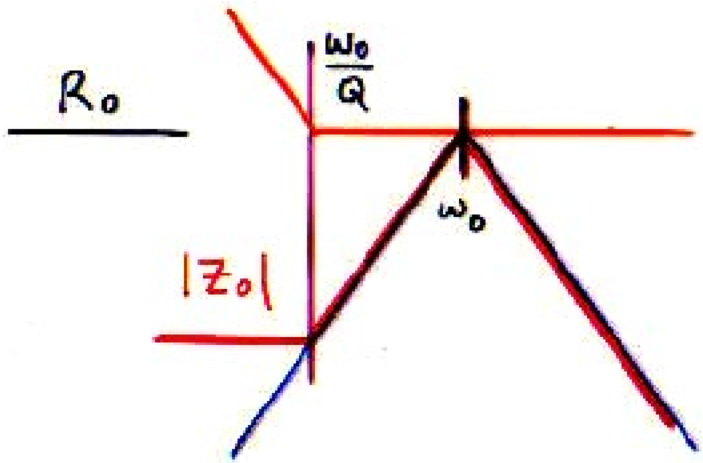
inverted zero

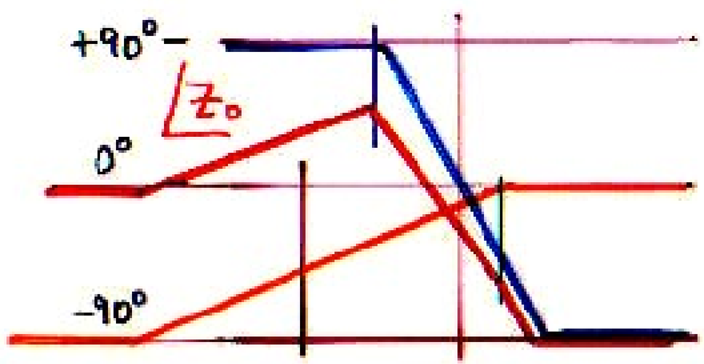


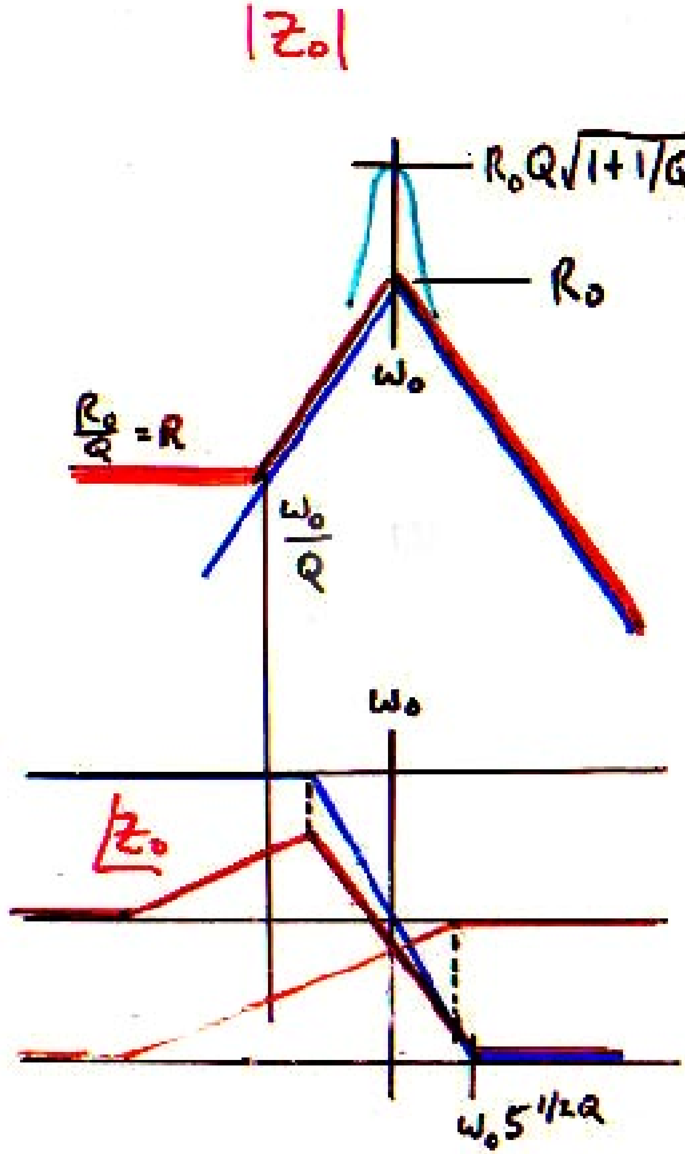


$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1+\frac{\omega_o/Q}{s}\right)$$

ref. value — $\frac{1}{Z_i}$ — inverted zero

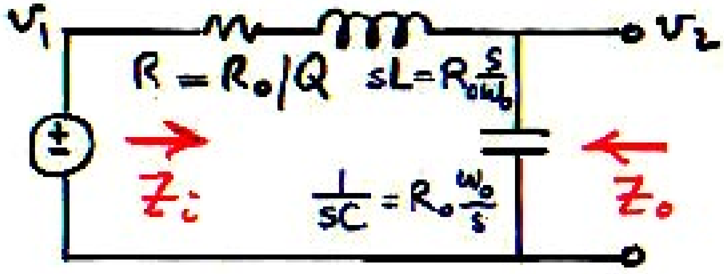






|Z₀|
$R_0 Q\sqrt{1+1/Q^2}$
R_0
ω_0
$\frac{R_0}{Q} = R$
$\frac{\omega_0}{Q}$
ω_0
$\angle Z_0$
$\omega_0 5^{1/2Q}$

Input and Output Impedances of low-pass filter



$$\omega_o \equiv \frac{1}{\sqrt{LC}}$$

$$R_o \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_o}{R}$$

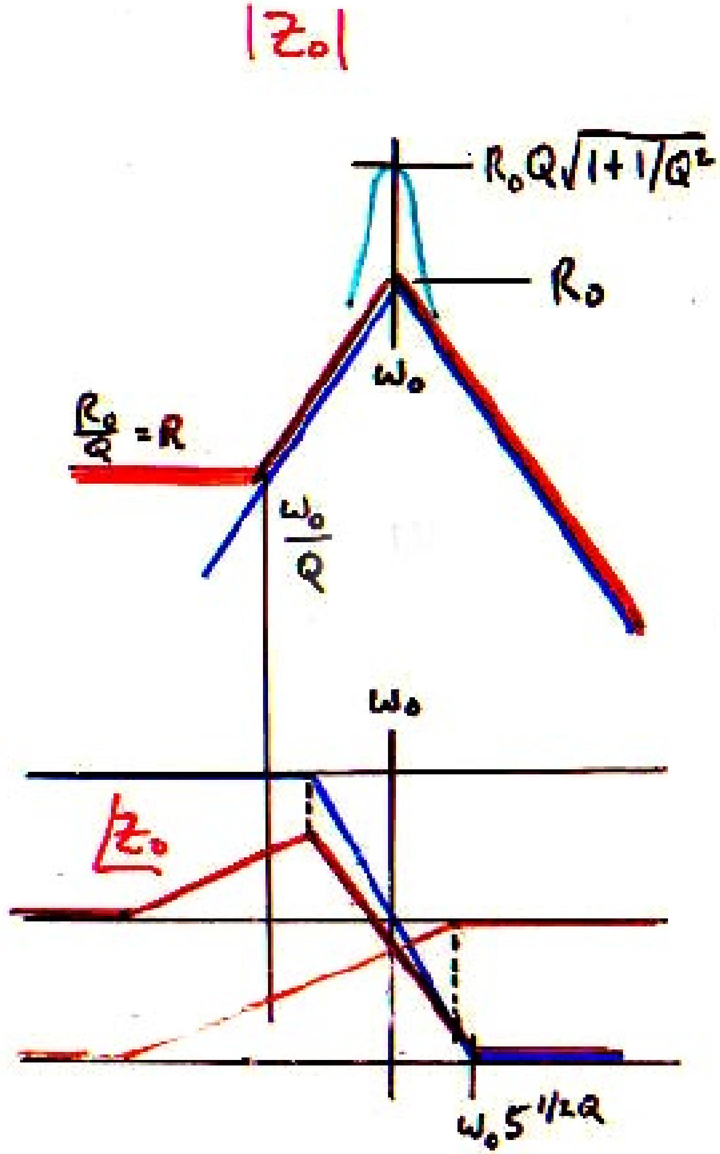
$$Z_i = \frac{R_o}{Q} + R_o \frac{s}{\omega_o} + R_o \frac{\omega_o}{s}$$

$$= R_o \frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$Z_o = \frac{\left(\frac{R_o}{Q} + R_o \frac{s}{\omega_o}\right) R_o \frac{\omega_o}{s}}{\frac{R_o}{Q} + R_o \frac{s}{\omega_o} + R_o \frac{\omega_o}{s}}$$

$$= R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Note how the algebra is shortened when the analysis starts with the normalized element values.

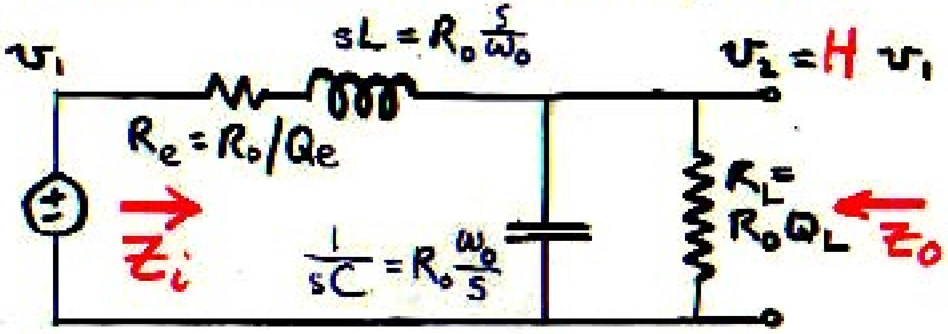


|Z₀|
$R_0 Q\sqrt{1+1/Q^2}$
R_0
ω_0
$\frac{R_0}{Q} = R$
$\frac{\omega_0}{Q}$
ω_0
$\angle Z_0$
$\omega_0 5^{1/2Q}$

Exercise

For the two-pole low-pass LC filter, sketch the magnitude and phase asymptotes of Z_i and Z_o for low Q ($\ll 0.5$).
(But take $Q > 0.1$)

Loaded low-pass LC filter



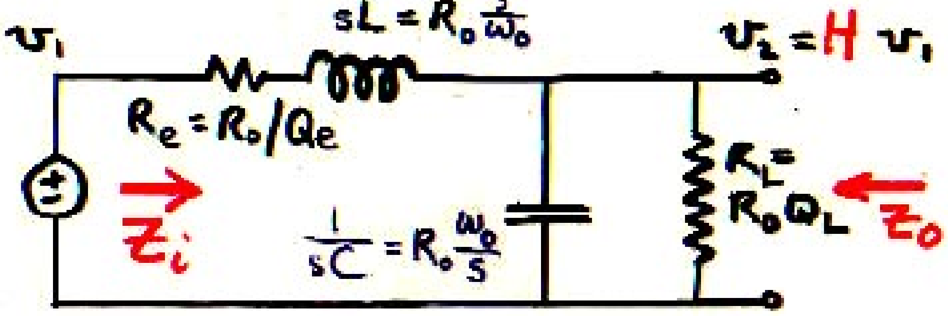
Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

Loaded low-pass LC filter

Note: $R \to R_e$, $Q \to Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

$$H = \frac{\dfrac{R_0 Q_L}{1 + Q_L \frac{s}{\omega_0}}}{\dfrac{R_0 Q_L}{1 + Q_L \frac{s}{\omega_0}} + \dfrac{R_0}{Q_e} + R_0 \dfrac{s}{\omega_0}} = \frac{Q_L}{Q_L + \frac{1}{Q_e} + \left(\frac{Q_L}{Q_e} + 1\right)\left(\frac{s}{\omega_0}\right) + Q_L \left(\frac{s}{\omega_0}\right)^2}$$

$$= \frac{1}{1 + 1/Q_e Q_L} \; \frac{1}{1 + \dfrac{\left(\frac{1}{Q_e} + \frac{1}{Q_L}\right)}{1 + 1/Q_e Q_L}\left(\dfrac{s}{\omega_0}\right) + \dfrac{1}{1 + 1/Q_e Q_L}\left(\dfrac{s}{\omega_0}\right)^2}$$

Result, compared with unloaded case:

1. Low-freq. asymptote is $\frac{1}{1+1/Q_e Q_L} = \frac{R_L}{R_L + R_e}$
(resistive divider)

2. The corner frequency is changed to
$\sqrt{1+1/Q_e Q_L}\,\omega_0$

3. The damping coefficient is changed to
$$\frac{\frac{1}{Q_e} + \frac{1}{Q_L}}{\sqrt{1+1/Q_e Q_L}}$$

For the high-Q case, $Q_e, Q_L \gg 0.5$, $Q_e Q_L \gg 1$ and the first two effects are negligible, and the damping coefficient becomes
$$\frac{1}{Q_e} + \frac{1}{Q_L}$$

Hence, for the high-Q case,

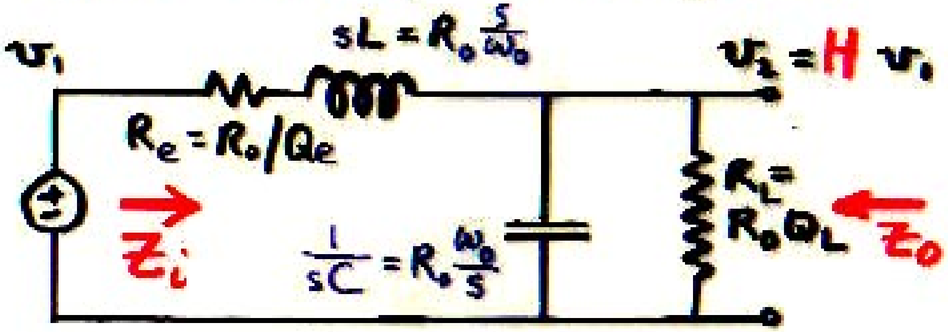
$$H \approx \frac{1}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

where Q_t is a "total" Q-factor given by the "parallel combination"

$$\frac{1}{Q_t} \equiv \frac{1}{Q_e} + \frac{1}{Q_L}$$

Loaded low-pass LC filter

Note: $R \to R_e$, $Q \to Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

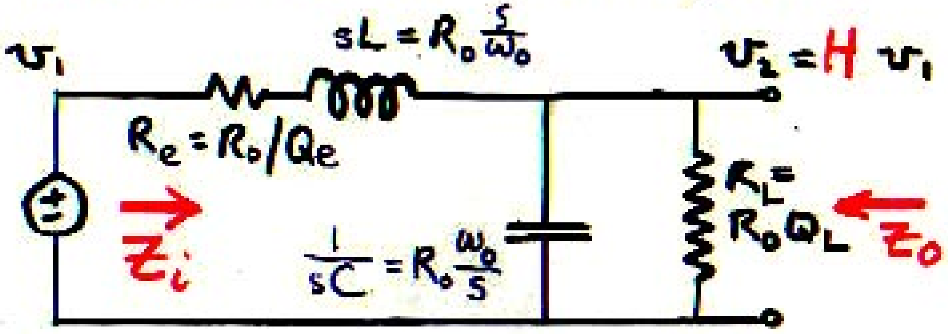
$$H = \frac{\dfrac{R_L}{1+sCR_L}}{\dfrac{R_L}{1+sCR_L} + R_e + sL} = \frac{R_L}{R_L + R_e + s(CR_L R_e + L) + s^2 LCR_L}$$

$$= \frac{R_L}{R_L + R_e} \; \frac{1}{1 + s\left(CR_e + \dfrac{L}{R_L}\right)\dfrac{R_L}{R_L + R_e} + s^2 LC \dfrac{R_L}{R_L + R_e}}$$

$$= \frac{R_L}{R_L + R_e} \; \frac{1}{1 + s\sqrt{LC}\left(R_e\sqrt{\dfrac{C}{L}} + \dfrac{1}{R_L}\sqrt{\dfrac{L}{C}}\right)\dfrac{R_L}{R_L + R_e} + s^2 LC \dfrac{R_L}{R_L + R_e}}$$

$$= \frac{1}{1 + 1/Q_e Q_L} \; \frac{1}{1 + \dfrac{\left(\dfrac{1}{Q_e} + \dfrac{1}{Q_L}\right)}{1 + 1/Q_e Q_L}\left(\dfrac{s}{\omega_0}\right) + \dfrac{1}{1 + 1/Q_e Q_L}\left(\dfrac{s}{\omega_0}\right)^2}$$

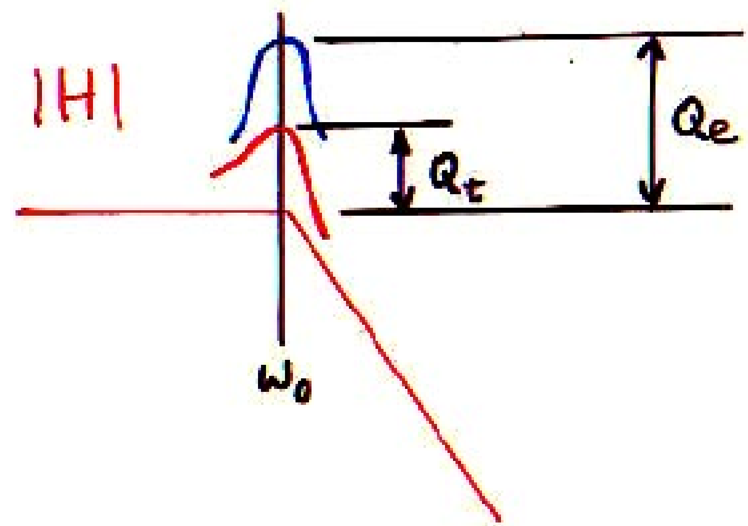
Loaded low-pass LC filter



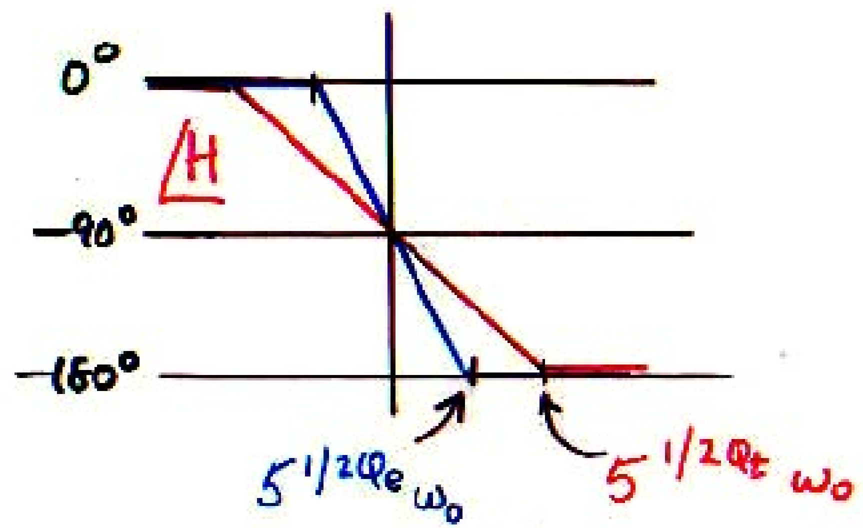
Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$



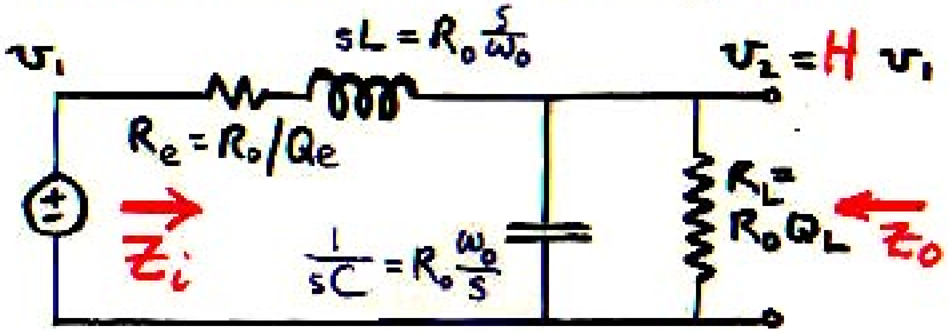
|H|
Qe
Qt
ωo



0°
∠H
-90°
-180°
5^{1/2Qe} ωo
5^{1/2Qt} ωo

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

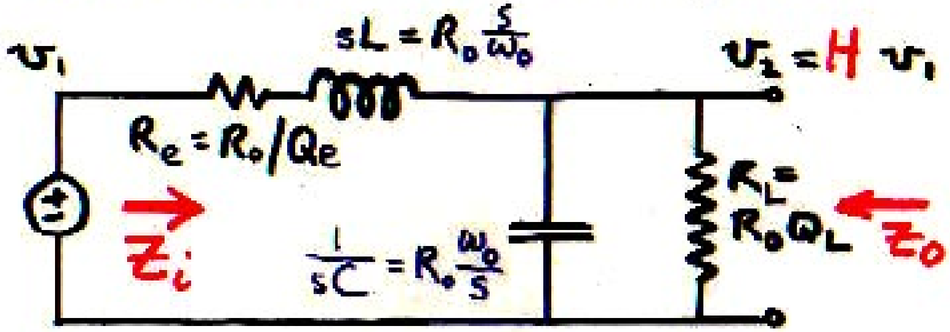
$$Z_i = \frac{R_0 Q_L}{1 + Q_L\left(\frac{s}{\omega_0}\right)} + \frac{R_0}{Q_e} + R_0 \frac{s}{\omega_0} = R_0 \frac{Q_L + \frac{1}{Q_e} + \left(\frac{Q_L}{Q_e} + 1\right)\left(\frac{s}{\omega_0}\right) + Q_L\left(\frac{s}{\omega_0}\right)^2}{1 + Q_L\left(\frac{s}{\omega_0}\right)}$$

$$= R_0\left(1 + 1/Q_e Q_L\right) \frac{1 + \frac{\frac{1}{Q_e} + \frac{1}{Q_L}}{1 + 1/Q_e Q_L}\left(\frac{s}{\omega_0}\right) + \frac{1}{1 + 1/Q_e Q_L}\left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

Same three effects as for H, but with addition of an inverted pole at ω_0/Q_L.

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



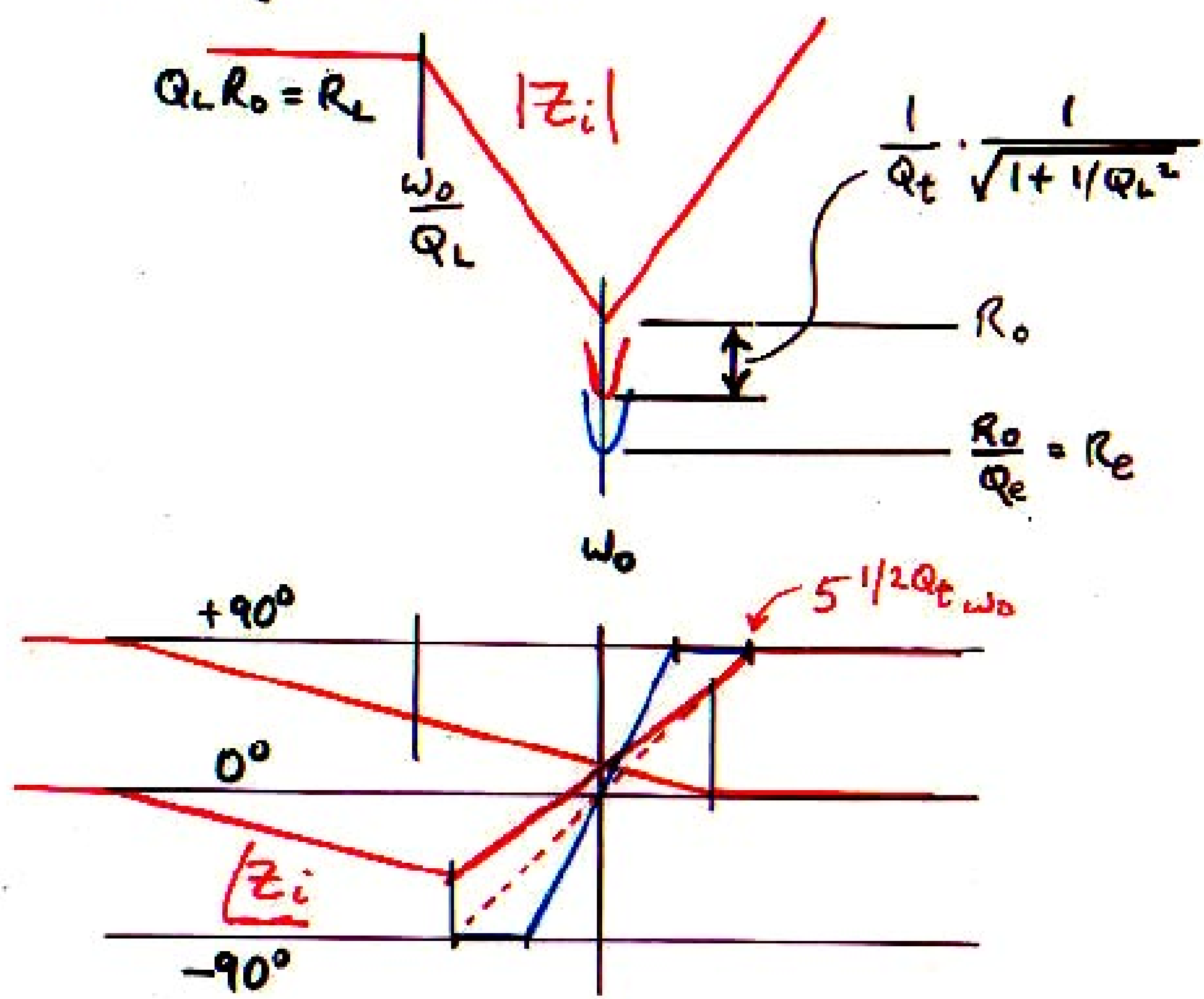
$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

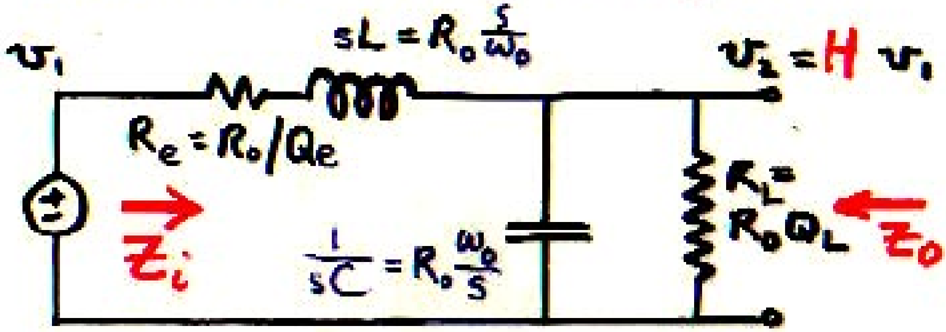
Hence, for the high-Q case,

$$Z_i \approx R_0 \, \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

For high-Q case:



Loaded low-pass LC filter



Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$

$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

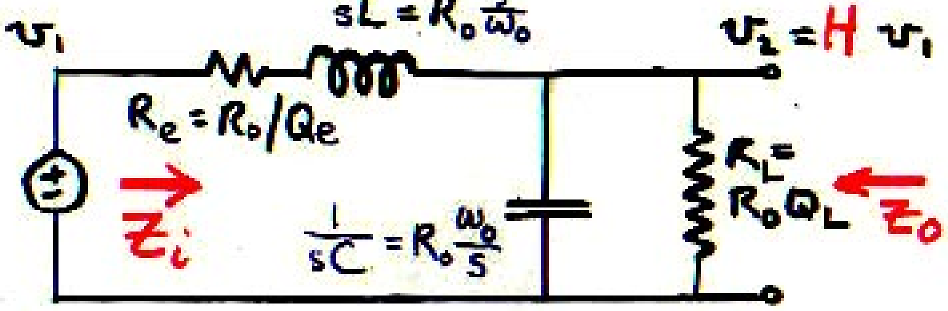
$$Z_o = \frac{\dfrac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)}\left(\dfrac{R_o}{Q_e} + R_o\dfrac{s}{\omega_o}\right)}{\dfrac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)} + \dfrac{R_o}{Q_e} + R_o\dfrac{s}{\omega_o}} = R_o Q_L \frac{\dfrac{1}{Q_e} + \dfrac{s}{\omega_o}}{Q_L + \dfrac{1}{Q_e} + \left(\dfrac{Q_L}{Q_e}+1\right)\left(\dfrac{s}{\omega_o}\right) + \left(\dfrac{s}{\omega_o}\right)^2}$$

$$= R_o \cdot \frac{1}{1+1/Q_eQ_L} \cdot \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_e}{s}\right)}{1 + \dfrac{\left(\frac{1}{Q_e}+\frac{1}{Q_L}\right)}{1+1/Q_eQ_L}\left(\dfrac{s}{\omega_o}\right) + \dfrac{1}{1+1/Q_eQ_L}\left(\dfrac{s}{\omega_o}\right)^2}$$

Same three effects as for H, so for high-Q case

$$Z_o \approx R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_e}{s}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Loaded low-pass LC filter

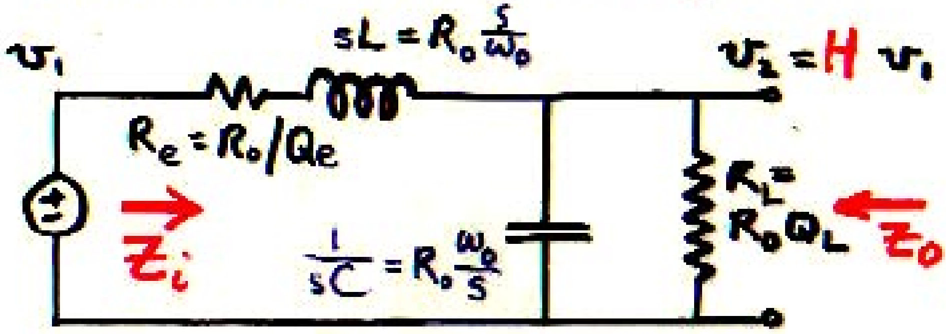


Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

Loaded low-pass LC filter



Note: $R \to R_e$, $Q \to Q_e$

$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

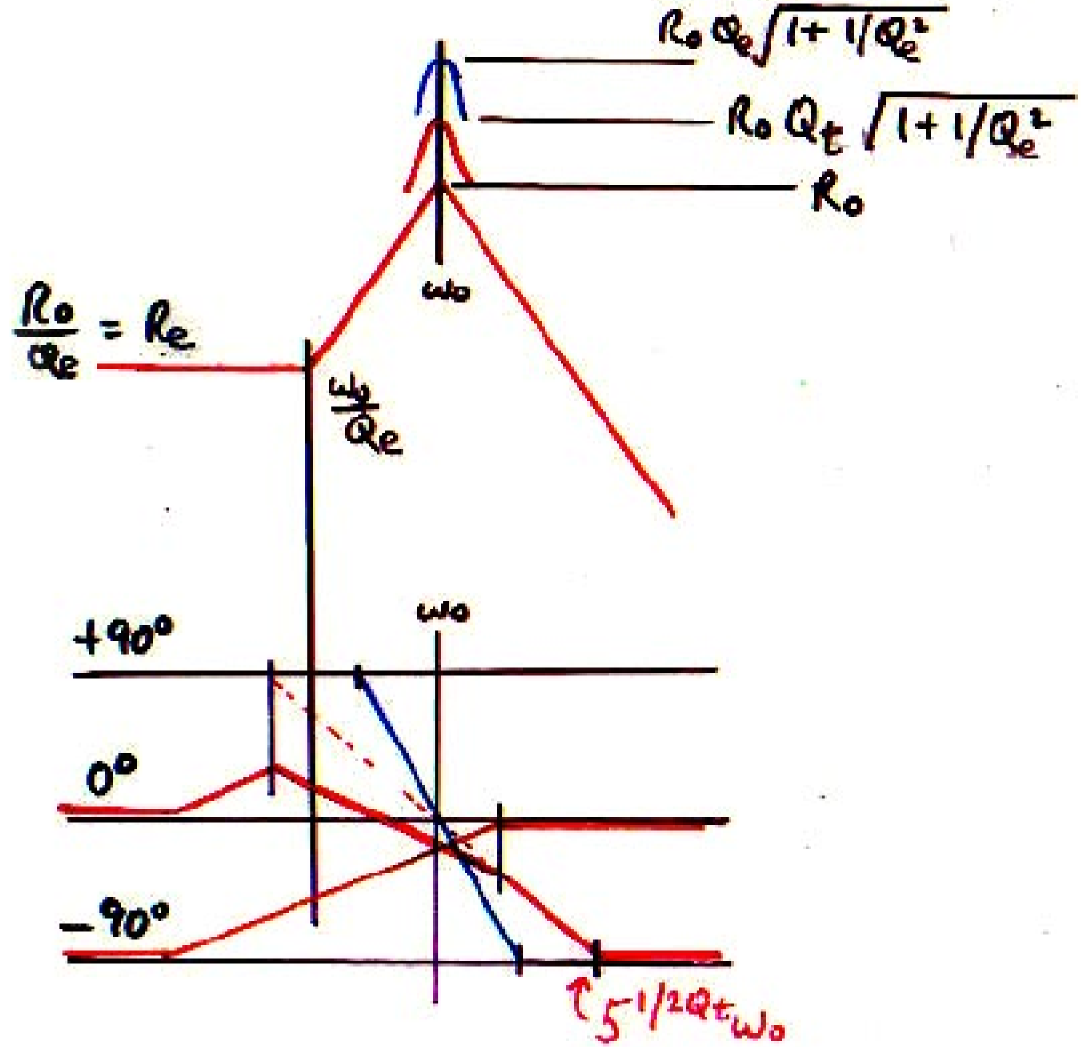
$$Z_o = \frac{\dfrac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)}\left(\dfrac{R_o}{Q_e} + R_o\dfrac{s}{\omega_o}\right)}{\dfrac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)} + \dfrac{R_o}{Q_e} + R_o\dfrac{s}{\omega_o}} = R_o Q_L \frac{\dfrac{1}{Q_e} + \dfrac{s}{\omega_o}}{Q_L + \dfrac{1}{Q_e} + \left(\dfrac{Q_L}{Q_e}+1\right)\left(\dfrac{s}{\omega_o}\right) + \left(\dfrac{s}{\omega_o}\right)^2}$$

$$= R_o \cdot \frac{1}{1+1/Q_eQ_L} \cdot \frac{\left(\frac{s}{\omega_o}\right)\left(1+\dfrac{\omega_o/Q_e}{s}\right)}{1 + \dfrac{\left(\frac{1}{Q_e}+\frac{1}{Q_L}\right)}{1+1/Q_eQ_L}\left(\dfrac{s}{\omega_o}\right) + \dfrac{1}{1+1/Q_eQ_L}\left(\dfrac{s}{\omega_o}\right)^2}$$

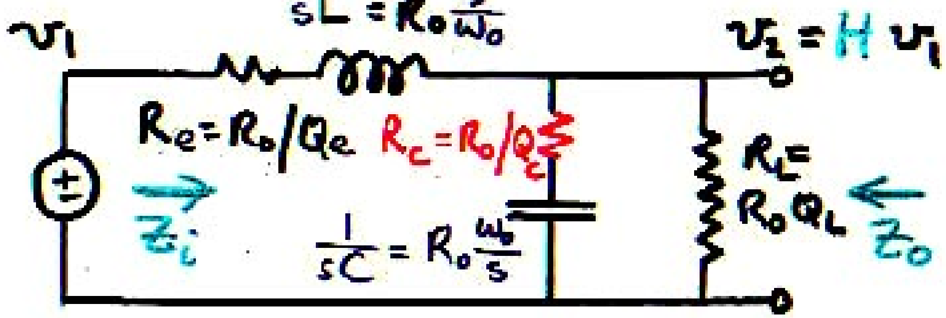
Same three effects as for H, so for high-Q case

$$Z_o \approx R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1+\dfrac{\omega_o/Q_e}{s}\right)}{1 + \dfrac{1}{Q_t}\left(\dfrac{s}{\omega_o}\right) + \left(\dfrac{s}{\omega_o}\right)^2}$$

For high-Q case:



Consider additional damping: Capacitor esr R_c



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

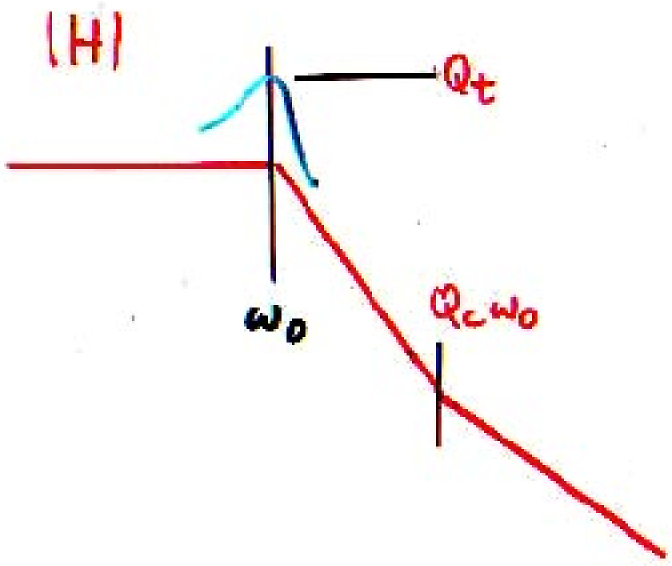
$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0} \qquad Q_c \equiv \frac{R_0}{R_c}$$

For the high-Q case, the previous results can be extended by inspection:

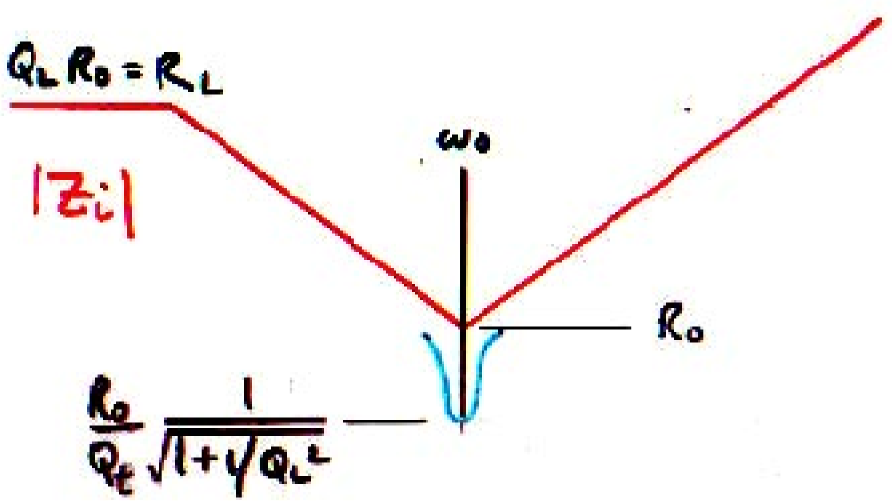
$$H = \frac{1 + \frac{1}{Q_c}\left(\frac{s}{\omega_0}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \Longleftarrow 1 + sR_cC$$

$$\frac{1}{Q_t} \equiv \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

$$Z_i = R_0 \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$



|H|
Qt
ω0
QLω0



QL R0 = RL
|Zi|
ω0
R0
R0/Qt · 1/√(1+1/QL²)

Principle for extension of results to a more complicated case:

1. Determine the new total Q_t.
2. Add any additional pole or zero factors (Is there any change in the $\omega \to 0$ or $\omega \to \infty$ asymptotes?)

Exercise:

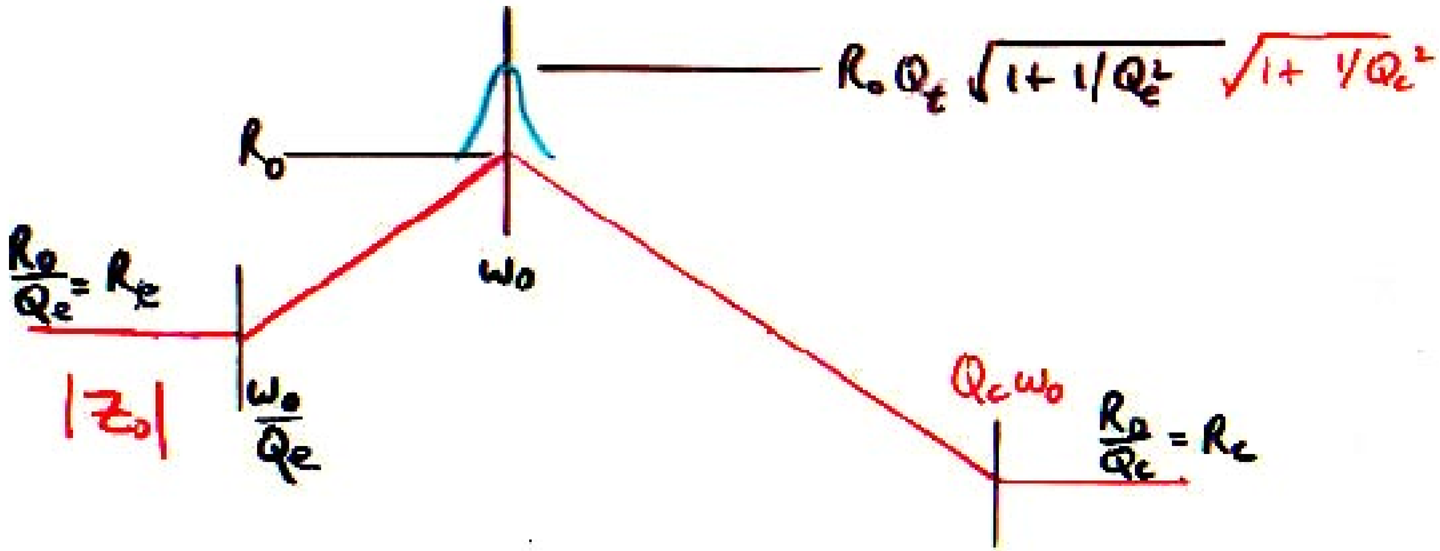
Obtain the corresponding results for Z_o.

Exercise solution:

$$Z_o = R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_e}{s}\right)\left(1 + \frac{1}{Q_c}\frac{s}{\omega_o}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Check high-frequency limit:

$$Z_o \xrightarrow{\omega \to \infty} \frac{R_o}{Q_c} = R_c$$



The key step is now to determine T12 and T22 from the small signal model for the condition $\hat{vi} = 0$:

$$T12 = \left[\frac{VO \times (RL - D^2 \times N^2 \times Z) \times (1 + sC \times RC)}{(D \times N \times \Delta)} \right] \quad [15]$$

$$T22 = \left[\frac{VO \times (RL - D^2 \times N^2 \times Z) \times (1 + sC \times RL)}{(D \times N \times RL \times \Delta)} \right] \quad [16]$$

where

$$Z = \left[\frac{\overbrace{(s^2L1 \times C1 \times RC1) + (sC1 \times RC1 \times RL1) + sL1 + RL1}^{(sL_1 + R_{L1})(sC_1 R_{C1} + 1)}}{(s^2L1 \times C1) + sC1 \times (RL1 + RC1) + 1} \right] \quad [17]$$

$$\Delta = \left[a_1 + (D^2 \times N^2 \times Z) \times (1 + sC \times RL) \right] \quad [18]$$

$$a_1 = \left[(s^2L \times C \times RL) + sC \times RL \times \left(RI + RC + \frac{L}{(C \times RL)}\right) + RL \right] \quad [19]$$

At the resonant frequency of the input filter, the impedance Z will attain a very high value, limited only by the series resistances RL1 and RC1. The peaking in the value of Z will affect both the numerators and denominators of the transfer functions T12 and T22, as shown in equations 15 and 16. The net effect will be a reduction in the loop gain G_T and a corresponding phase margin reduction.