10

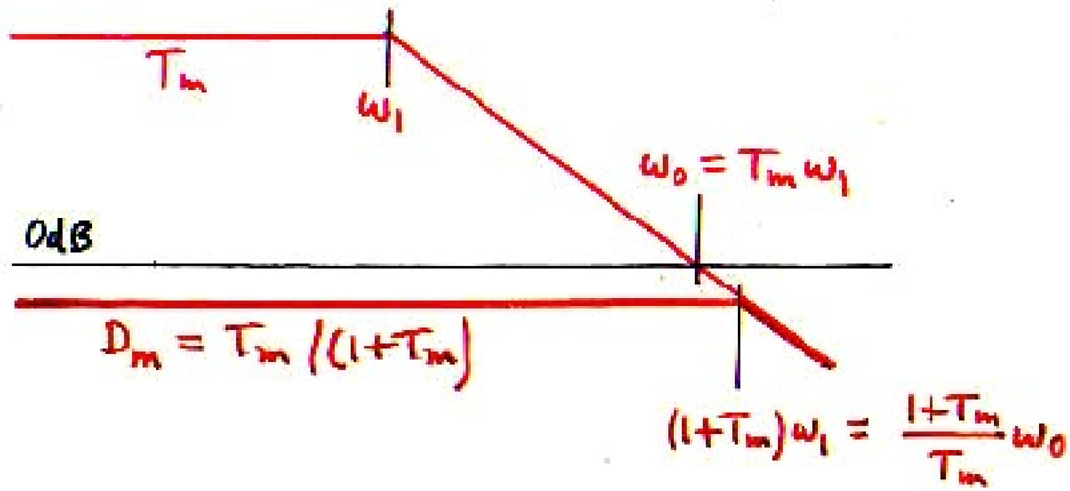
HOW MUCH PHASE MARGIN IS NECESSARY?

How much phase margin is necessary?

Determine the effect of the phase margin ϕ_m on the closed loop gain G. Since $G = G_\infty D$, the discrepancy factor $D = T/(1+T)$ is the parameter of interest.

Consider the relation between D and T.

1-pole response:

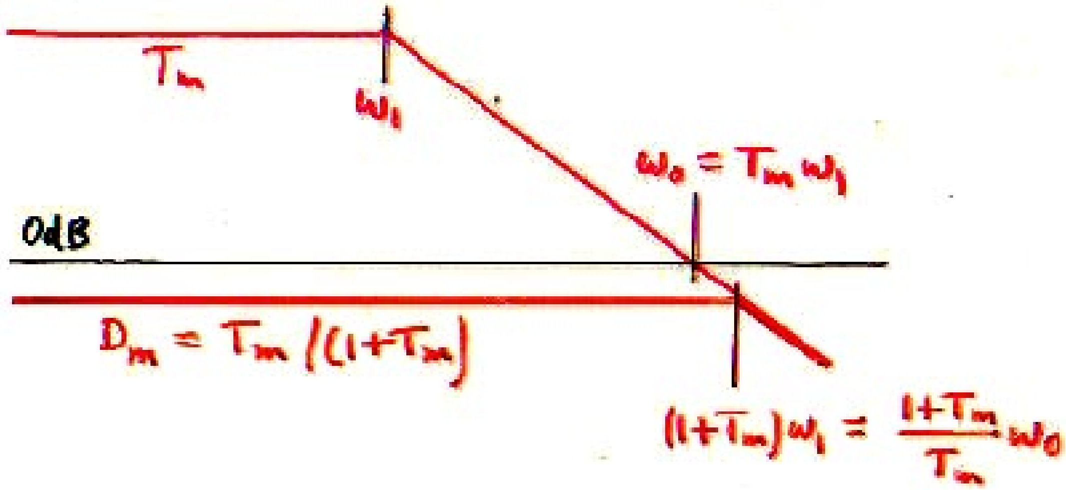


$$T = T_m \frac{1}{1+\frac{s}{\omega_1}} = \frac{1}{\frac{s}{\omega_0}\left(1+\frac{\omega_0}{T_m s}\right)}$$

$$D = \frac{T_m}{1+T_m}\frac{1}{1+\frac{s}{(1+T_m)\omega_1}} = \frac{T_m}{1+T_m}\frac{1}{1+\frac{T_m s}{(1+T_m)\omega_0}}$$

The phase margin is at least 90°, and the discrepancy factor also has one pole.

1-pole response:

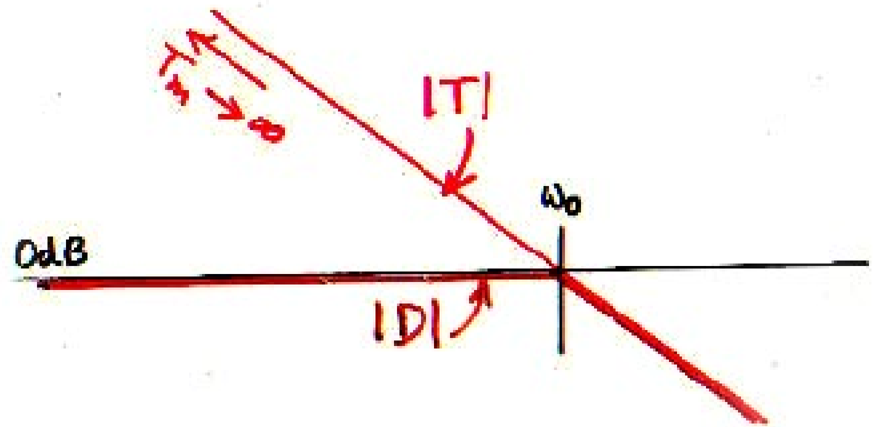


$$T = T_m \frac{1}{1+\frac{s}{\omega_1}} = \frac{1}{\frac{s}{\omega_0}\left(1+\frac{\omega_0}{T_m s}\right)} \xrightarrow{T_m\to\infty} \frac{1}{\frac{s}{\omega_0}}$$

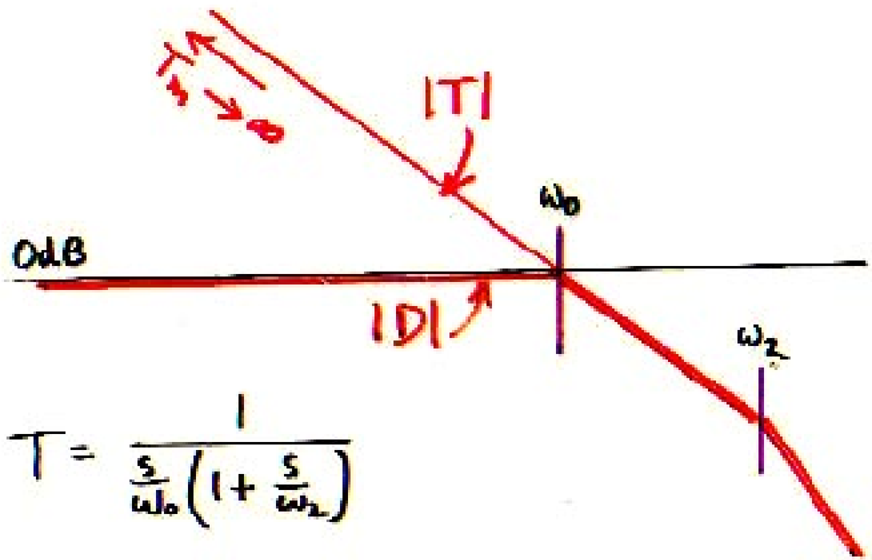
$$D = \frac{T_m}{1+T_m}\frac{1}{1+\frac{s}{(1+T_m)\omega_1}} = \frac{T_m}{1+T_m}\frac{1}{1+\frac{T_m s}{(1+T_m)\omega_0}} \xrightarrow{T_m\to\infty} \frac{1}{1+\frac{s}{\omega_0}}$$

The phase margin is at least 90°, and the discrepancy factor also has one pole.

In the limiting case $T_m \to \infty$, the phase margin is 90°; and D still has one pole.



|T|
0dB
|D|
ω_0



$$T = \frac{1}{\frac{s}{\omega_0}\left(1+\frac{s}{\omega_2}\right)}$$

If ω_2 is much higher than ω_0, obviously D has the same second pole.

Investigate what happens to D if ω_2 is close to, or even below, ω_0:

$$D = \frac{T}{1+T} = \frac{1}{1+\frac{s}{\omega_0}\left(1+\frac{s}{\omega_2}\right)} = \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_c}\right)+\left(\frac{s}{\omega_c}\right)^2}$$

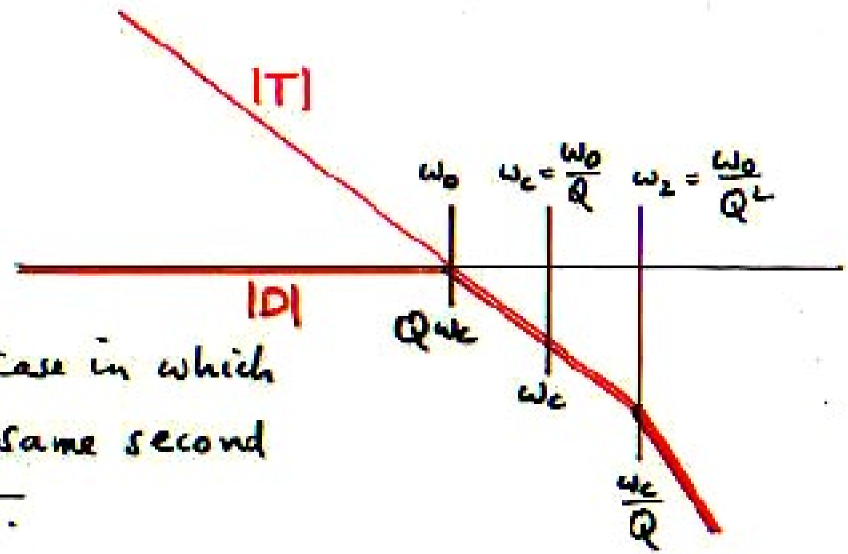
where

$$\omega_c \equiv \sqrt{\omega_0\omega_2} \qquad Q \equiv \sqrt{\frac{\omega_0}{\omega_2}}$$

Consider ω_0 fixed, and ω_2 variable (Q variable)

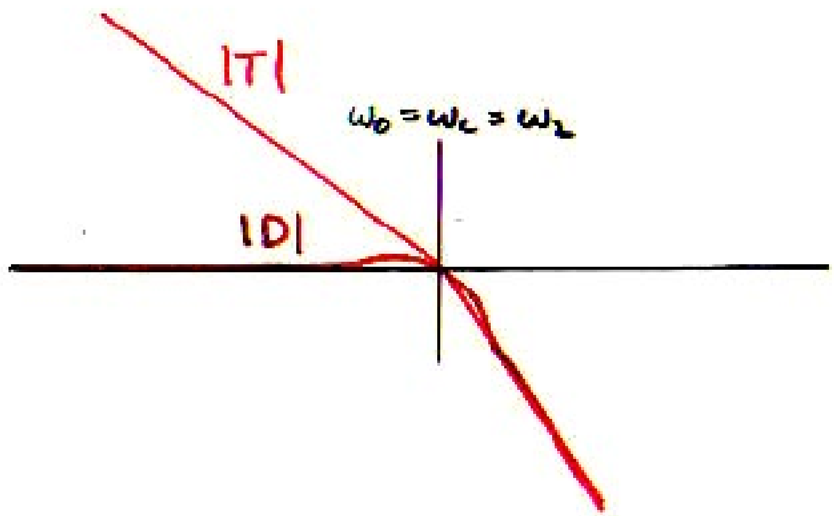
$$\omega_2 = \frac{\omega_0}{Q^2} \qquad \omega_c = \frac{\omega_0}{Q}$$

$Q \ll 0.5$:

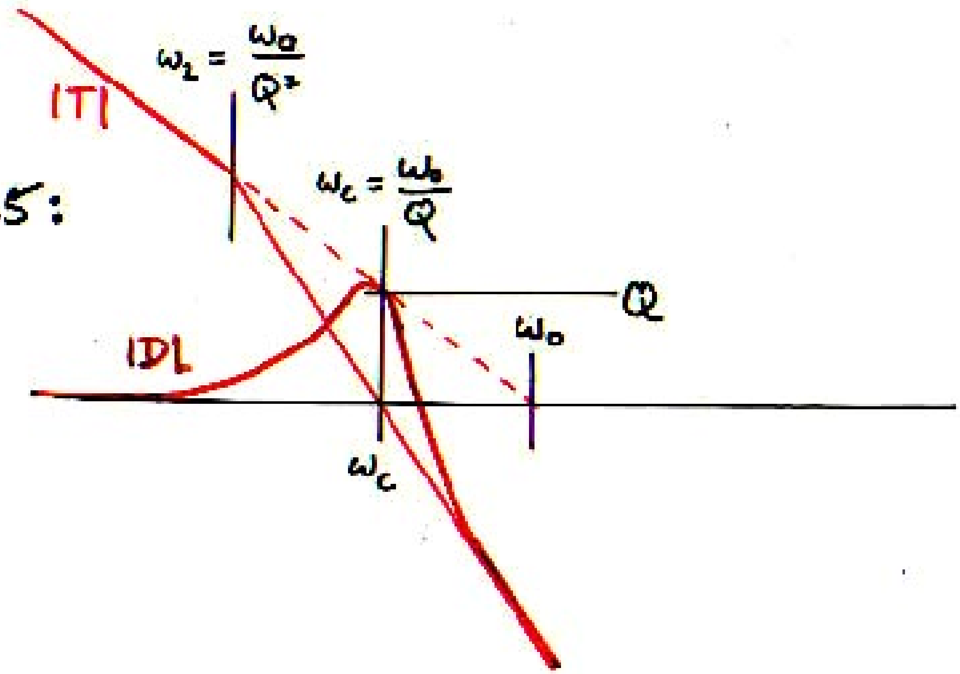


This is the case in which D has the same second pole as T.

$Q = 1$:

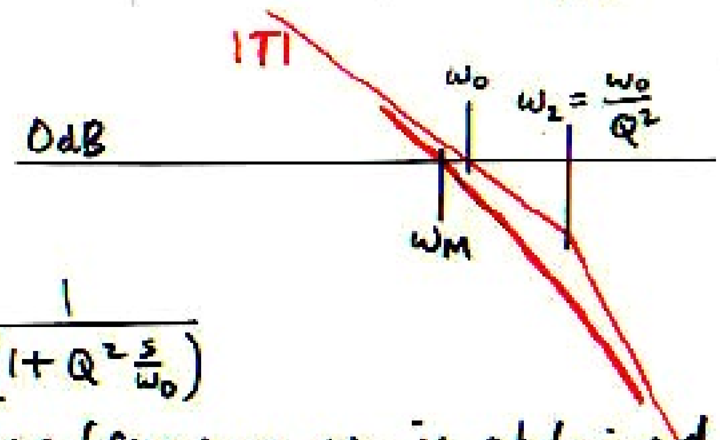


$Q \gg 0.5$:



Hence: discrepancy factor D peaks if second pole is not sufficiently far above ω_0, and D is characterized by the Q-factor of its quadratic, which is related to ω_2 (and ω_0) by $Q = \sqrt{\frac{\omega_0}{\omega_2}}$.

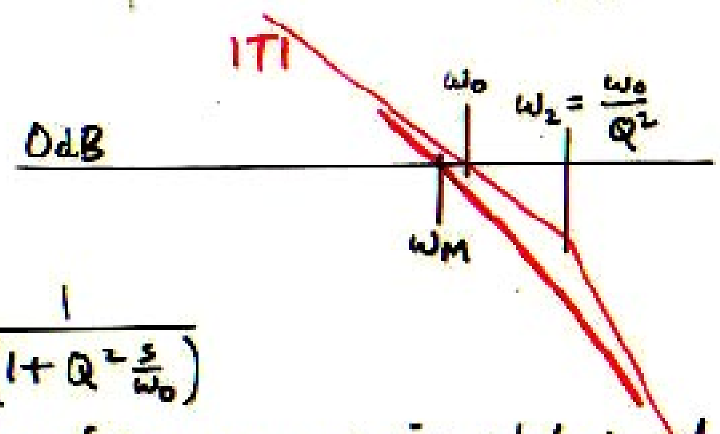
Now, investigate what happens to the phase margin ϕ_M if ω_2 is close to, or even below, ω_0: that is, investigate the relation between Q and ϕ_M.



$$T = \frac{1}{\frac{s}{\omega_0}\left(1 + Q^2 \frac{s}{\omega_0}\right)}$$

The crossover frequency ω_M is obtained by setting $|T| = 1$:

Now, investigate what happens to the phase margin ϕ_M if ω_z is close to, or even below, ω_0: that is, investigate the relation between Q and ϕ_M.



$$T = \frac{1}{\frac{s}{\omega_0}\left(1 + Q^2 \frac{s}{\omega_0}\right)}$$

The crossover frequency ω_M is obtained by setting $|T| = 1$:

$$1 = \frac{1}{\left(\frac{\omega_M}{\omega_0}\right)^2 \left[1 + Q^4 \left(\frac{\omega_M}{\omega_0}\right)^2\right]}$$

$$Q^4 \left(\frac{\omega_M}{\omega_0}\right)^4 + \left(\frac{\omega_M}{\omega_0}\right)^2 - 1 = 0$$

$$\left(\frac{\omega_M}{\omega_0}\right)^2 = -\frac{c}{b}\frac{1}{F} = \frac{1}{\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4Q^4}}$$

$$\frac{\omega_M}{\omega_0} = \sqrt{\frac{2}{1 + \sqrt{1 + 4Q^4}}}$$

The phase margin ϕ_M is found from $\angle T$ evaluated at the crossover frequency ω_M:

$$\phi_M \equiv 180^\circ + \angle T\big|_{\omega_M}$$

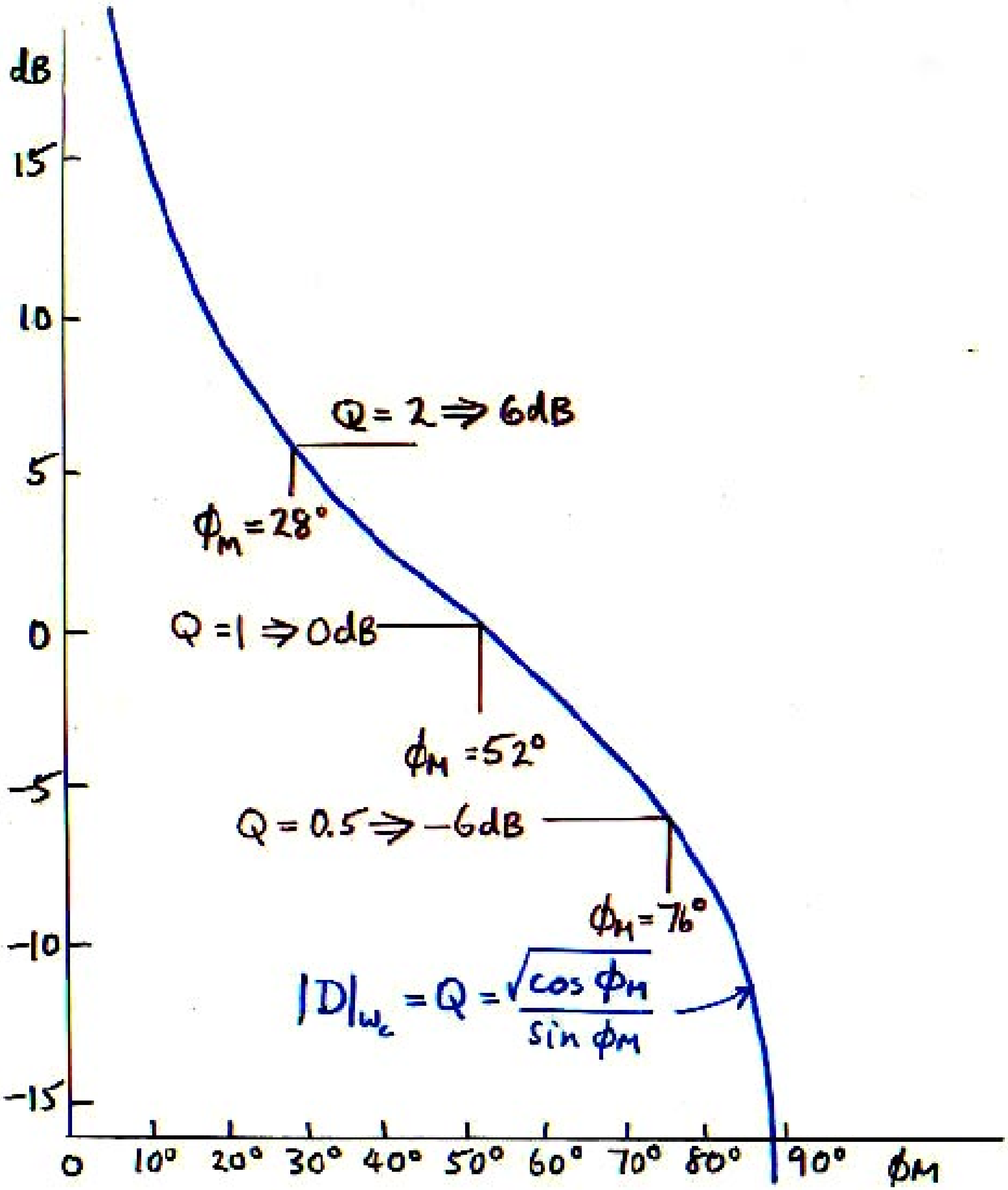
$$= 180 + \left(-90^\circ - \tan^{-1}\frac{\omega_M}{\omega_0/Q^2}\right)$$

which leads to

$$\phi_M = \tan^{-1}\sqrt{\frac{1+\sqrt{1+4Q^4}}{2Q^4}}$$

By inversion:

$$Q = \frac{\sqrt{\cos\phi_M}}{\sin\phi_M}$$

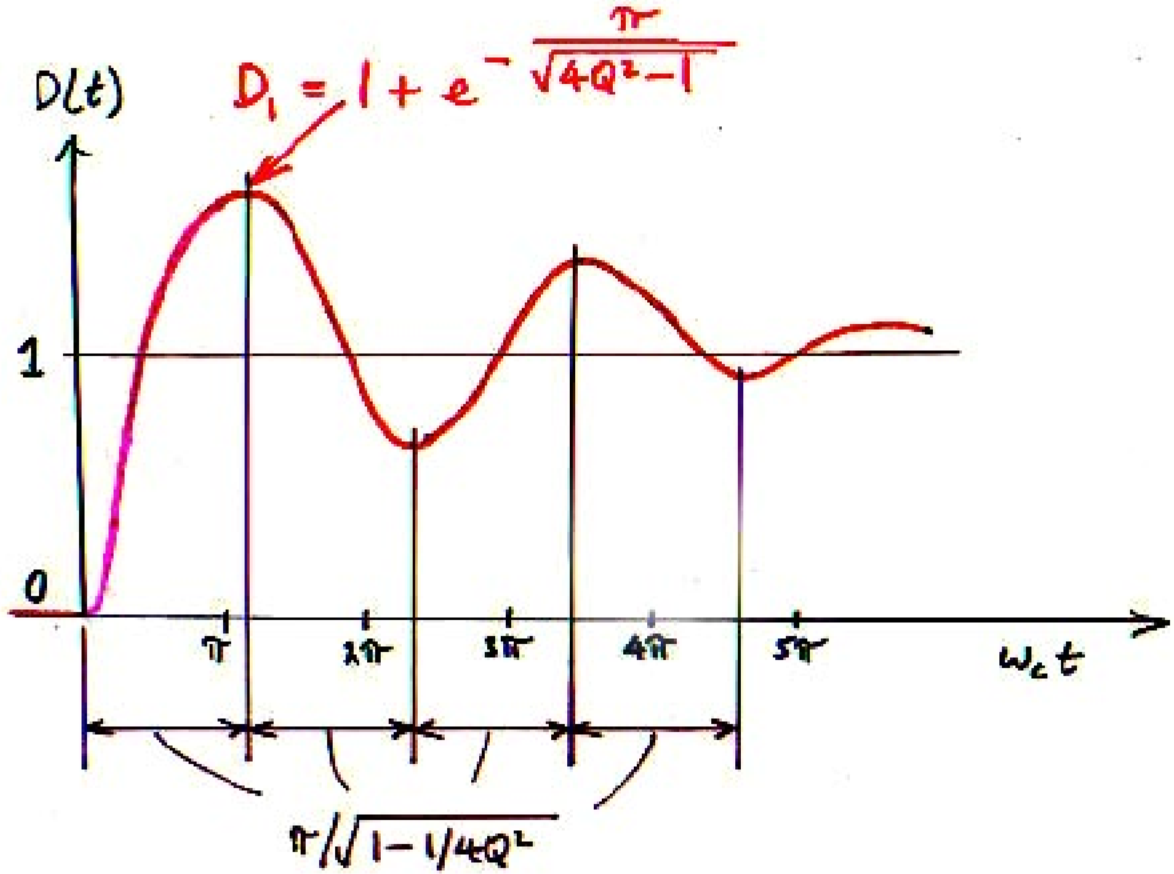


dB
15
10
5
0
-5
-10
-15
0
10°
20°
30°
40°
50°
60°
70°
80°
90°
ϕ_M
$Q = 2 \Rightarrow 6\,\mathrm{dB}$
$\phi_M = 28°$
$Q = 1 \Rightarrow 0\,\mathrm{dB}$
$\phi_M = 52°$
$Q = 0.5 \Rightarrow -6\,\mathrm{dB}$
$\phi_M = 76°$
$|D|_{\omega_c} = Q = \frac{\sqrt{\cos\phi_M}}{\sin\phi_M}$

A phase margin ϕ_M less than 76° causes complex roots in D, and therefore in the closed-loop gain G. In the frequency domain, this results in peaked high-frequency response; in the time domain, it results in transient overshoot. For constant G_∞, $G \propto D$ and the response to a step input is

$$D(t) = \mathcal{L}^{-1} \frac{1}{s} \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_c}\right) + \left(\frac{s}{\omega_c}\right)^2}$$

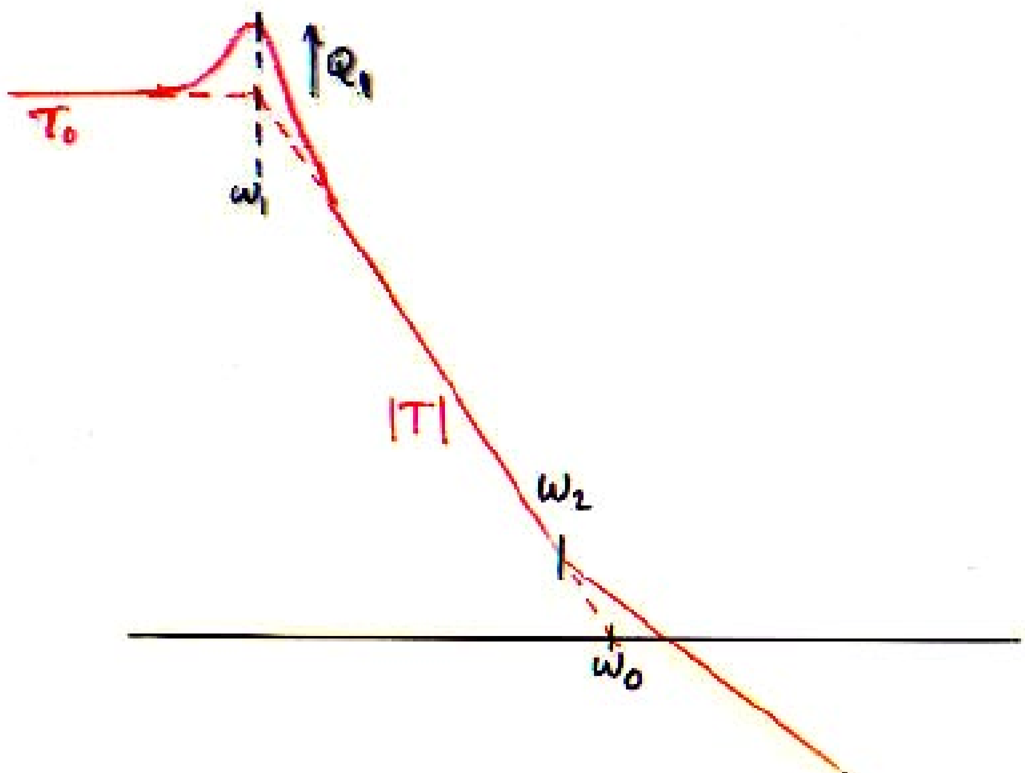
$$\stackrel{Q > 0.5}{=} 1 - \frac{1}{\sqrt{1 - 1/4Q^2}} e^{-\frac{\omega_c t}{2Q}} \sin\left[\sqrt{1-1/4Q^2}\,\omega_c t + \sin^{-1}\sqrt{1-1/4Q^2}\right]$$



Q	ϕ_M	D_1
$1 \Rightarrow 0$dB	52°	1.16 : 16% overshoot
$2 \Rightarrow 6$dB	28°	1.44 : 44% overshoot

Exercise

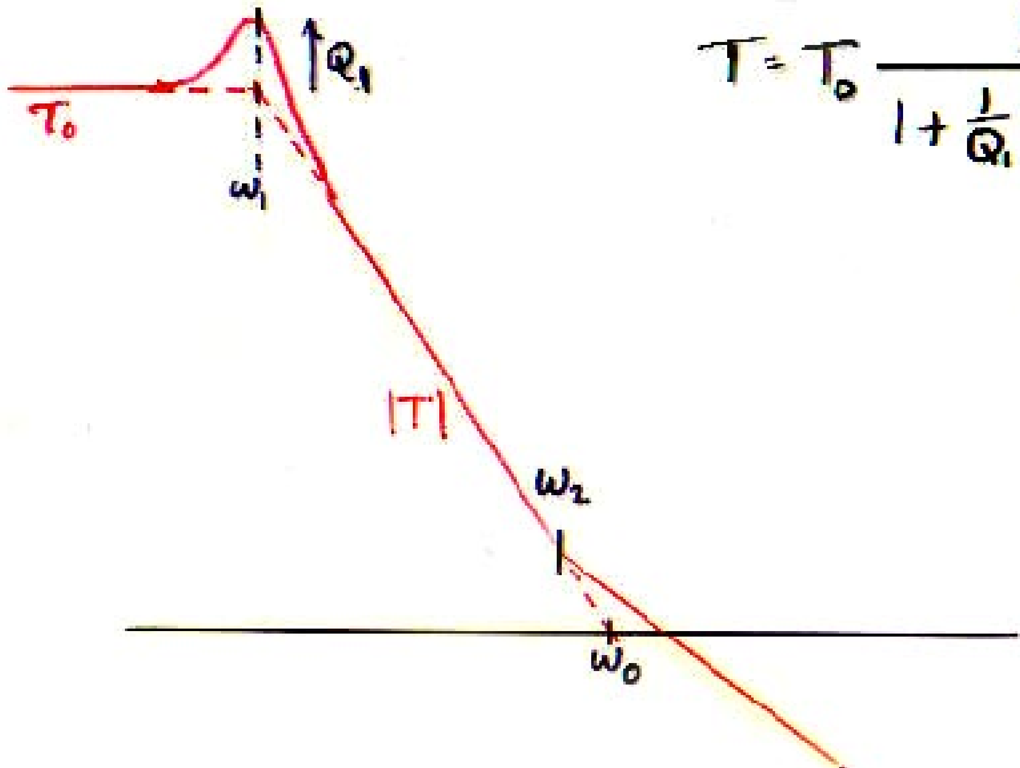
Consider a loop gain that approaches extrapolated crossover frequency ω_0 at a double slope, -40dB/dec, and has a zero at ω_2:



Express the discrepancy factor $D = T/(1+T)$ in normalized form, and identify a Q.

Exercise

Consider a loop gain that approaches extrapolated crossover frequency ω_0 at a double slope, -40dB/dec, and has a zero at ω_2:



$$T = T_0 \frac{1+\frac{s}{\omega_2}}{1+\frac{1}{Q_1}\left(\frac{s}{\omega_1}\right)+\left(\frac{s}{\omega_1}\right)^2}$$

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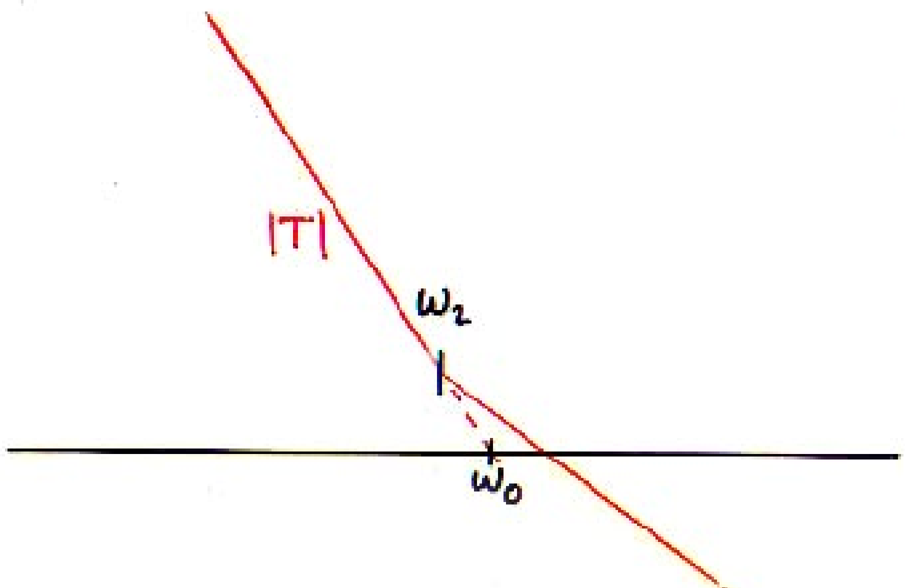
Consider a loop gain that approaches extrapolated crossover frequency ω_0 at a double slope, -40dB/dec, and has a zero at ω_z:



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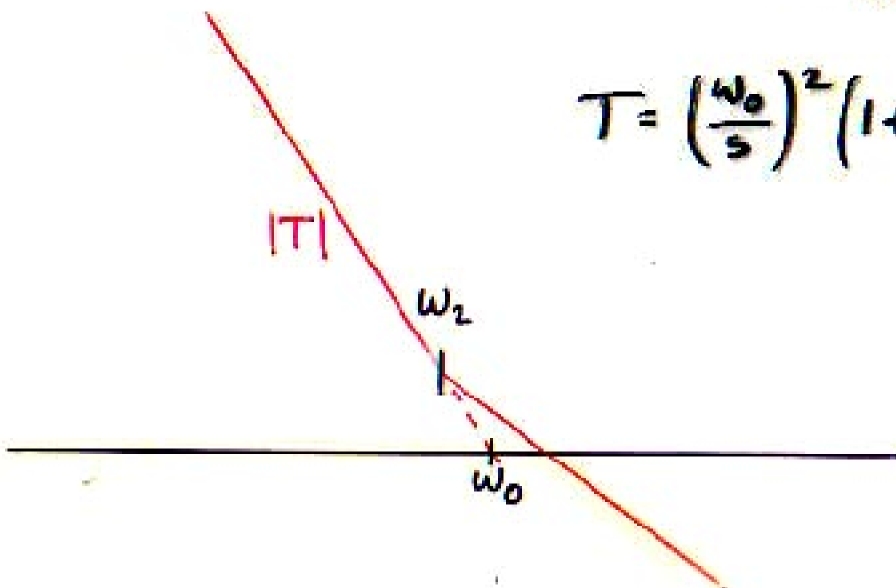

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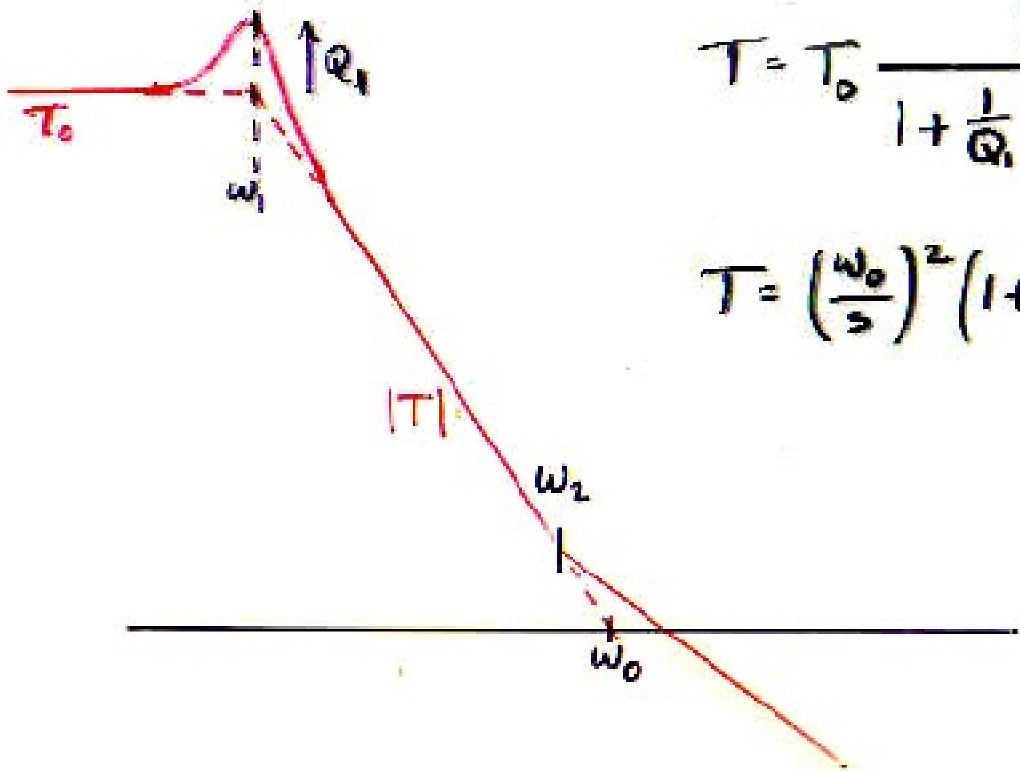

$$T = T_0 \frac{1+\frac{s}{\omega_2}}{1+\frac{1}{Q_1}\left(\frac{s}{\omega_1}\right)+\left(\frac{s}{\omega_1}\right)^2}$$

$$T = \left(\frac{\omega_0}{s}\right)^2\left(1+\frac{s}{\omega_2}\right)$$

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$$T = \left(\frac{\omega_0}{s}\right)^2 \left(1 + \frac{s}{\omega_2}\right)$$

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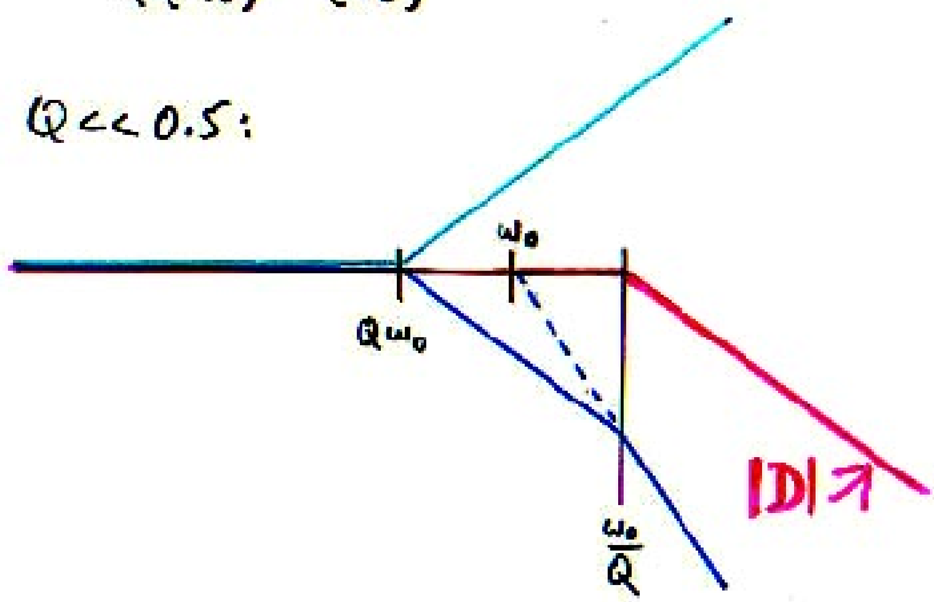
$$= \frac{1+\frac{1}{Q}\frac{s}{\omega_0}}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2} \qquad \text{where } Q \equiv \frac{\omega_2}{\omega_0}$$

$Q << 0.5$:



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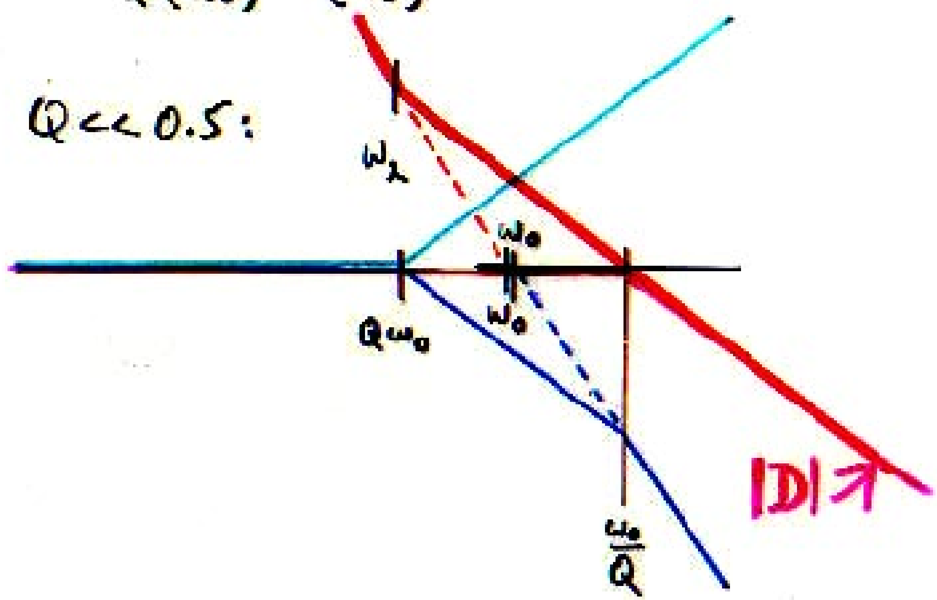

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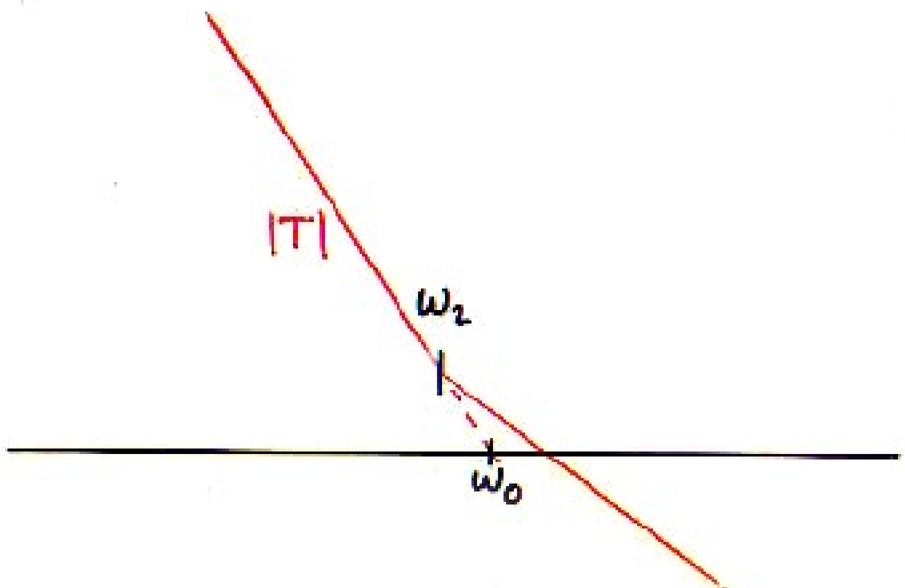

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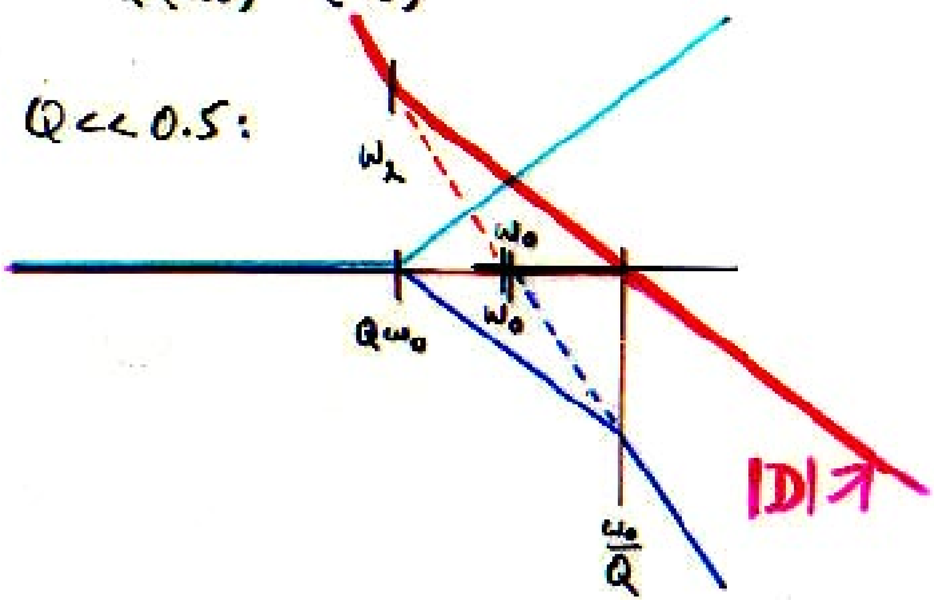
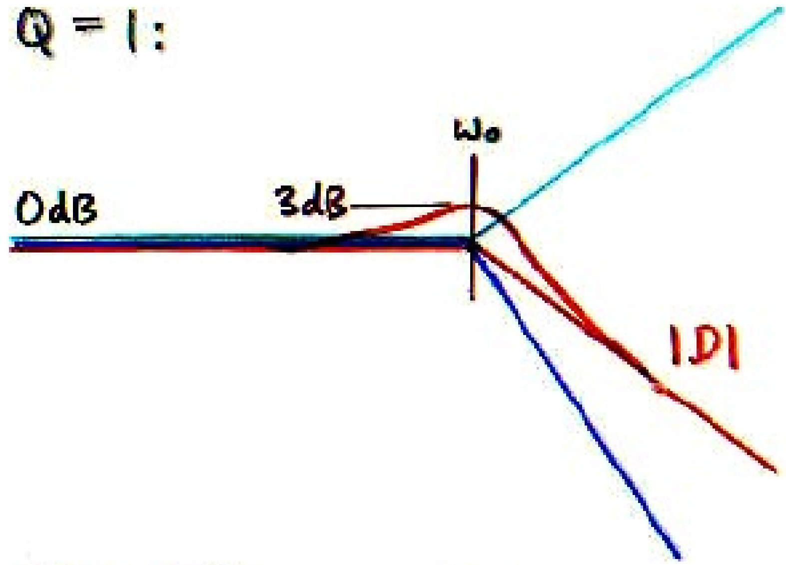
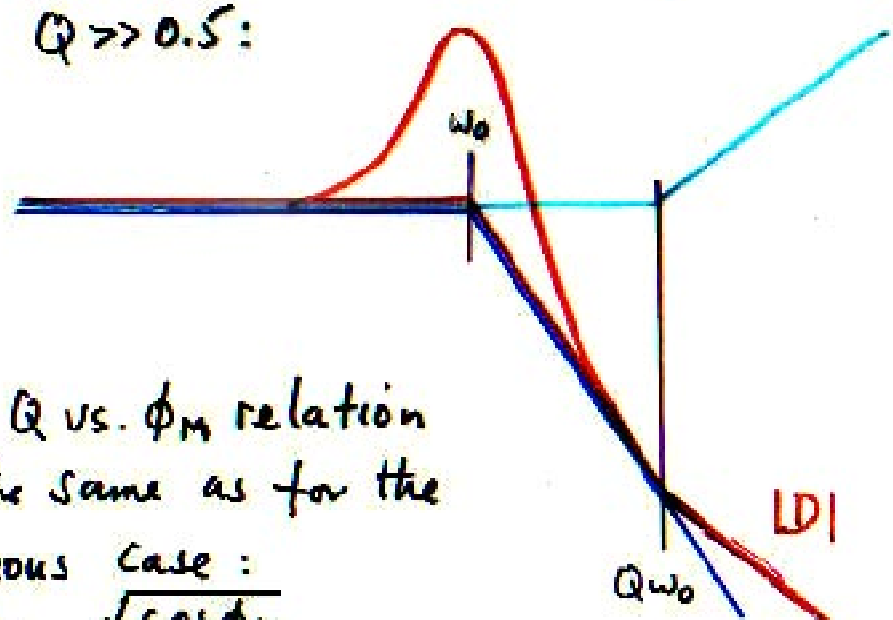


Q = 1:



Q >> 0.5:



The Q vs. ϕ_M relation is the same as for the previous case:

$$Q = \frac{\sqrt{\cos\phi_M}}{\sin\phi_M}$$

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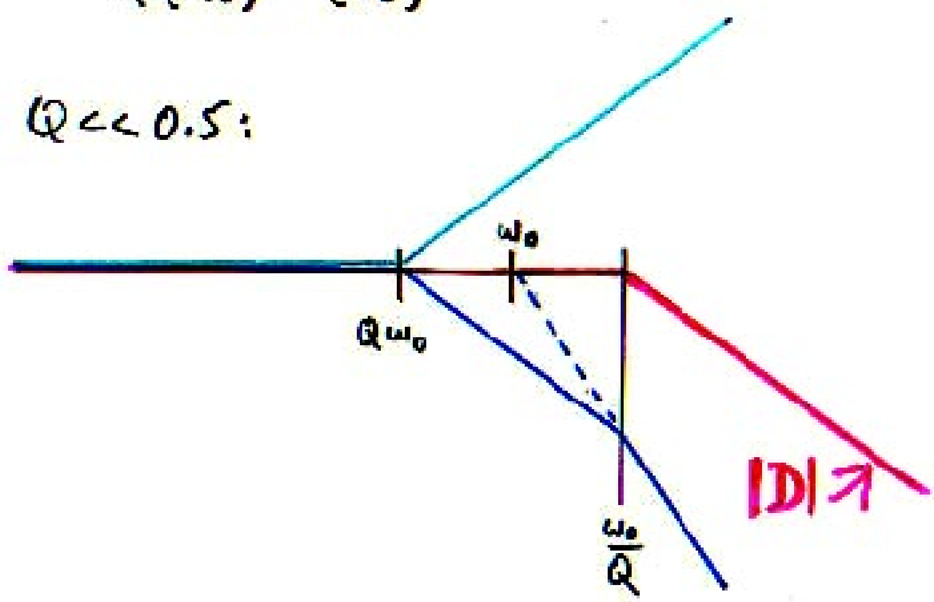

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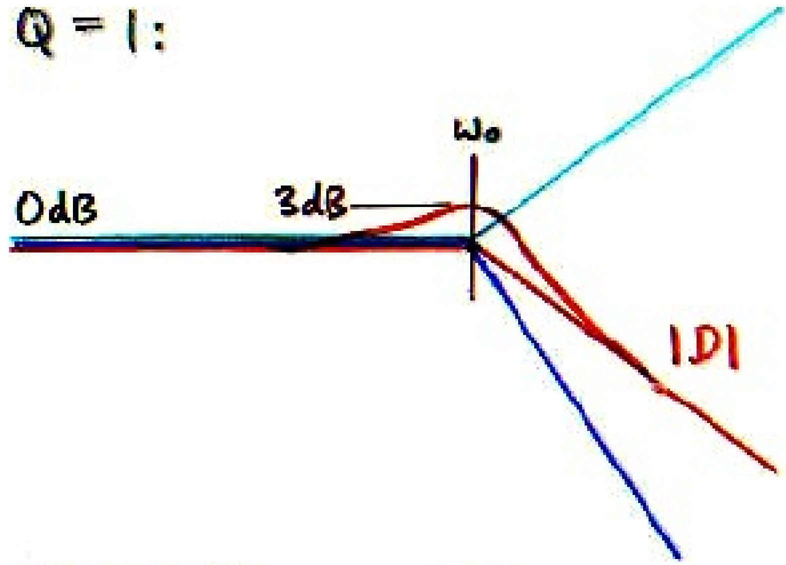

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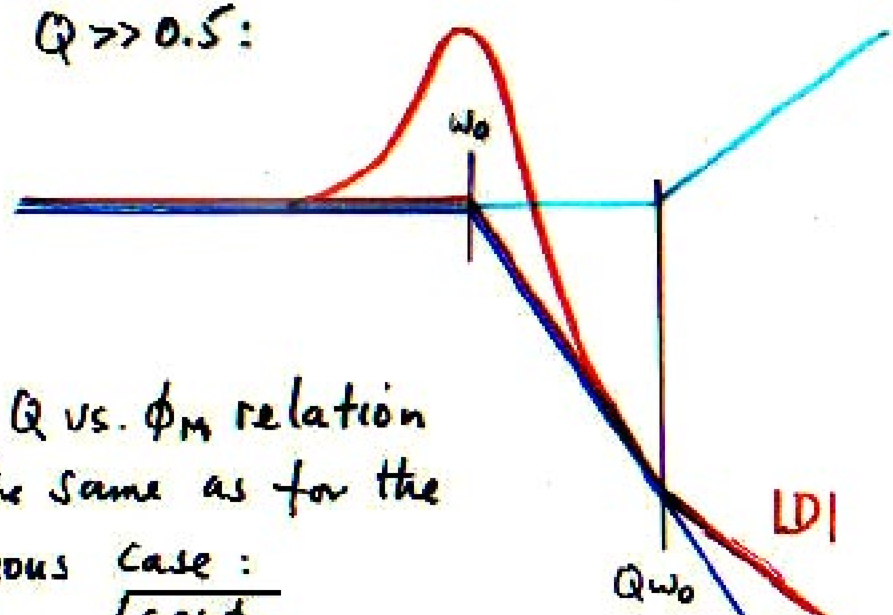

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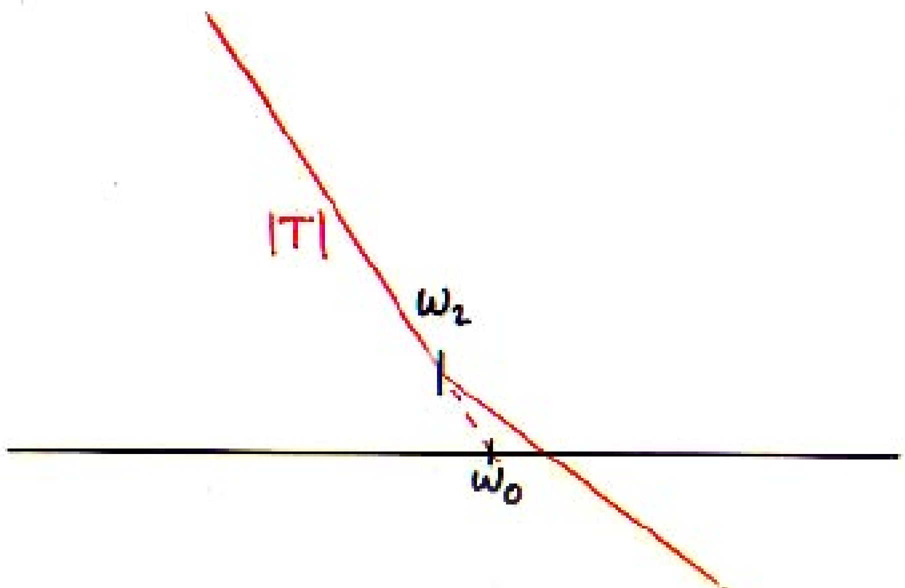
Consider a loop gain that approaches extrapolated crossover frequency ω_0 at a double slope, -40dB/dec, and has a zero at ω_z:



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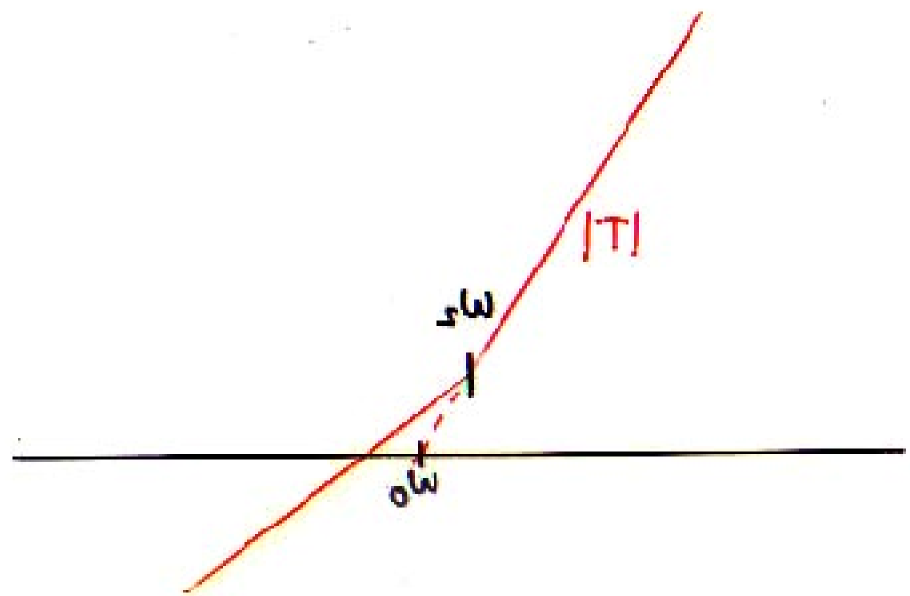

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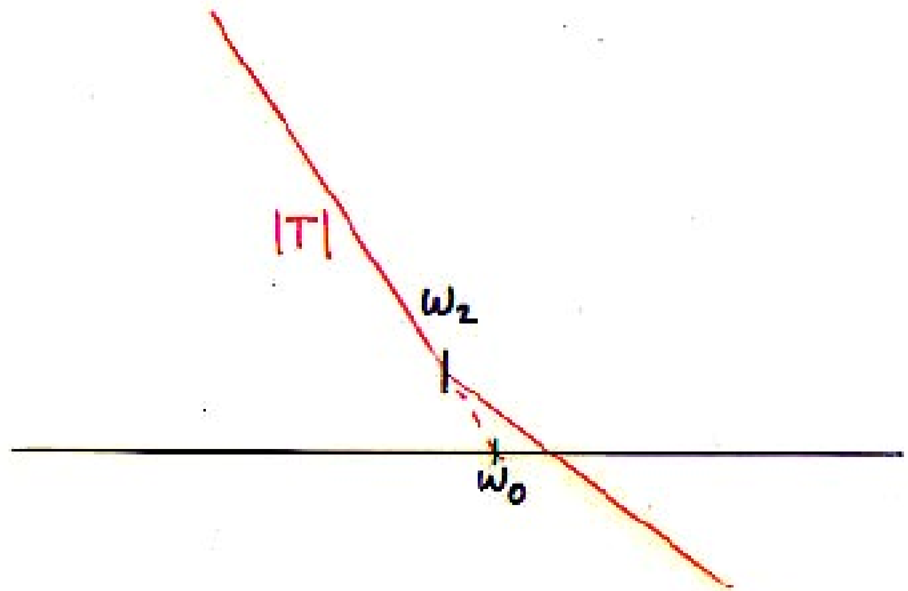

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Generalization: How Much Phase Margin is Needed?

Depends on two considerations:

1. Effect of phase margin ϕ_M on closed-loop response via the Discrepancy Factor D. Too small a ϕ_M causes peaking in D.

2. The sensitivity of ϕ_M to variations (worst-case). Avoid making ϕ_M strongly dependent on highly variable parameters.

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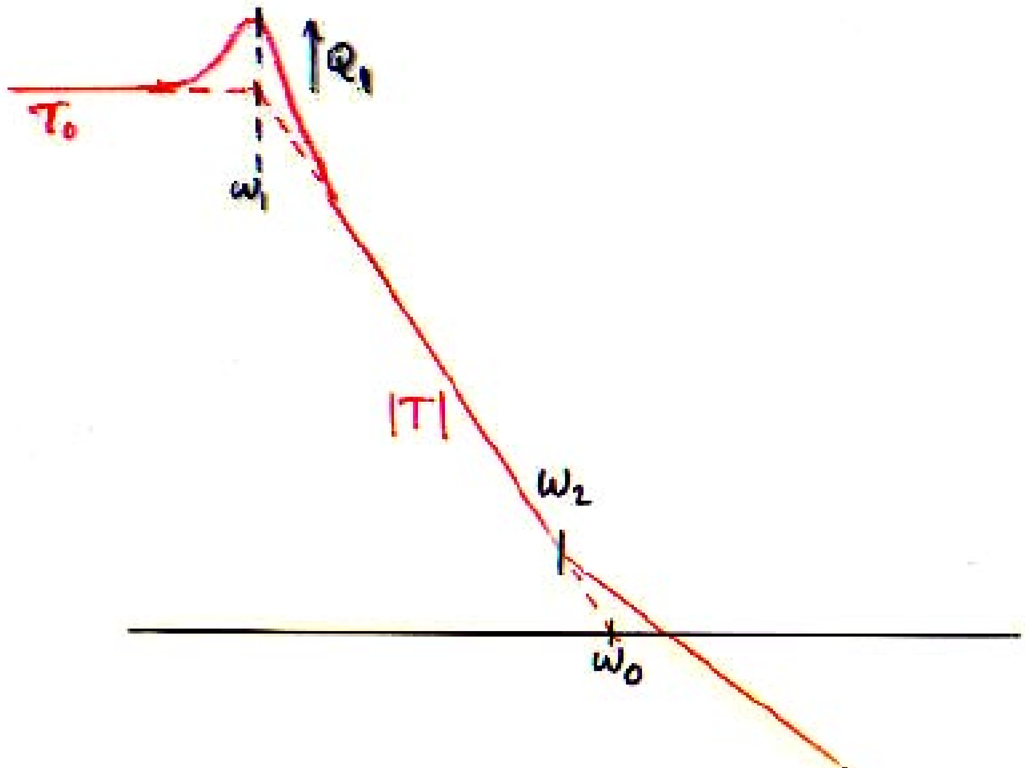

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