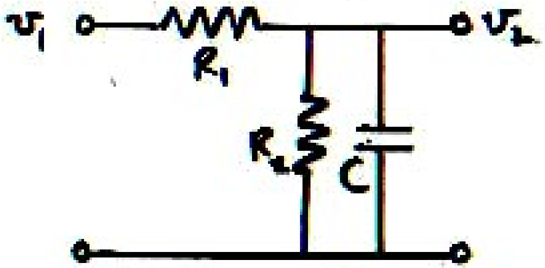
3

NORMAL AND INVERTED POLES AND ZEROS

"Flat gain"



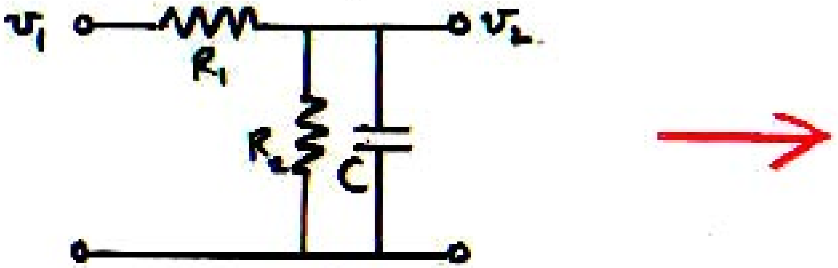
The hard way:

$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

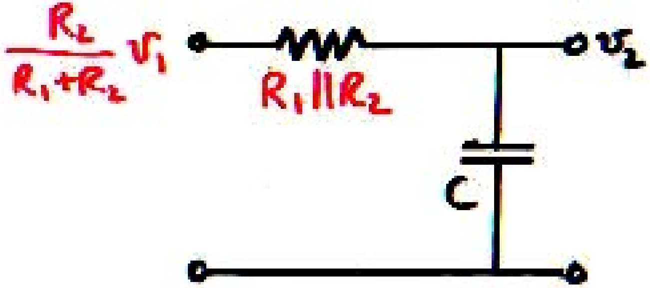
$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1\|R_2)}$$

"Flat gain"



The easy way:



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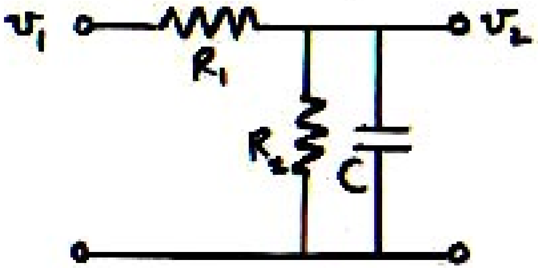
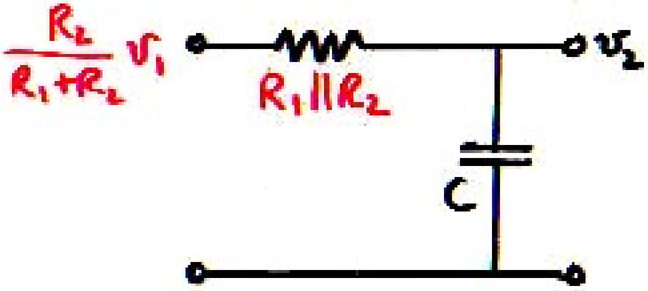
$$\frac{v_2}{v_1} = \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

"Flat gain"



→

The easy way:



The hard way:

$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

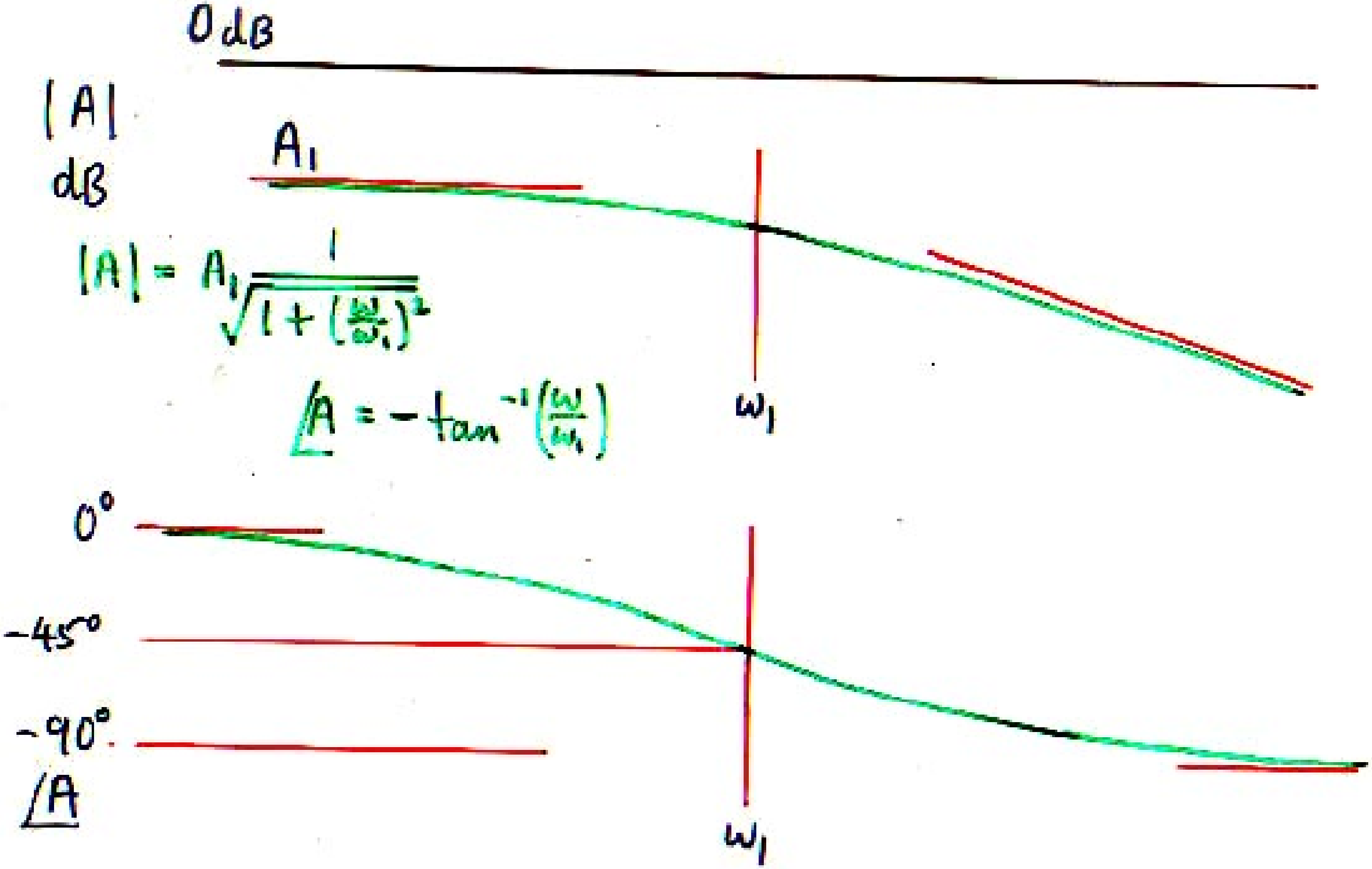
$$\frac{v_2}{v_1} = \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

Result:

$$\frac{v_2}{v_1} \equiv A = A_1 \frac{1}{1+\frac{s}{\omega_1}} \quad \text{where} \quad A_1 \equiv \frac{R_2}{R_1+R_2} \qquad \omega_1 \equiv \frac{1}{C(R_1 \| R_2)}$$

Single-pole response:

$$A = \underset{\text{flatgain}\nearrow}{A_1}\,\frac{1}{1+\frac{s}{\omega_1}} \leftarrow \text{normal pole}$$



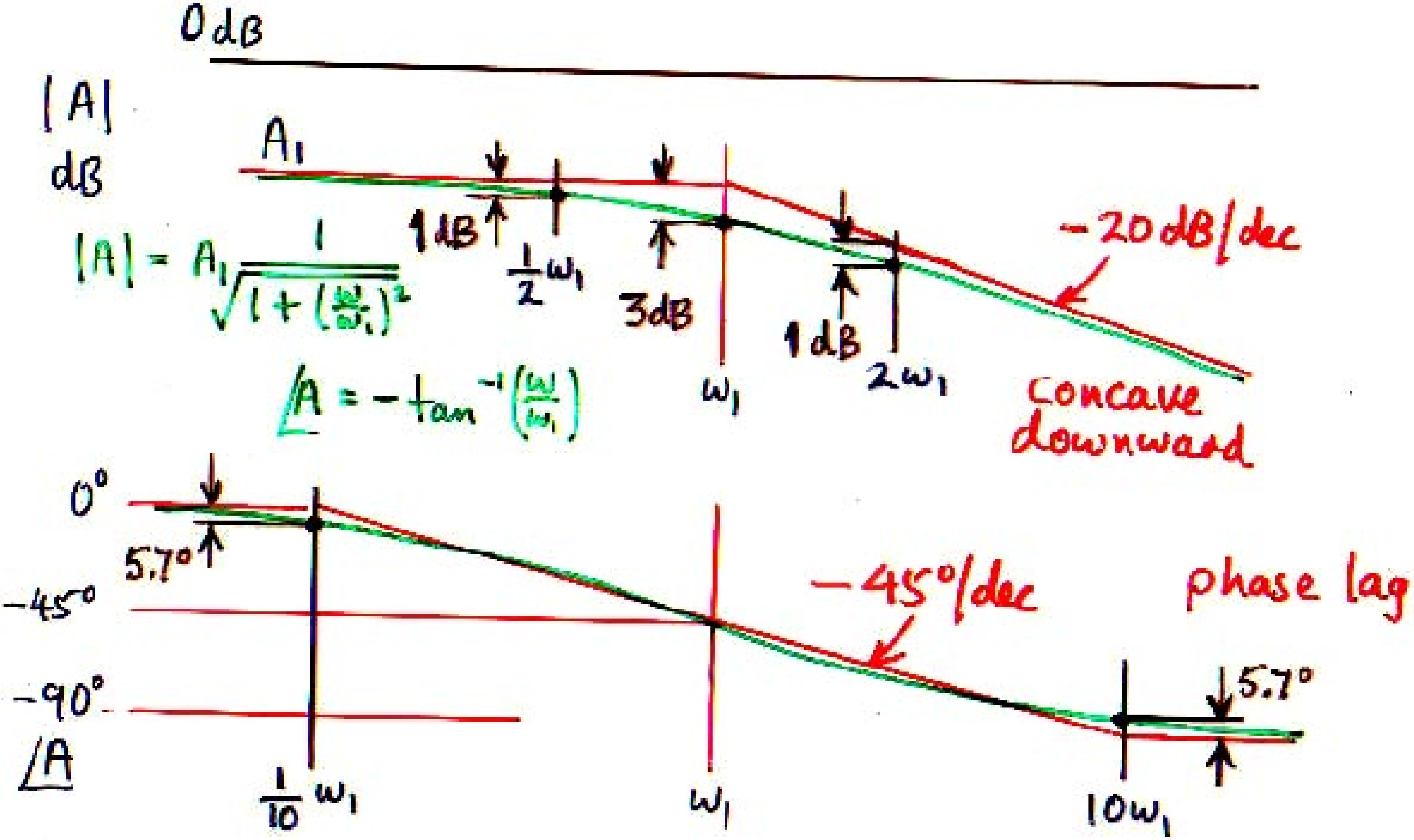
$$|A| = A_1 \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}$$

$$\angle A = -\tan^{-1}\left(\frac{\omega}{\omega_1}\right)$$

Single-pole response:

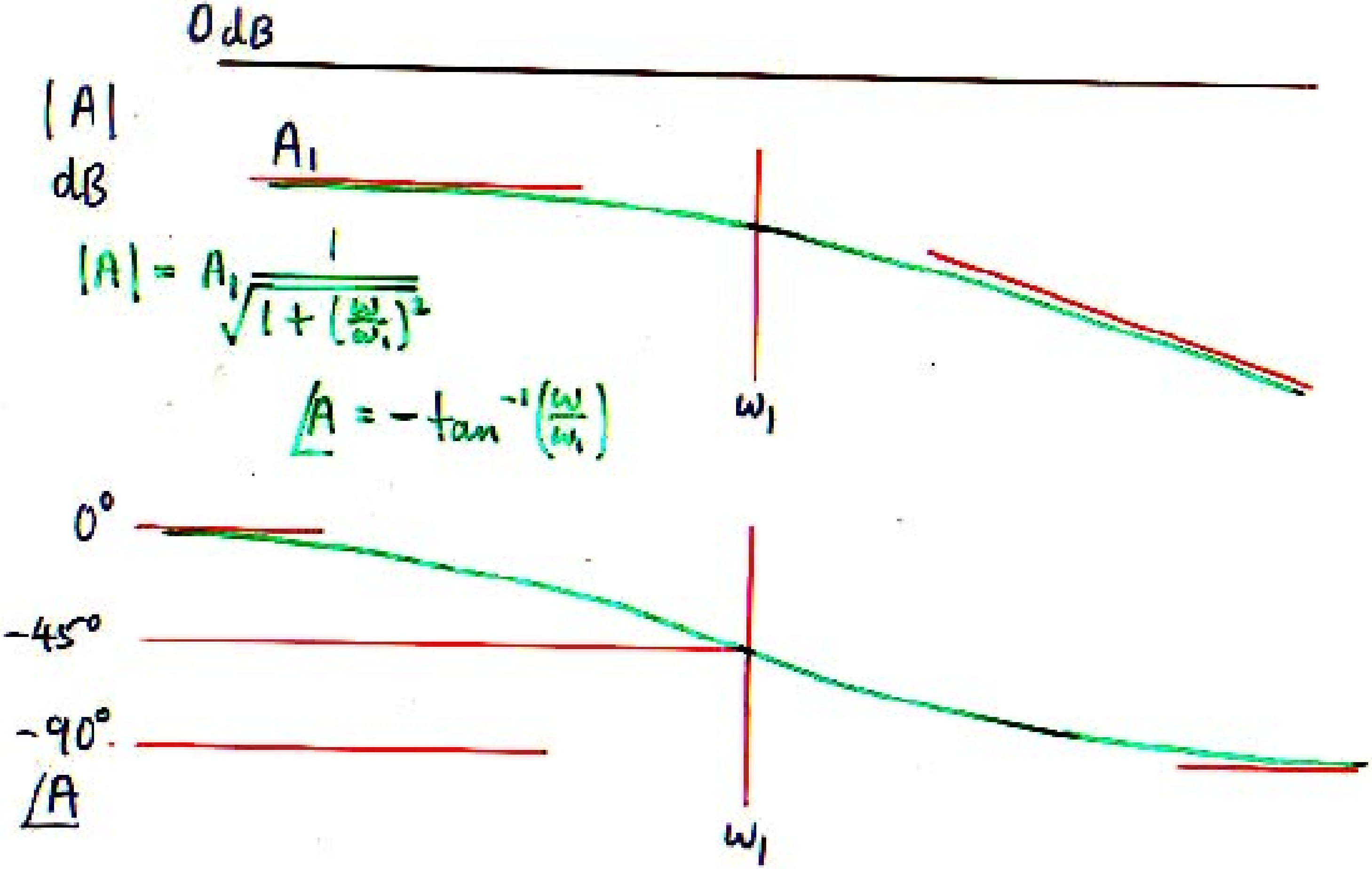
$$A = A_1 \frac{1}{1 + \frac{s}{\omega_1}}$$

flatgain ↗ ↖ normal pole



Single-pole response:

$$A = \underset{\text{flatgain}\nearrow}{A_1} \frac{1}{1+\frac{s}{\omega_1}} \Leftarrow \text{normal pole}$$



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Single-pole response:

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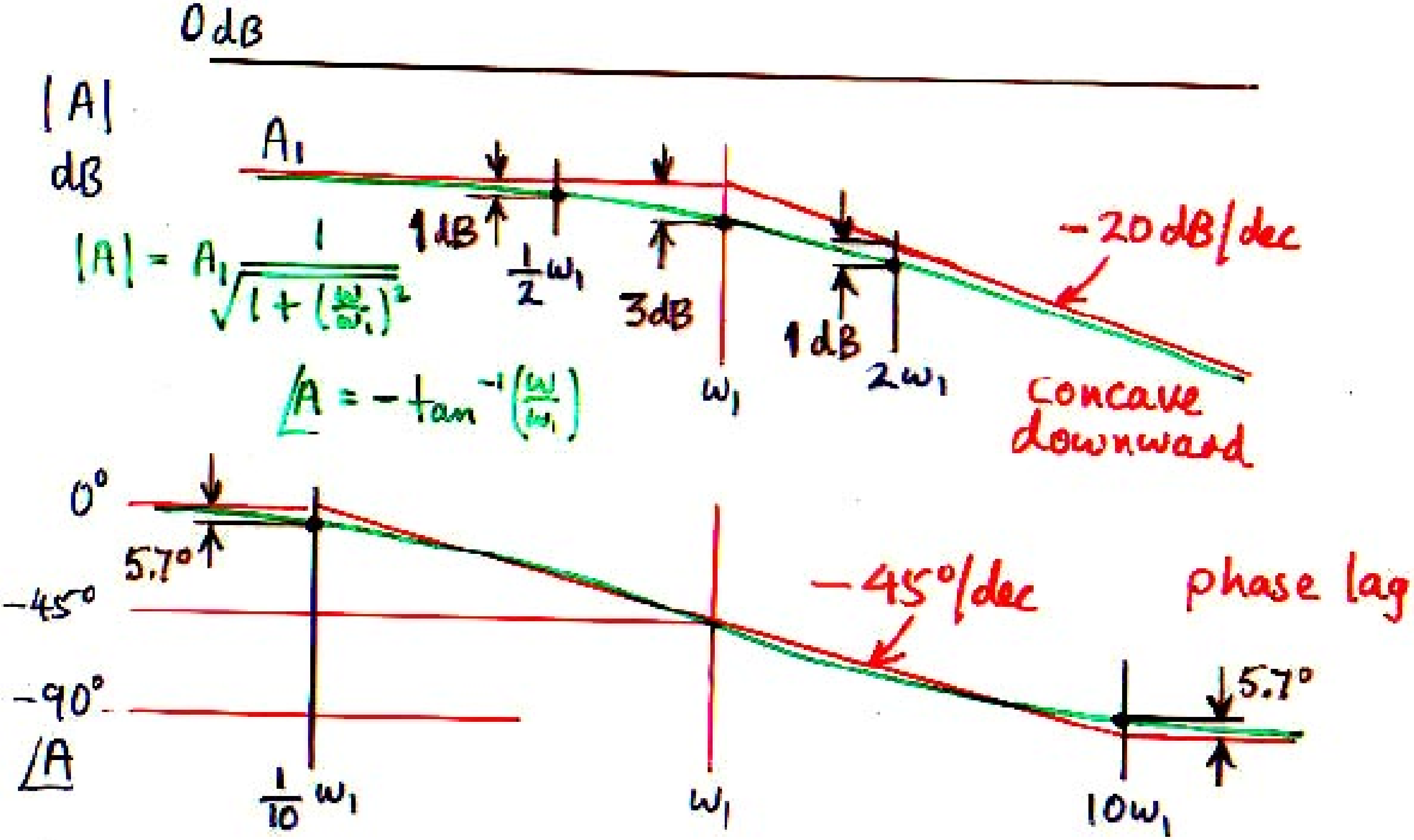
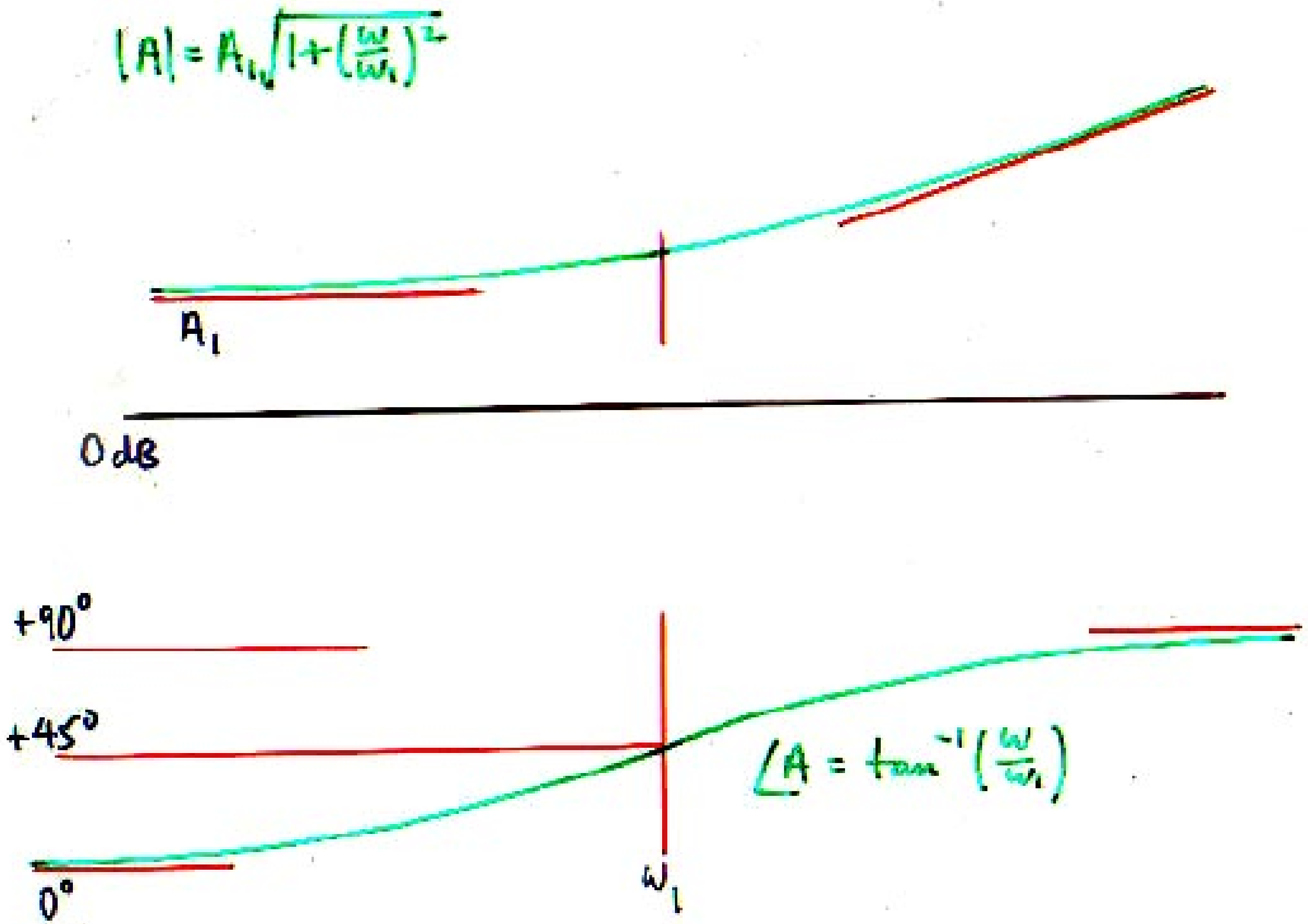


Single-zero response:

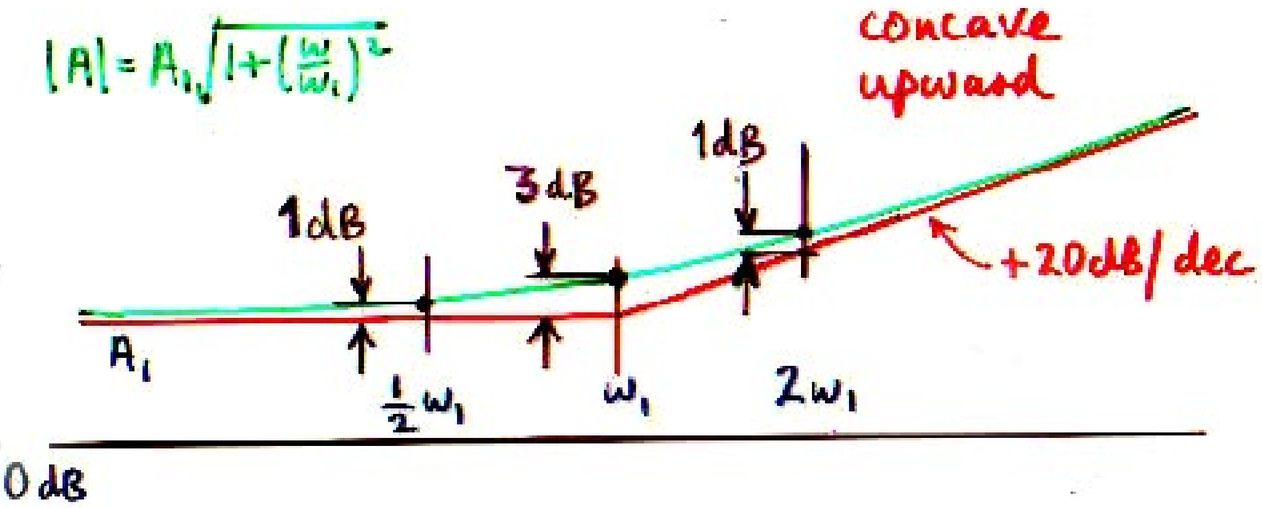
$$A = \underset{\text{flat gain}}{A_1}\left(1+\underset{\text{normal zero}}{\frac{s}{\omega_1}}\right)$$

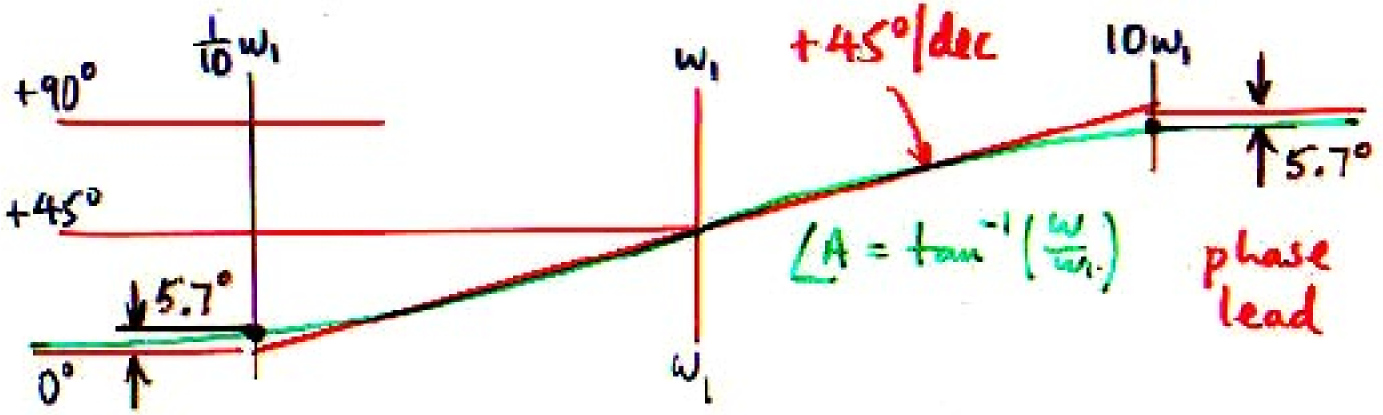


Single-zero response:

$$A = A_1 \left(1 + \frac{s}{\omega_1}\right)$$

flat gain ↑ ↖ normal zero





Generalization: Property of Magnitude and Phase Graphs

A corner can be "seen" from further away on the phase graph than on the magnitude graph.

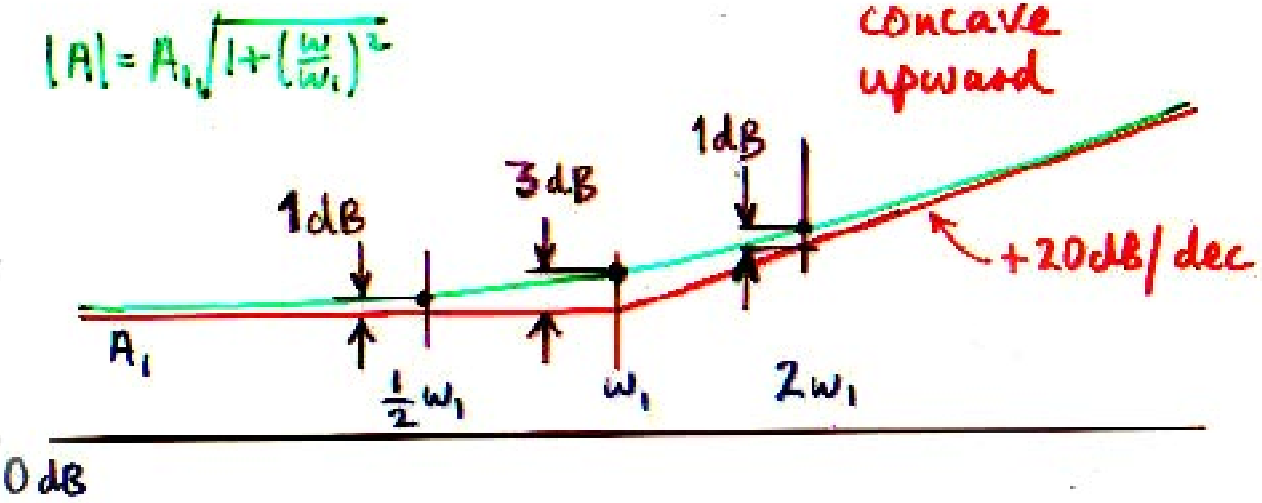
OR:

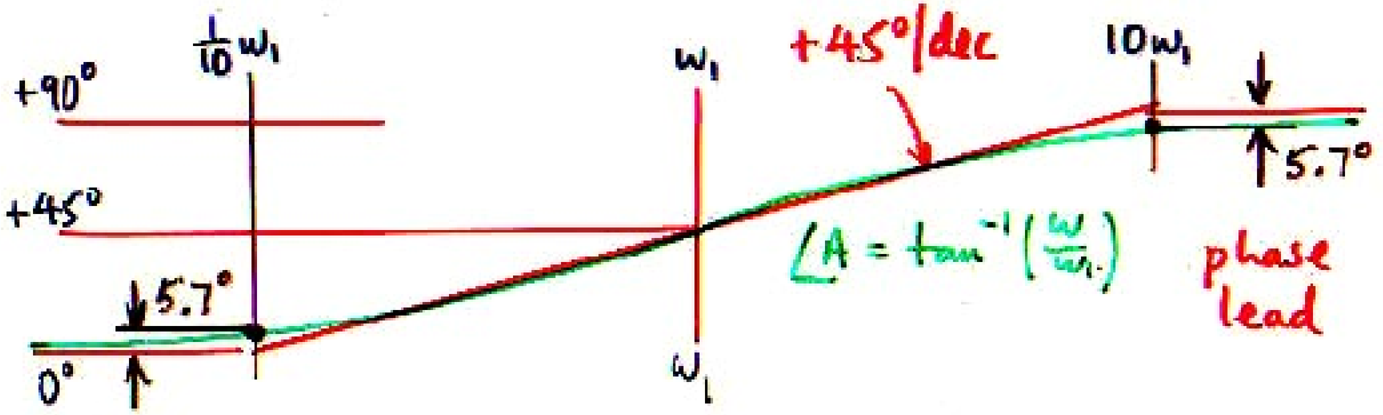
The phase gives a more accurate value of a nearby corner frequency than does the magnitude.

Single-zero response:

$$A = A_1 \left(1 + \frac{s}{\omega_1}\right)$$

flat gain ↗ ↖ normal zero





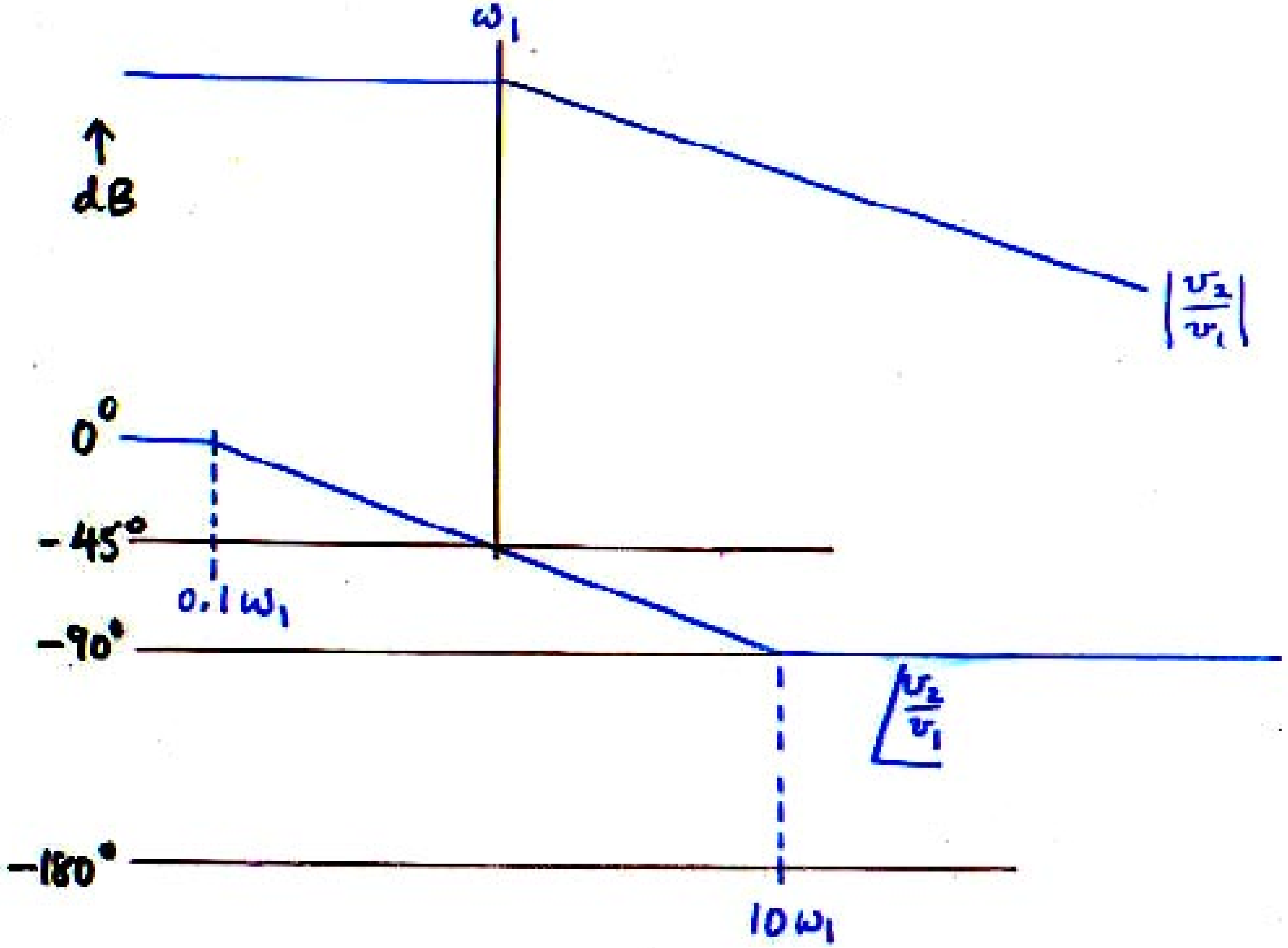
Double-pole low-pass RC filters

$$\frac{v_3}{v_1} = \frac{1}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)} \quad \text{where} \quad \omega_1 \equiv \frac{1}{C_1R_1} \quad \omega_2 \equiv \frac{1}{C_2R_2}$$

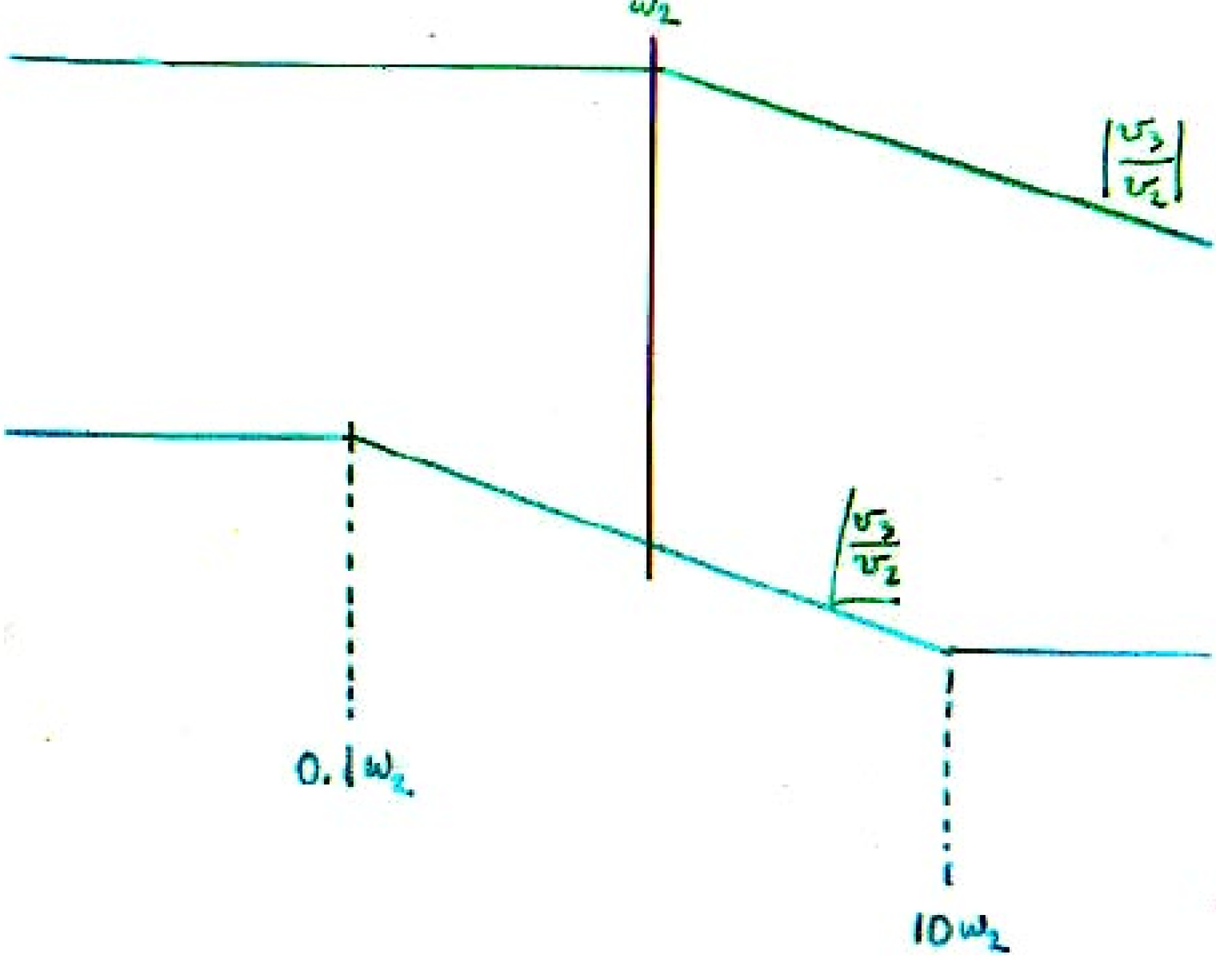
$$\left|\frac{v_3}{v_1}\right|_{dB} = -20\log\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2} - 20\log\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2}$$

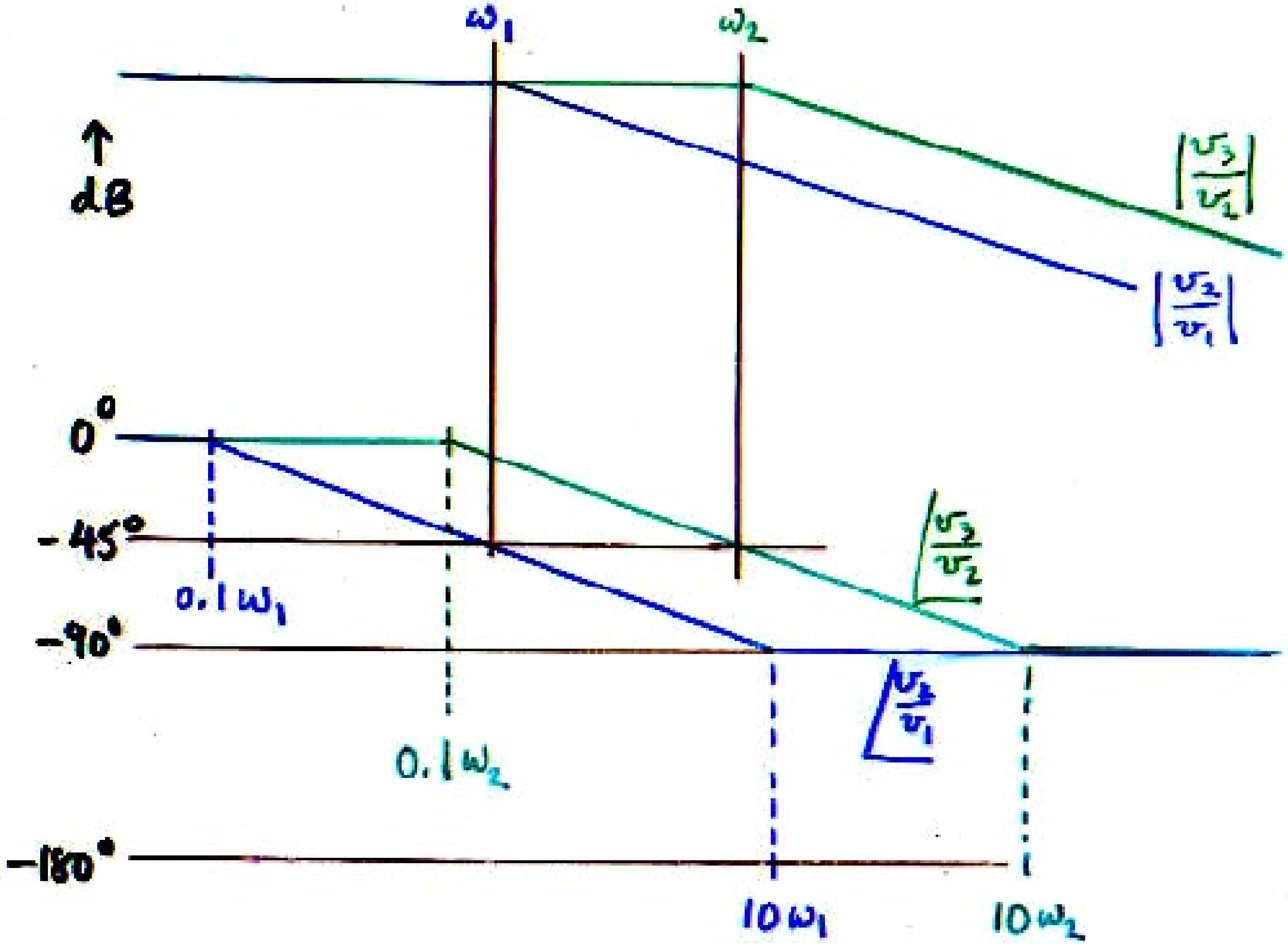
superposition

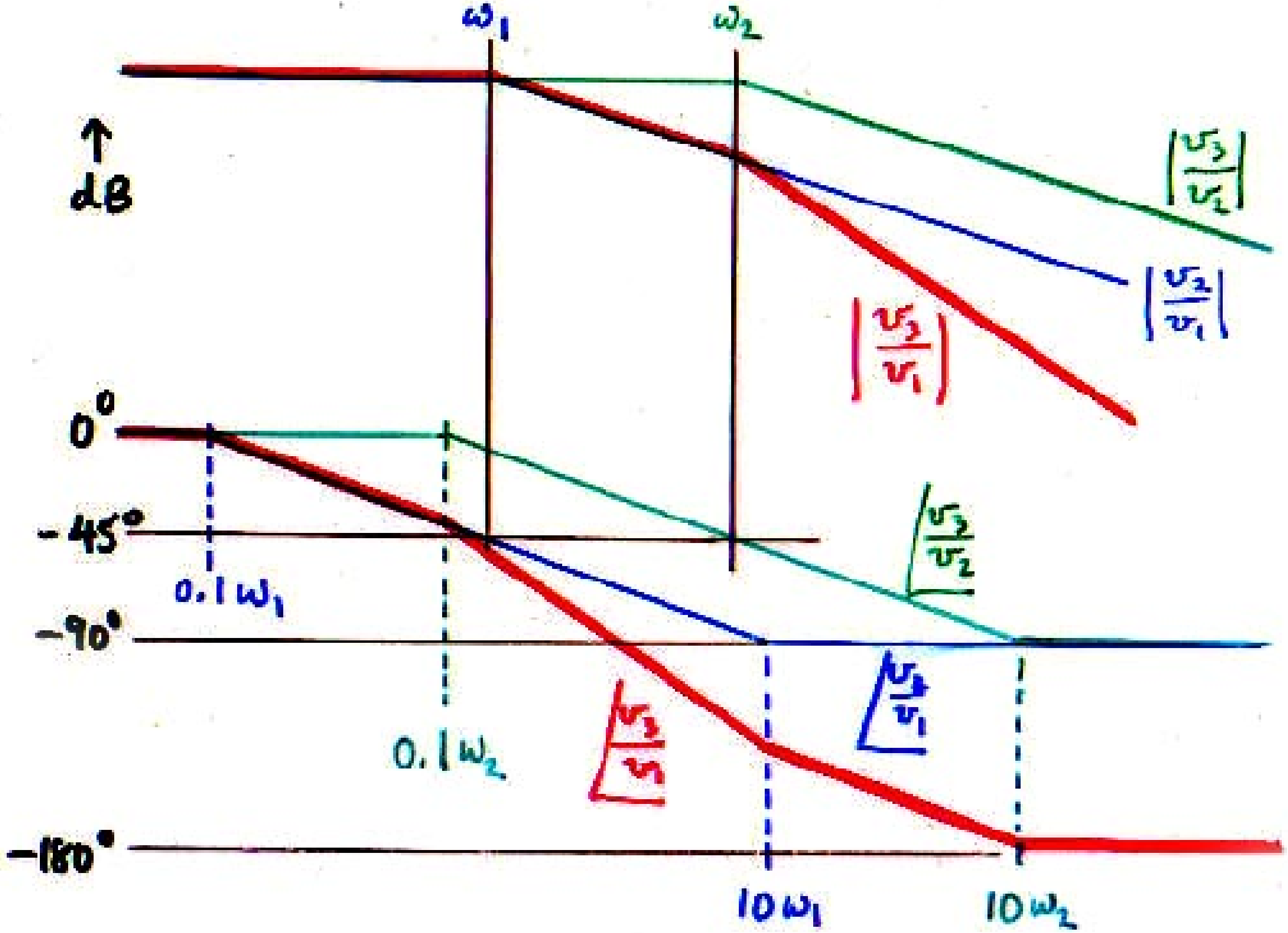
$$\angle\frac{v_3}{v_1} = -\tan^{-1}\left(\frac{\omega}{\omega_1}\right) - \tan^{-1}\left(\frac{\omega}{\omega_2}\right)$$



ω_1
↑ dB
$\left|\frac{v_2}{v_1}\right|$
0^0
$-45°$
$0.1\omega_1$
$-90°$
$\angle\frac{v_2}{v_1}$
$-180°$
$10\omega_1$

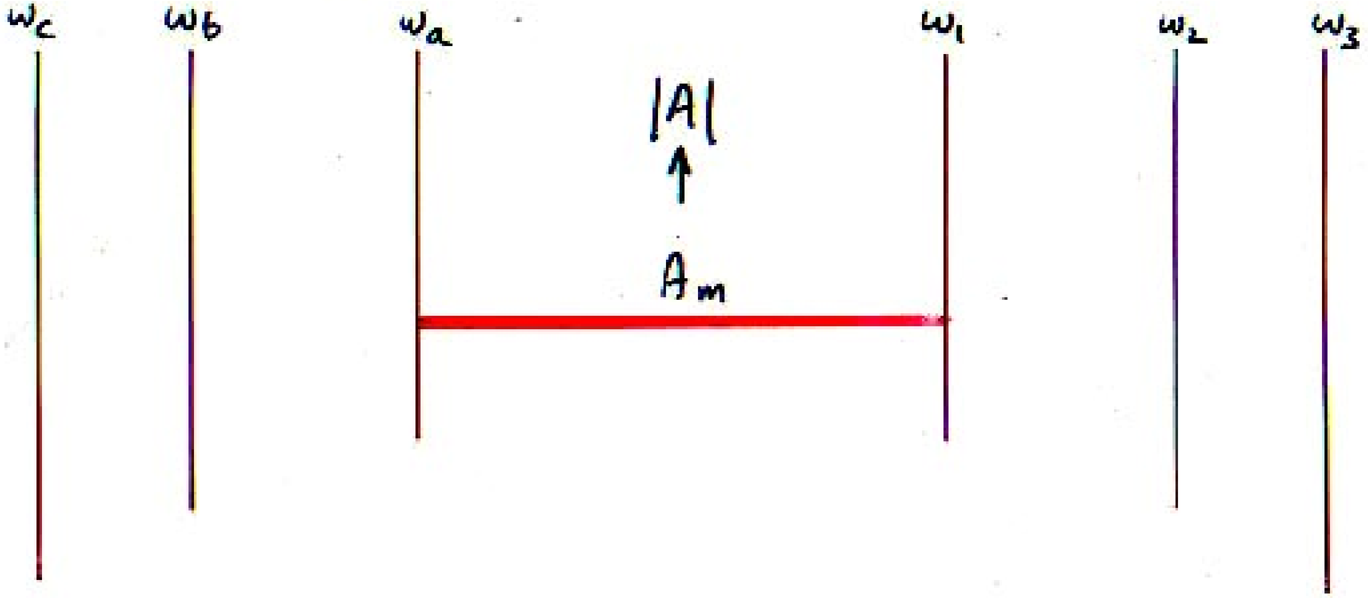

ω_2
$\left|\frac{v_3}{v_2}\right|$
$\angle\frac{v_3}{v_2}$
$0.1\omega_2$
$10\omega_2$


ω1
ω2
↑ dB
|v3/v1|
|v2/v1|
0°
-45°
0.1ω1
-90°
∠v3/v2
0.1ω2
∠v2/v1
-180°
10ω1
10ω2



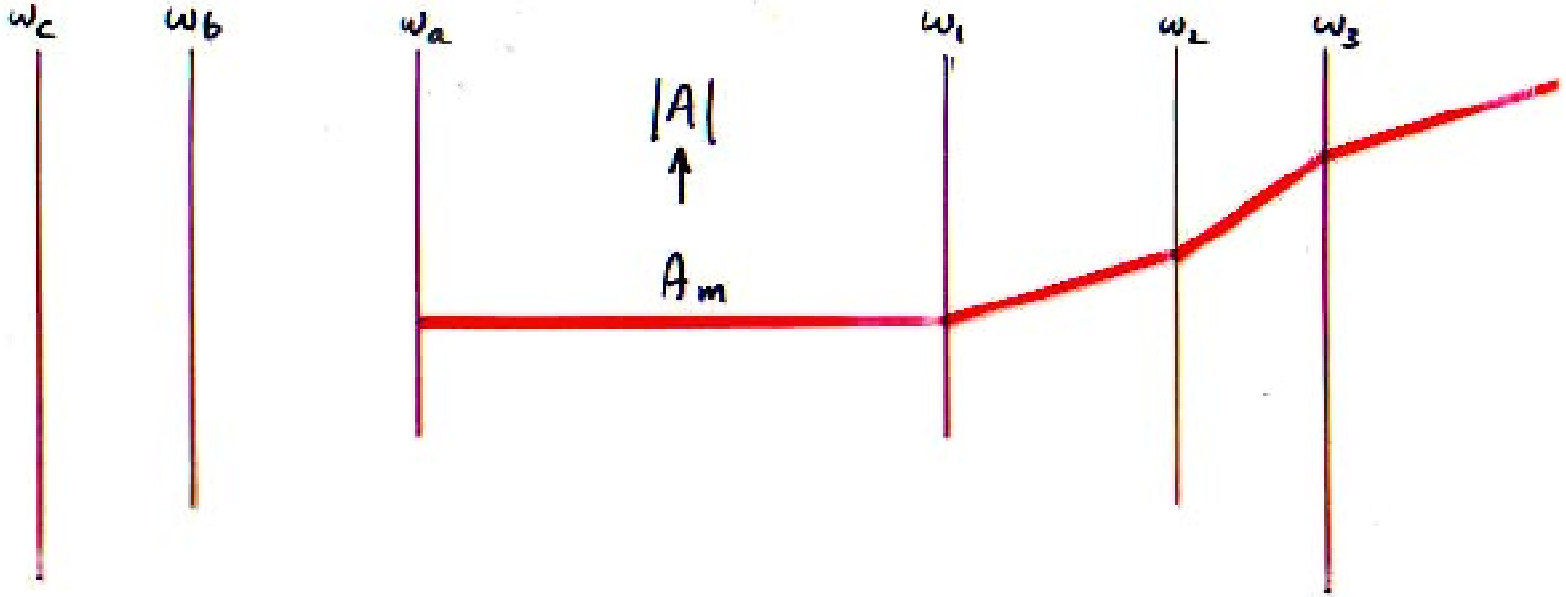
ω_1
ω_2
↑ dB
$\left|\frac{v_3}{v_2}\right|$
$\left|\frac{v_2}{v_1}\right|$
$\left|\frac{v_3}{v_1}\right|$
0°
-45°
-90°
-180°
$0.1\omega_1$
$0.1\omega_2$
$10\omega_1$
$10\omega_2$
$\angle\frac{v_3}{v_2}$
$\angle\frac{v_2}{v_1}$
$\angle\frac{v_3}{v_1}$

Normal and Inverted poles and zeros



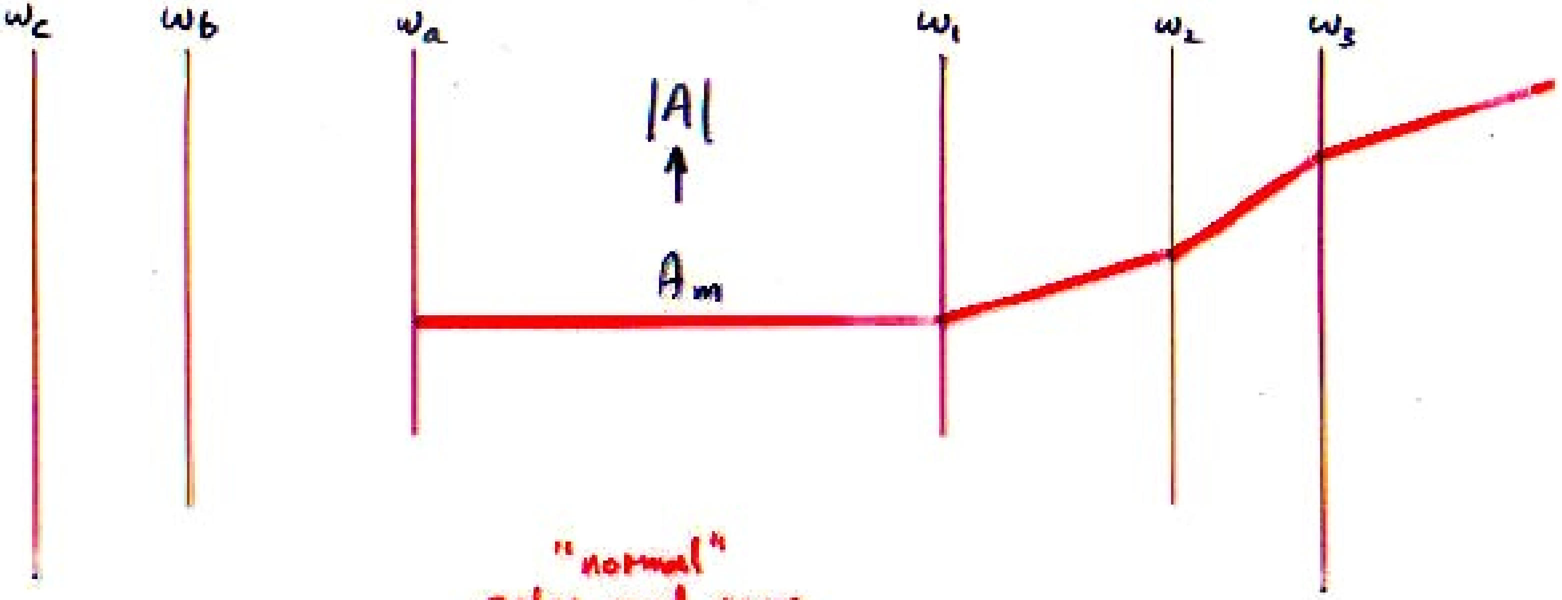
$$A = A_m$$

Normal and Inverted poles and zeros



$A = A_m$

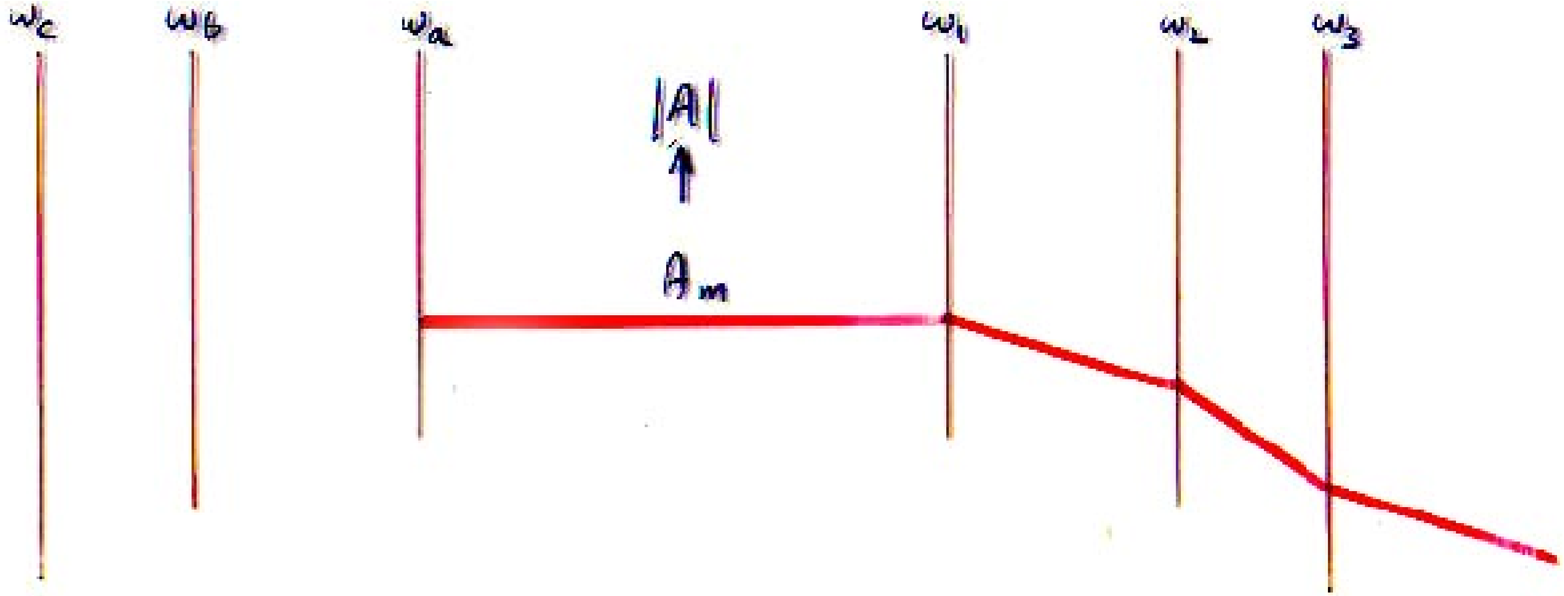
Normal and Inverted poles and zeros



"normal"
poles and zeros

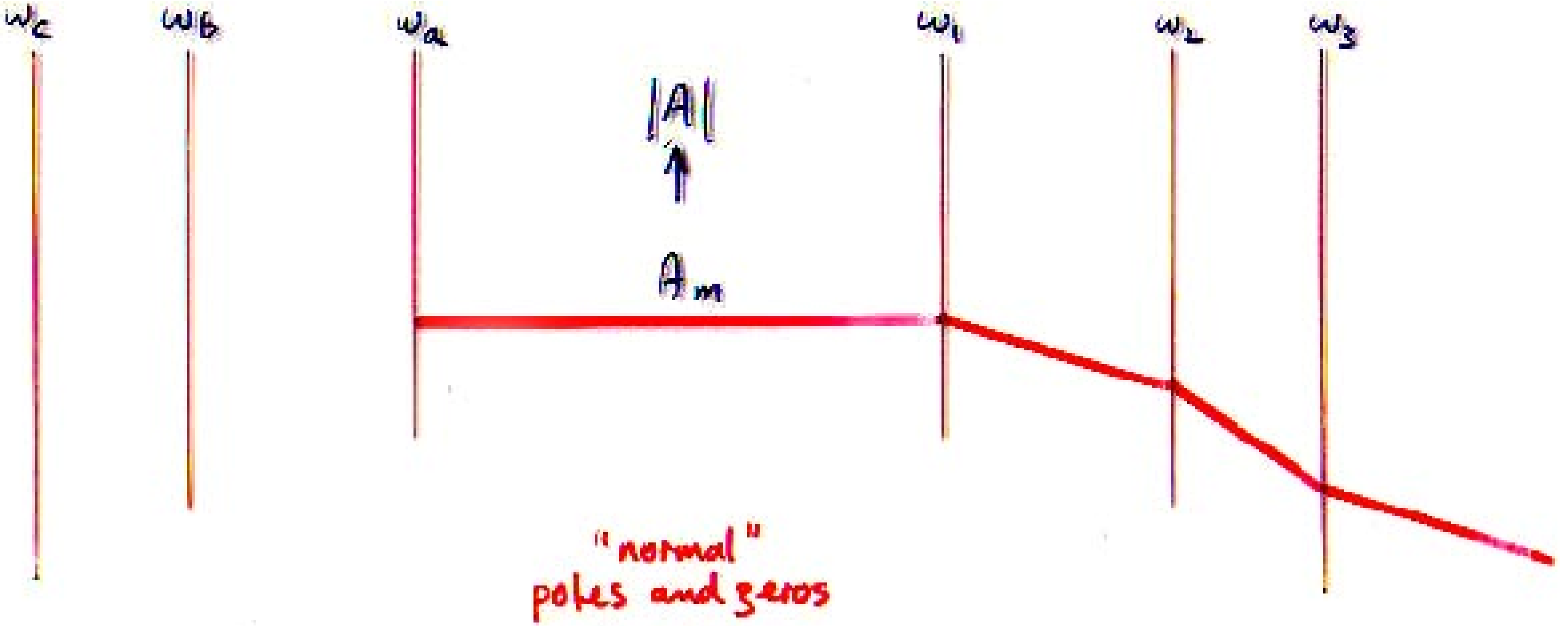
$$A = A_m \frac{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_3}\right)}$$

Normal and Inverted poles and zeros



$$A = A_m$$

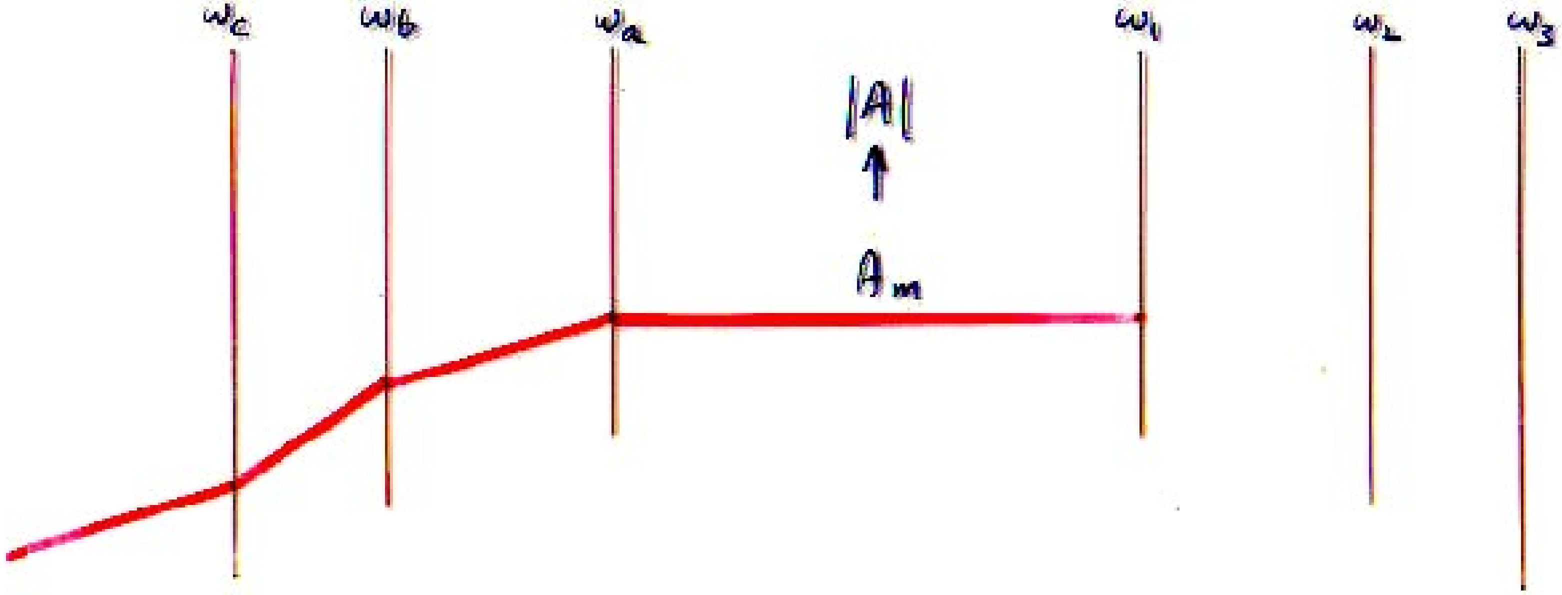
Normal and Inverted poles and zeros



$$A = A_m \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

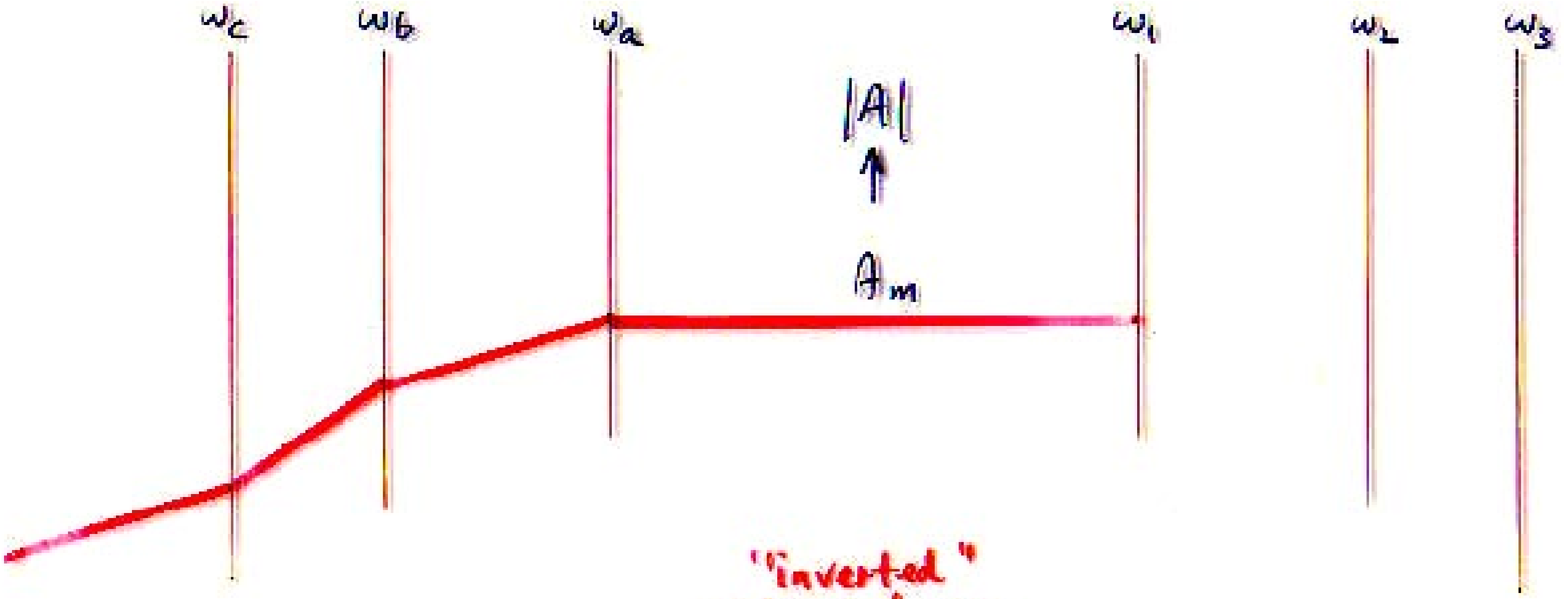
Inversion of pole-zero factors ⟺ vertical inversion of magnitude graph

Normal and Inverted poles and zeros



$$A = A_m$$

Normal and Inverted poles and zeros

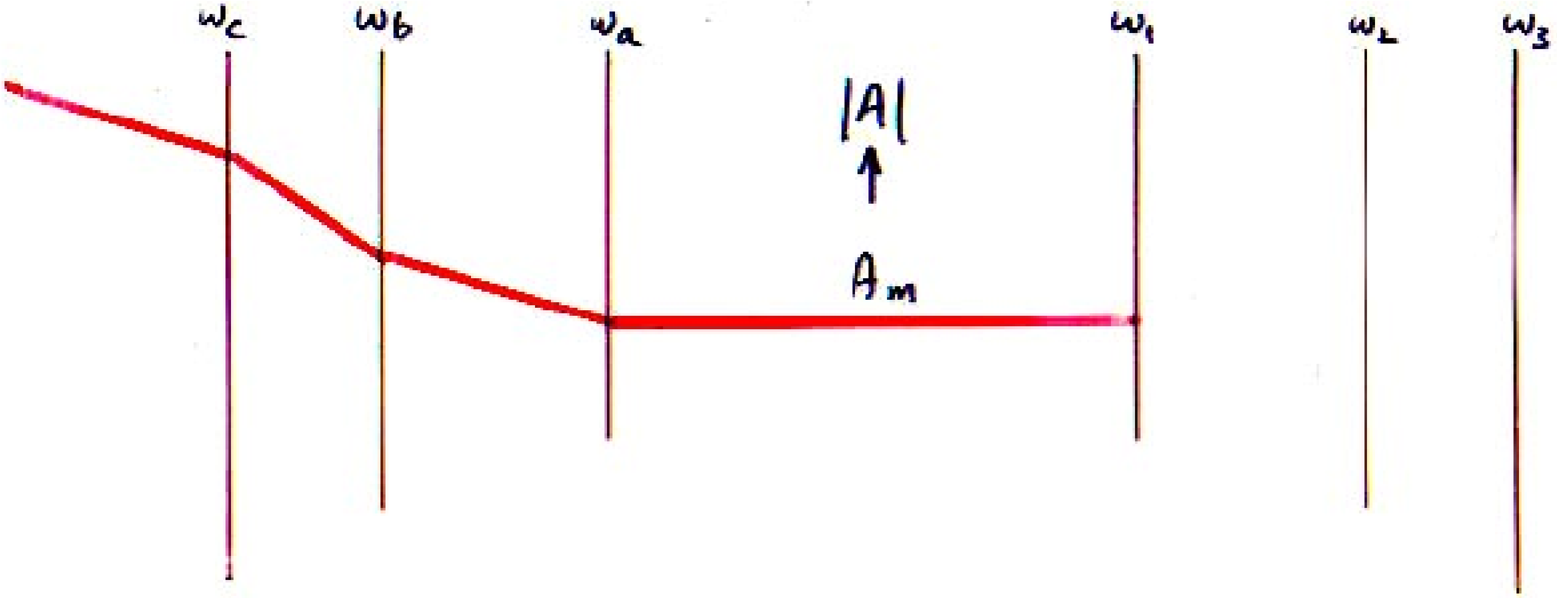


"inverted" poles and zeros

$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)}$$

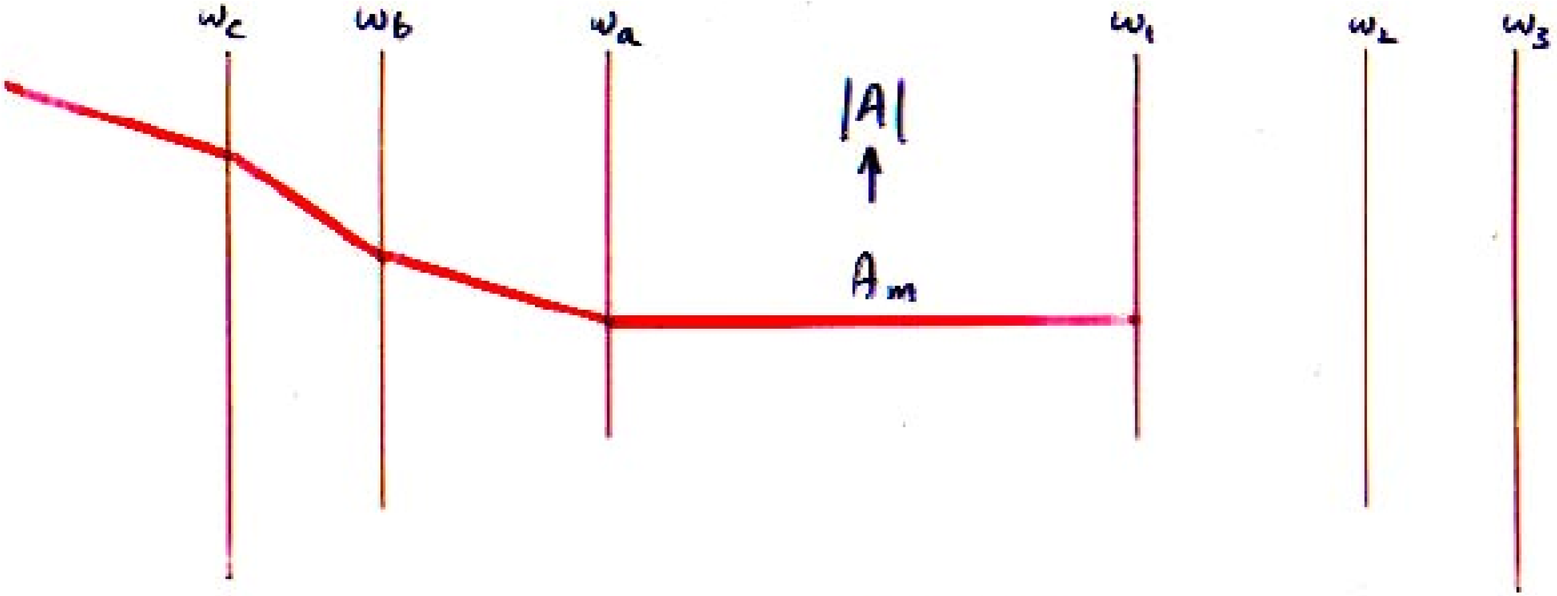
Inversion of frequency terms ⇔ horizontal reversal of magnitude graph

Normal and Inverted poles and zeros



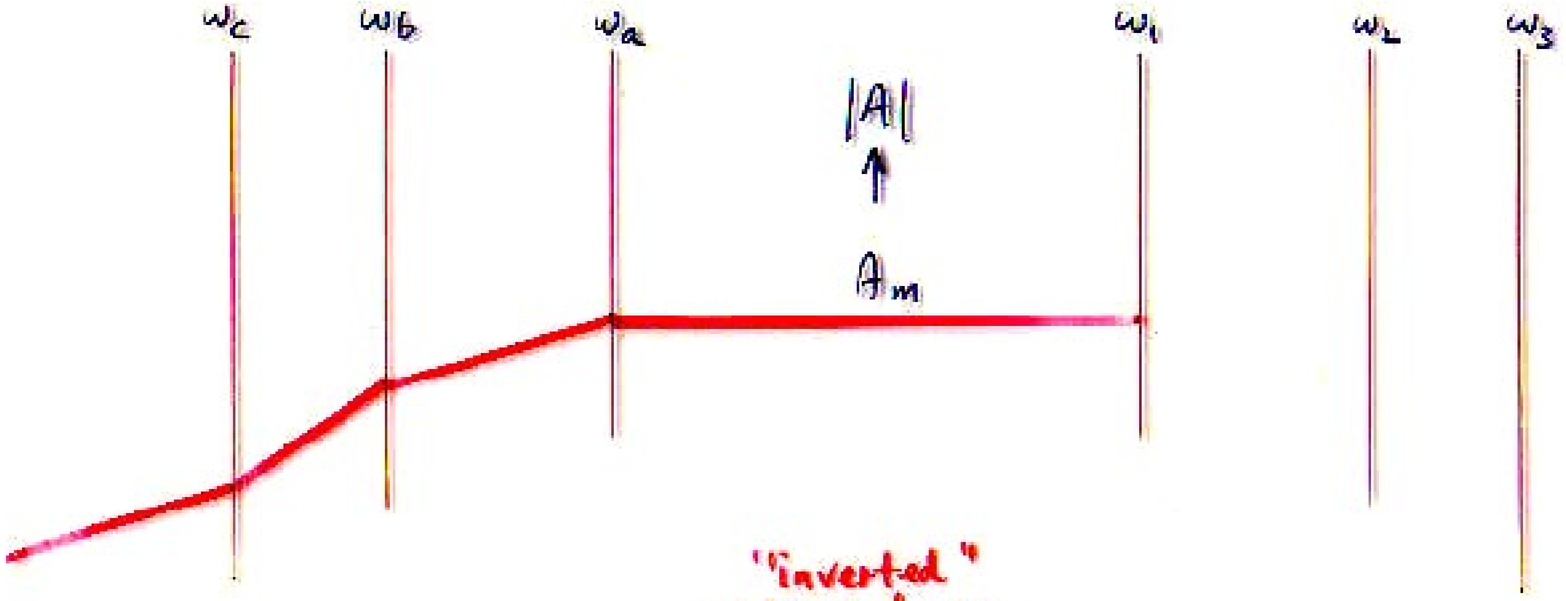
$A = A_m$

Normal and Inverted poles and zeros



$$A = A_m \frac{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)}{1+\frac{\omega_c}{s}}$$

Normal and Inverted poles and zeros

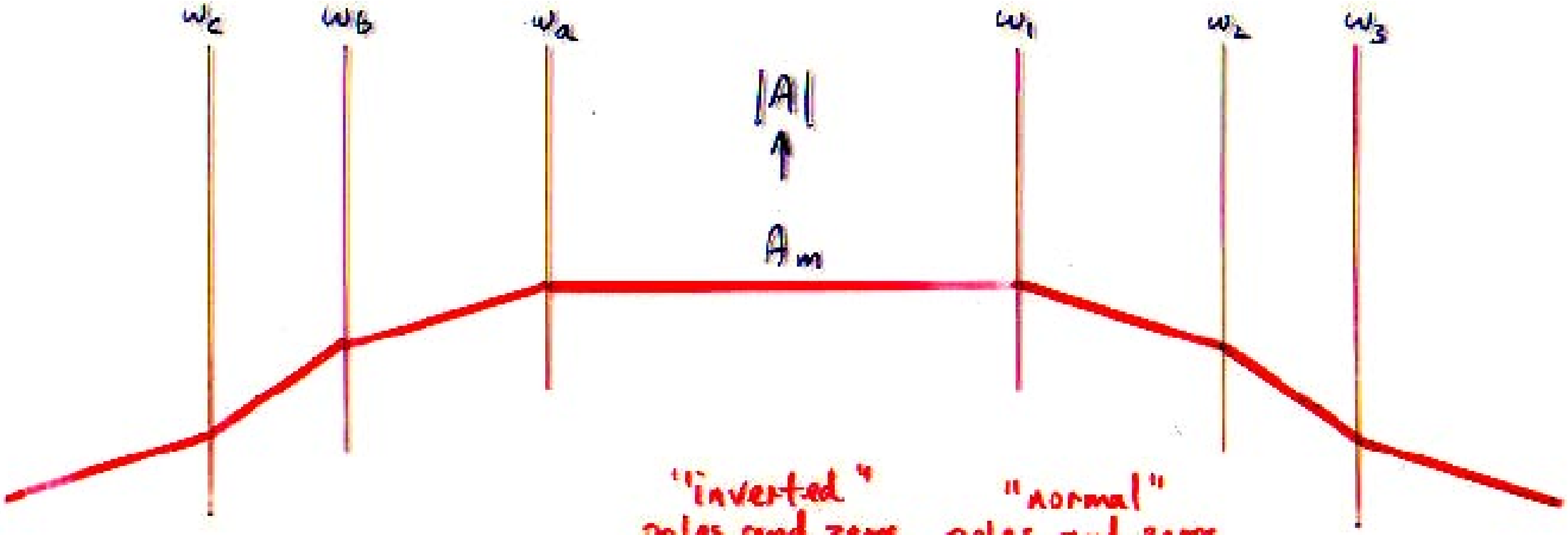


"inverted"
poles and zeros

$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)}$$

Inversion of frequency terms $\Leftrightarrow$ horizontal reversal of magnitude graph

Normal and Inverted poles and zeros



"inverted" poles and zeros

"normal" poles and zeros

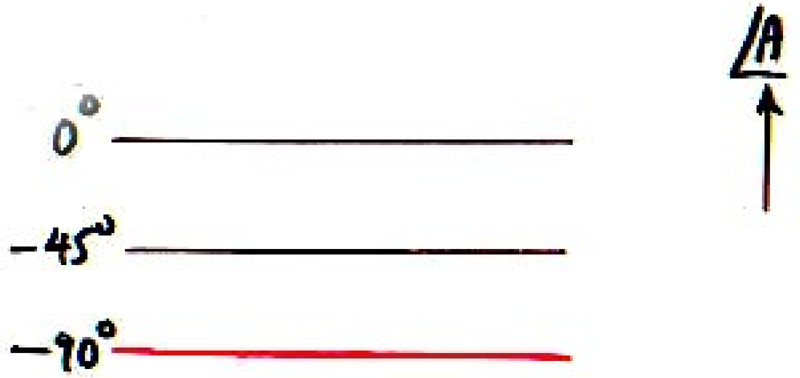
$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)} \cdot \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

Inversion of frequency terms ⇔ horizontal reversal of magnitude graph

If there is no "flat gain", use a reference value:

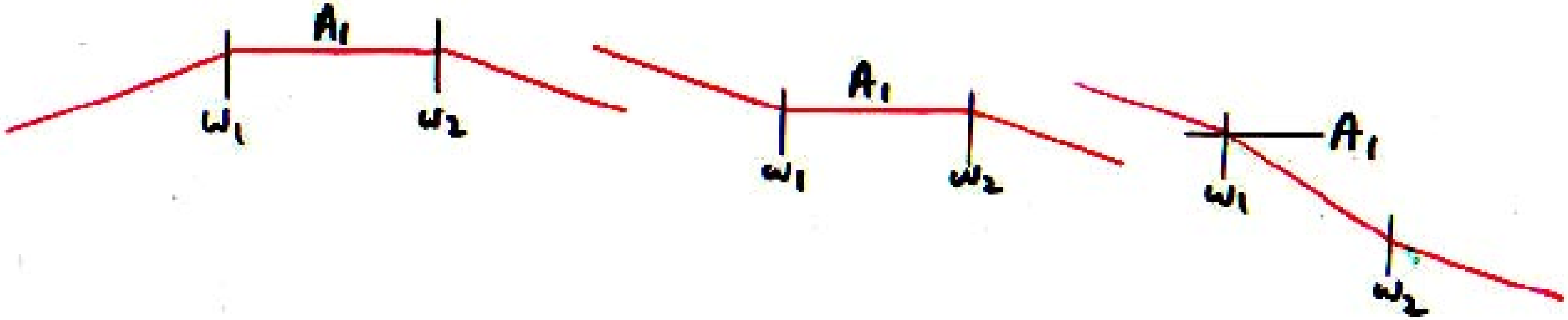
$$A = A_3 \frac{1}{\frac{s}{\omega_3}} = A_3 \frac{\omega_3}{s}$$

$$A = A_4 \frac{s}{\omega_4}$$



Exercises

Express the gains in factored pole-zero form



Exercises

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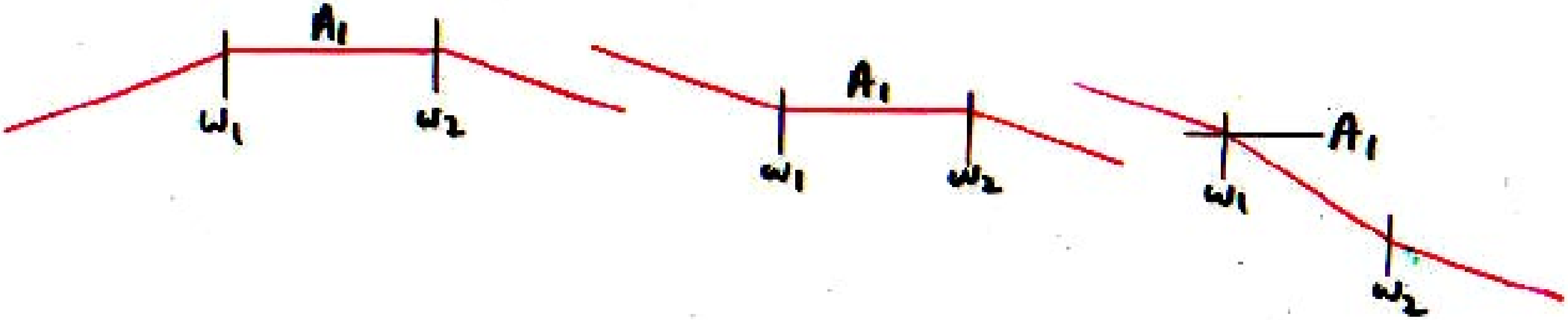
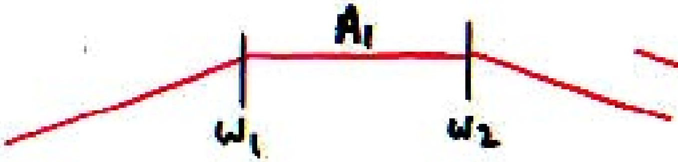
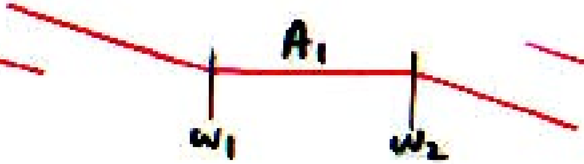


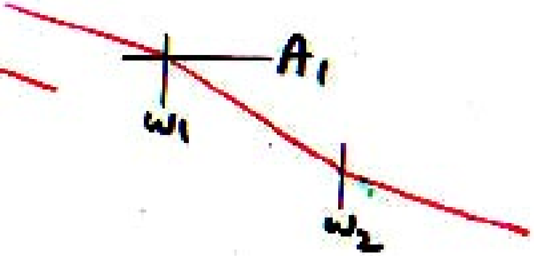
$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

Exercises

Express the gains in factored pole-zero form





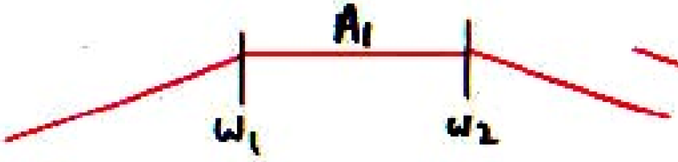


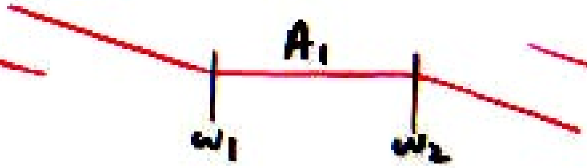
$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

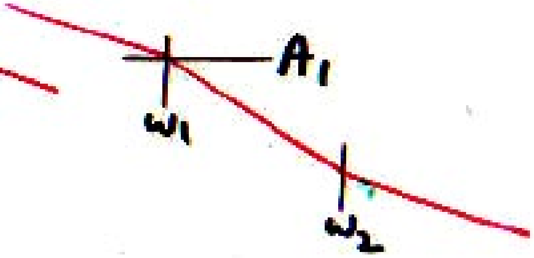
$$A = A_1 \frac{1+\frac{\omega_1}{s}}{1+\frac{s}{\omega_2}}$$

Exercises

Express the gains in factored pole-zero form







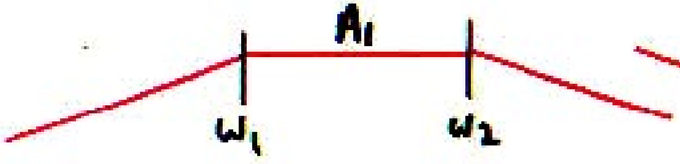
$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

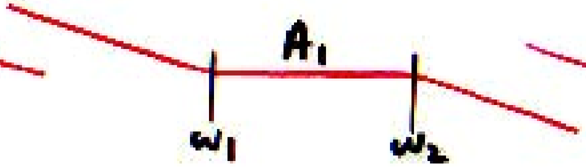
$$A = A_1 \frac{1+\frac{\omega_1}{s}}{1+\frac{s}{\omega_2}}$$

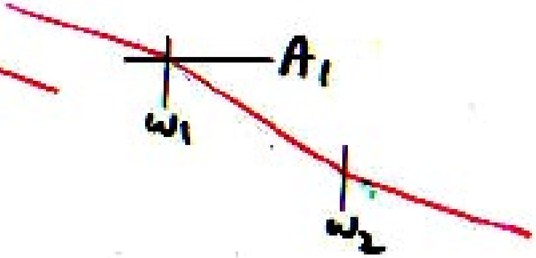
$$A = A_1\left(\frac{\omega_1}{s}\right) \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

Exercises

Express the gains in factored pole-zero form







$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A = A_1 \frac{1+\frac{\omega_1}{s}}{1+\frac{s}{\omega_2}}$$

$$A = A_1 \left(\frac{\omega_1}{s}\right) \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

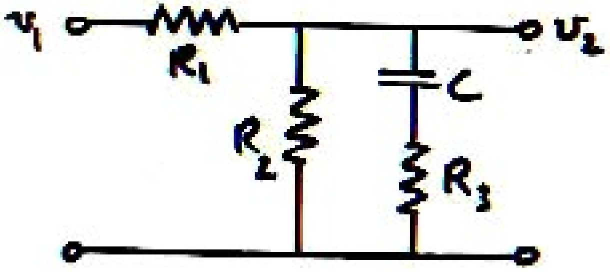
$$A = A_1 \left(\frac{\omega_1}{s}\right)^2 \frac{1+\frac{s}{\omega_2}}{1+\frac{\omega_1}{s}}$$

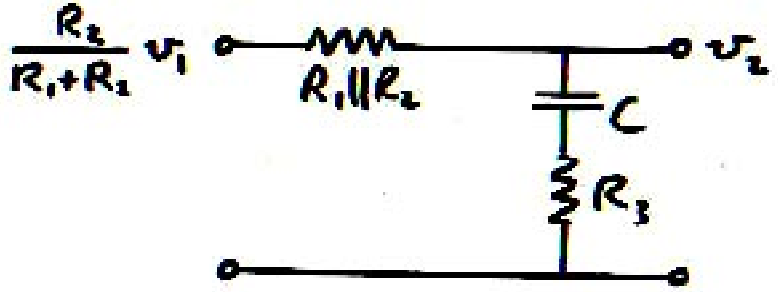
DESIGN-ORIENTED ANALYSIS

Techniques of Design-Oriented Analysis

Lowering the Entropy of an expression
Doing the algebra on the circuit diagram.
Doing the algebra on the graph.
Using inverted poles and zeros.
Using numerical values to justify analytic approximations.
Improved formulas for quadratic roots
The Input/Output Impedance Theorem
The Feedback Theorem
Loop gain by injection of a test signal into the closed loop
Measurement of an unstable loop gain
The Extra Element Theorem (EET)

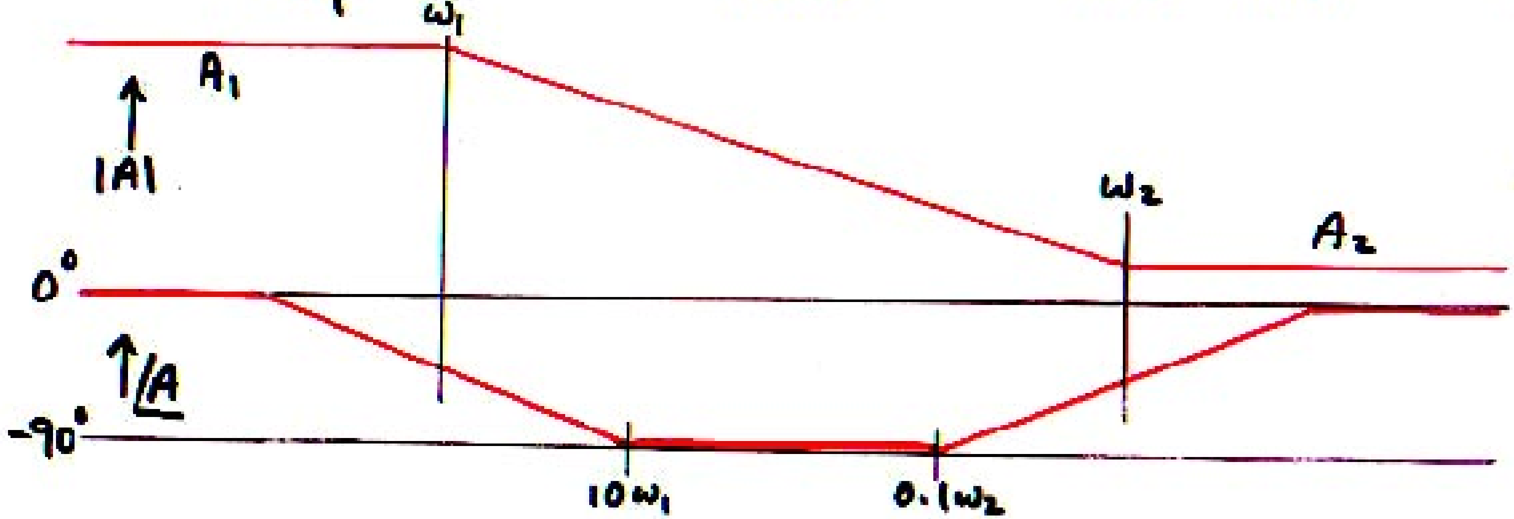
Lag-lead network





$$\frac{v_2}{v_1} \equiv A = \frac{R_2}{R_1+R_2} \cdot \frac{\frac{1}{sC} + R_3}{\frac{1}{sC} + R_3 + R_1 \| R_2}$$

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} \quad \text{where} \quad A_1 \equiv \frac{R_2}{R_1+R_2} \quad \omega_1 \equiv \frac{1}{C(R_3+R_1 \| R_2)} \quad \omega_2 \equiv \frac{1}{CR_3}$$



In this case, there are two flat gains. As derived, the *low-frequency* flat gain A_1 appears as coefficient, together with *normal* pole and zero:

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

Equally well, directly from the $|A|$ asymptotes, the result could be written with the *high-frequency* flat gain A_2 as coefficient, together with *inverted* zero and pole:

$$A = A_2 \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

What is the relation between A_1 and A_2? One form of the result can be derived from the other algebraically:

$$A = \underset{\substack{\uparrow \\ \text{This is } A|_{s\to 0}}}{\boxed{A_1}} \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_1 \frac{\frac{s}{\omega_2}}{\frac{s}{\omega_1}} \frac{\frac{\omega_2}{s}+1}{\frac{\omega_1}{s}+1} = A_1 \frac{\omega_1}{\omega_2} \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

In this case, there are two flat gains. As derived, the *low-frequency* flat gain A_1 appears as coefficient, together with *normal* pole and zero:

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

Equally well, directly from the $|A|$ asymptotes, the result could be written with the *high-frequency* flat gain A_2 as coefficient, together with *inverted* zero and pole:

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What is the relation between A_1 and A_2? One form of the result can be derived from the other algebraically:

$$A = \boxed{A_1} \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_1 \frac{\frac{s}{\omega_2}}{\frac{s}{\omega_1}} \, \frac{\frac{\omega_2}{s}+1}{\frac{\omega_1}{s}+1} = \boxed{A_1 \frac{\omega_1}{\omega_2}} \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

This is $A|_{s\to 0}$

This is $A|_{s\to\infty}$, so must be A_2

Result:

$$\frac{A_2}{A_1} = \frac{\omega_1}{\omega_2}$$

For the lag-lead network:

$$A_2 = A_1 \frac{\omega_1}{\omega_2} = \frac{R_2}{R_1 + R_2} \frac{\cancel{C}R_3}{\cancel{C}(R_3 + R_1 \| R_2)}$$

which is obvious from the reduced model.

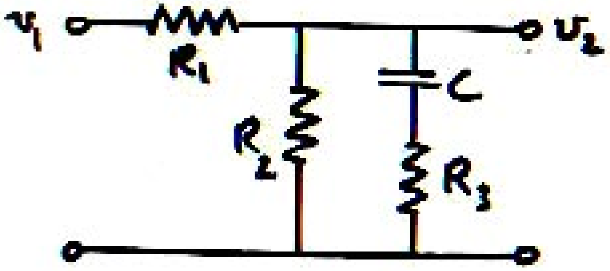
Generalization: Gain–Bandwidth Trade-Off

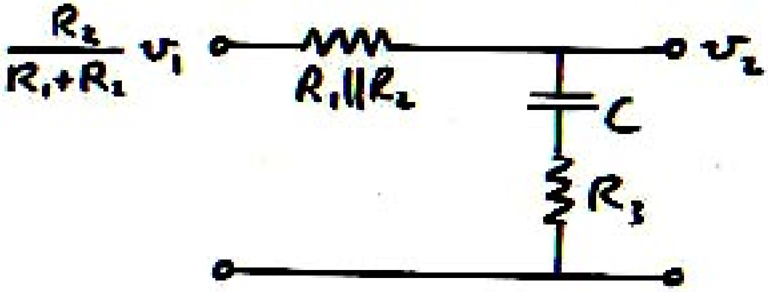
For a single-slope (± 20dB/dec)

Ratio of flat gains = Ratio of corner frequencies that separate them

This is a form of gain-bandwidth trade-off.

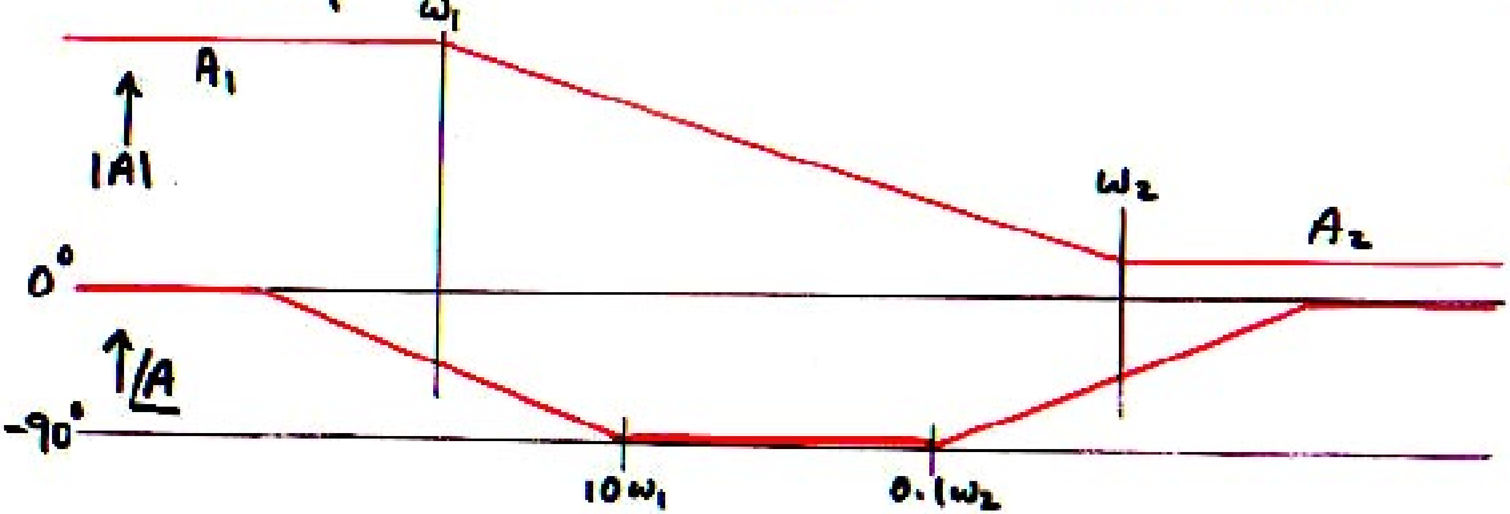
Lag-lead network





$$\frac{v_2}{v_1} \equiv A = \frac{R_2}{R_1+R_2}\cdot\frac{\frac{1}{sC}+R_3}{\frac{1}{sC}+R_3+R_1\|R_2}$$

$$A = A_1\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$ where $A_1 \equiv \frac{R_2}{R_1+R_2}$ $\omega_1 \equiv \frac{1}{C(R_3+R_1\|R_2)}$ $\omega_2 \equiv \frac{1}{CR_3}$



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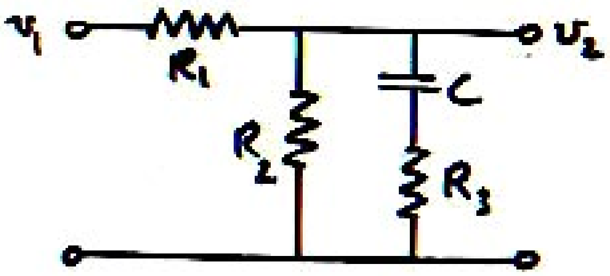
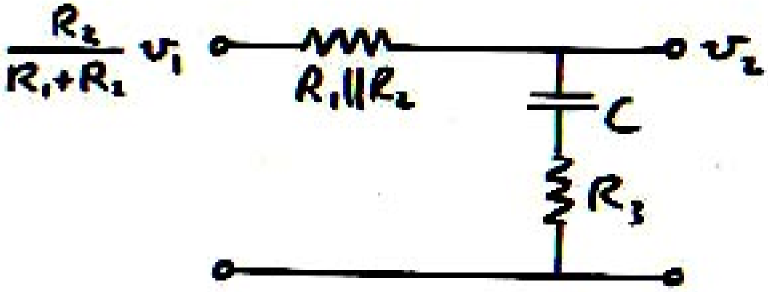
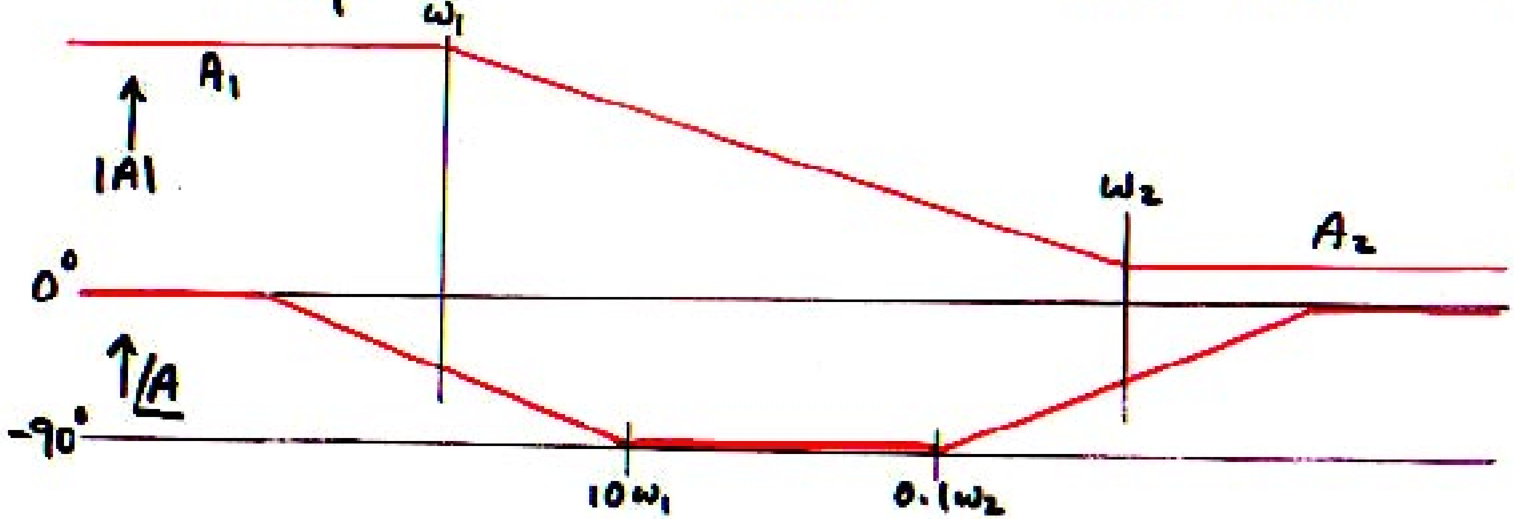




$$\frac{v_2}{v_1} \equiv A = \frac{R_2}{R_1+R_2} \cdot \frac{\frac{1}{sC} + R_3}{\frac{1}{sC} + R_3 + R_1 \| R_2}$$

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} \quad \text{where} \quad A_1 \equiv \frac{R_2}{R_1+R_2} \quad \omega_1 \equiv \frac{1}{C(R_3+R_1\|R_2)} \quad \omega_2 \equiv \frac{1}{CR_3}$$



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For a single-slope ($\pm$ 20dB/dec)

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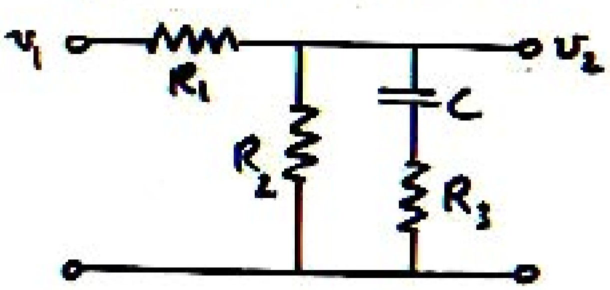

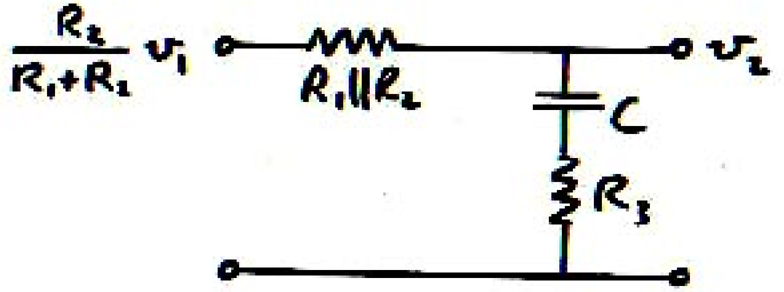

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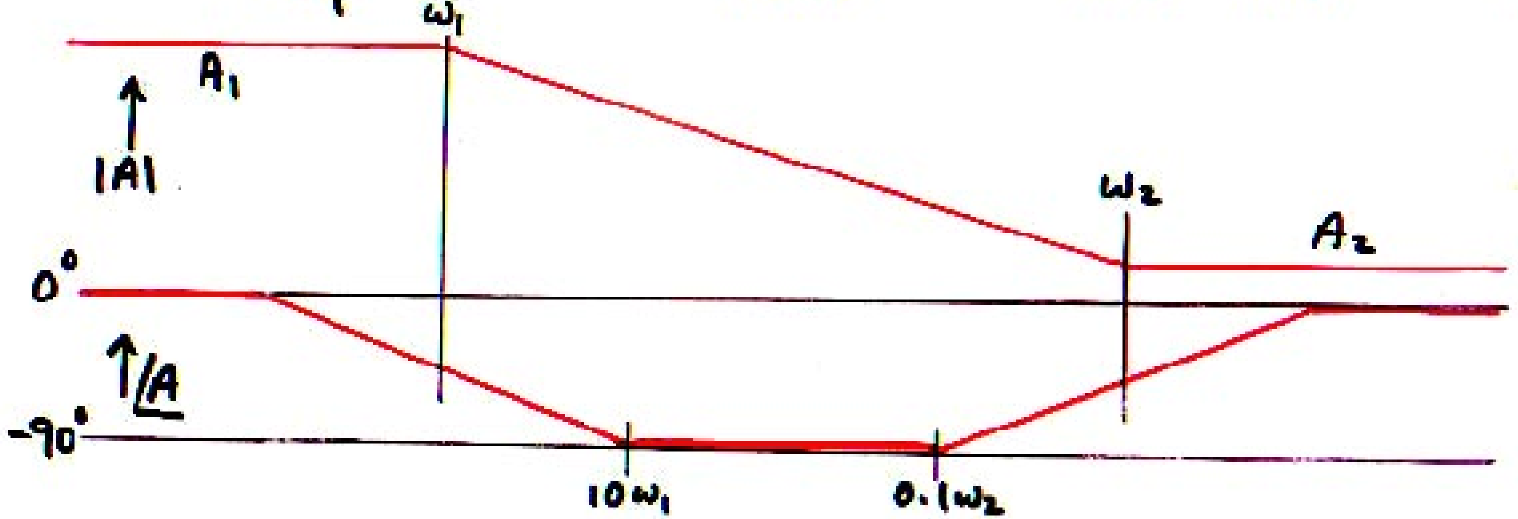
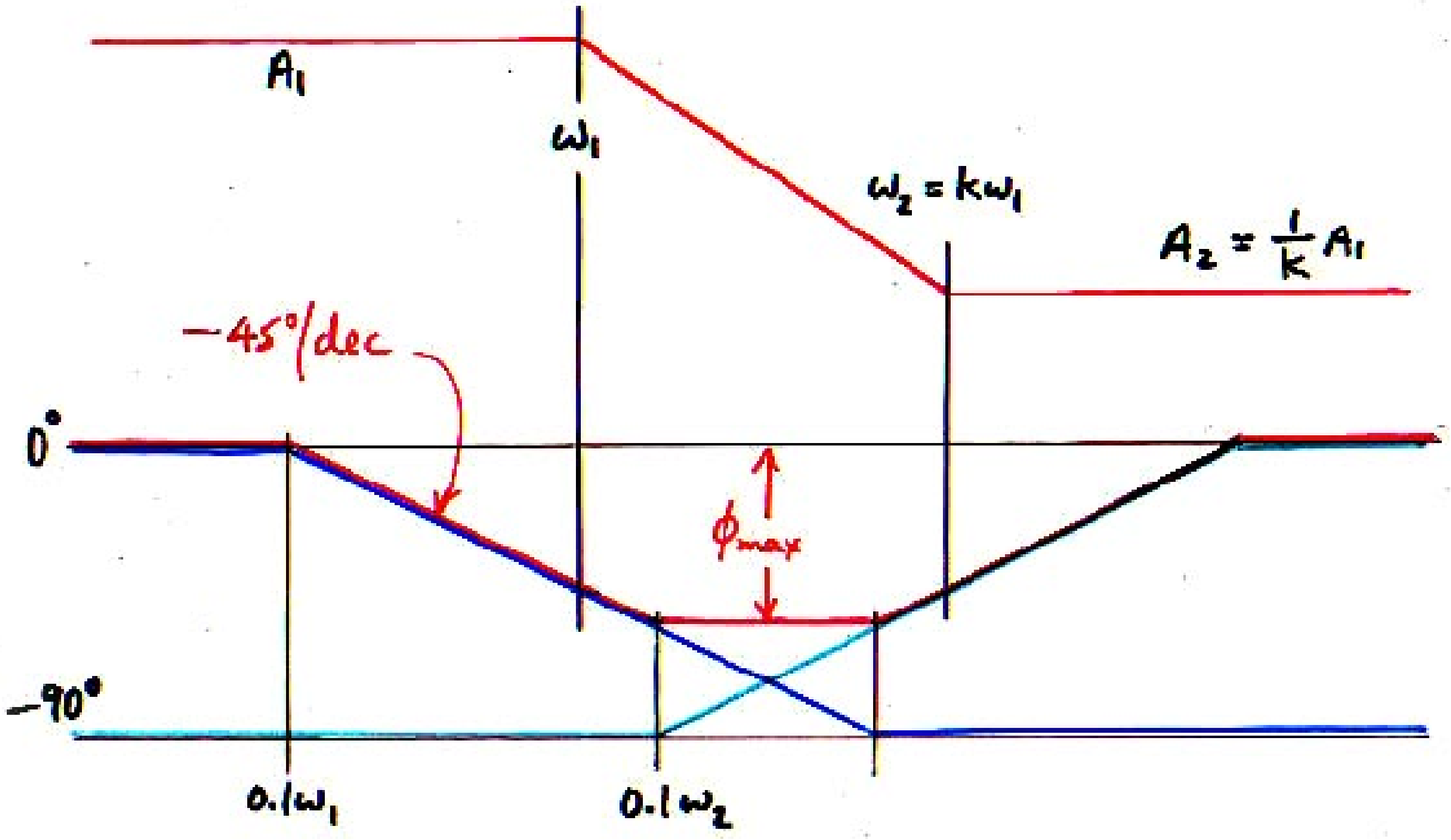


If $w_2 > 100 w_1$, phase asymptotes do not overlap and the phase lag reaches 90° before returning to zero.

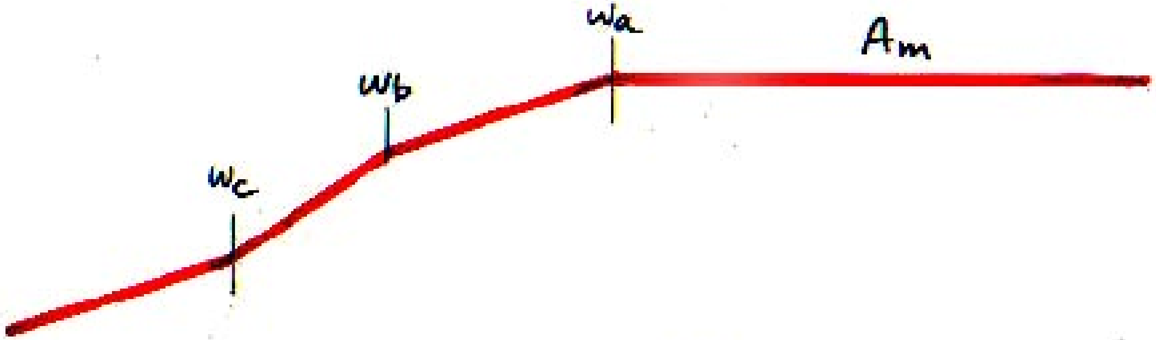
If $w_2 < 100 w_1$, the phase asymptotes do overlap, and the phase lag reaches a maximum, less than 90°, which is a function of the ratio of the flat gains.

Find the maximum phase lag ϕ_{max} as a function of the gain ratio $k \equiv A_1/A_2 = \omega_2/\omega_1$



$$\phi_{max} = -45° \log \frac{0.1\omega_2}{0.1\omega_1} = -45° \log k \quad (k < 100)$$

Relationships to conventional forms:



$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)} = A_m \frac{\frac{\omega_c}{s}}{\frac{\omega_a}{s}\,\frac{\omega_b}{s}} \frac{\left(\frac{s}{\omega_c}+1\right)}{\left(\frac{s}{\omega_a}+1\right)\left(\frac{s}{\omega_b}+1\right)}$$

$$= \frac{A_m \omega_c s}{\omega_a \omega_b} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)} = \frac{s}{\omega_x} \frac{\left(1+\frac{f}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)}$$

conventional form (normal poles and zeros)

Where is ω_x on the graph? Where is A_m in the formula?

ω_x is not a useful parameter.

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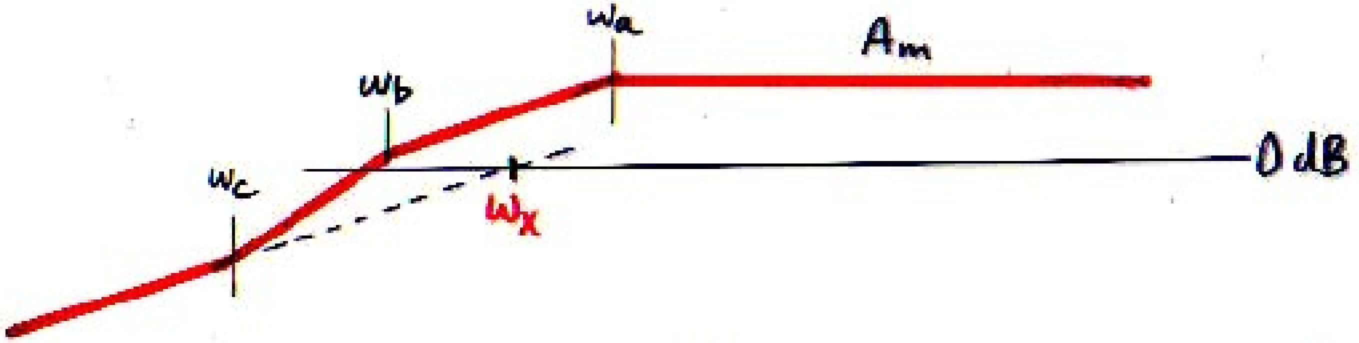

$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)} = A_m \frac{\frac{\omega_c}{s}}{\frac{\omega_a}{s}\,\frac{\omega_b}{s}} \frac{\left(\frac{s}{\omega_c}+1\right)}{\left(\frac{s}{\omega_a}+1\right)\left(\frac{s}{\omega_b}+1\right)}$$

$$= \frac{A_m \omega_c s}{\omega_a \omega_b} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)} = \frac{s}{\omega_x} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)} \swarrow$$

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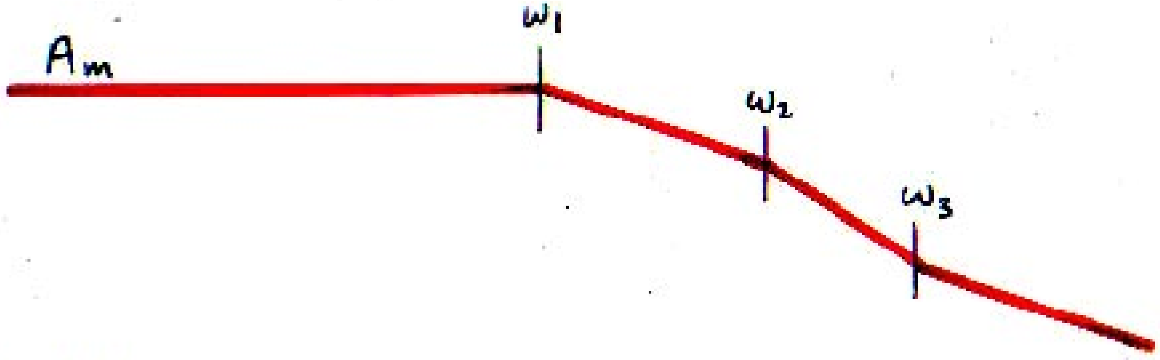


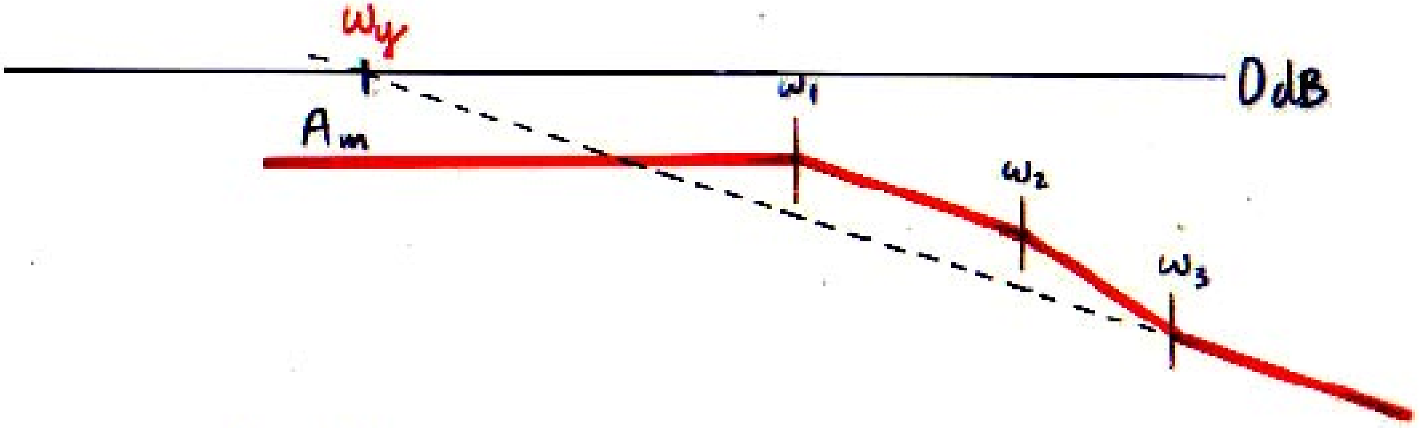
$$A = A_m \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)} = A_m \frac{\frac{1}{\omega_3}}{\frac{1}{\omega_1}\,\frac{1}{\omega_2}} \frac{(\omega_3+s)}{(\omega_1+s)(\omega_2+s)}$$

$$= \frac{A_m \omega_1 \omega_2}{\omega_3} \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)} = \omega_y \cdot \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)}$$

conventional form

Where is ω_y on the graph? Where is A_m in the formula?

ω_y is not a useful parameter.



$$A = A_m \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)} = A_m \frac{\frac{1}{\omega_3}}{\frac{1}{\omega_1}\frac{1}{\omega_2}} \frac{(\omega_3+s)}{(\omega_1+s)(\omega_2+s)}$$

$$= \frac{A_m \omega_1 \omega_2}{\omega_3} \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)} = \omega_y \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)}$$

conventional form

Where is ω_y on the graph? Where is A_m in the formula?

ω_y is not a useful parameter.

More than one flat gain

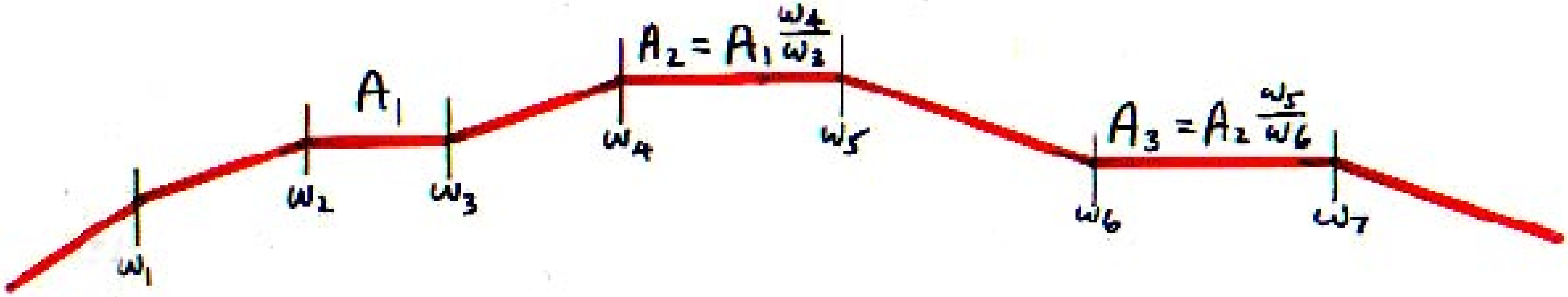
$$A = A_1 \frac{1+\frac{s}{\omega_1}}{1+\frac{s}{\omega_2}} = A_1 \frac{\omega_2}{\omega_1} \frac{1+\frac{\omega_1}{s}}{1+\frac{\omega_2}{s}} = A_2 \frac{1+\frac{\omega_1}{s}}{1+\frac{\omega_2}{s}}$$

Hence: "gain-bandwidth tradeoff":

$$\frac{A_2}{A_1} = \frac{\omega_2}{\omega_1}$$

Either flat gain can be used as "reference" gain.

Any flat gain can be used as "reference" gain Aref. With respect to Aref, poles and zeros above Aref are normal, those below Aref are inverted.



$$A = A_1 \frac{\left(1+\frac{s}{\omega_3}\right)\left(1+\frac{s}{\omega_6}\right)}{\left(1+\frac{\omega_2}{s}\right)\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_4}\right)\left(1+\frac{s}{\omega_5}\right)\left(1+\frac{s}{\omega_7}\right)}$$

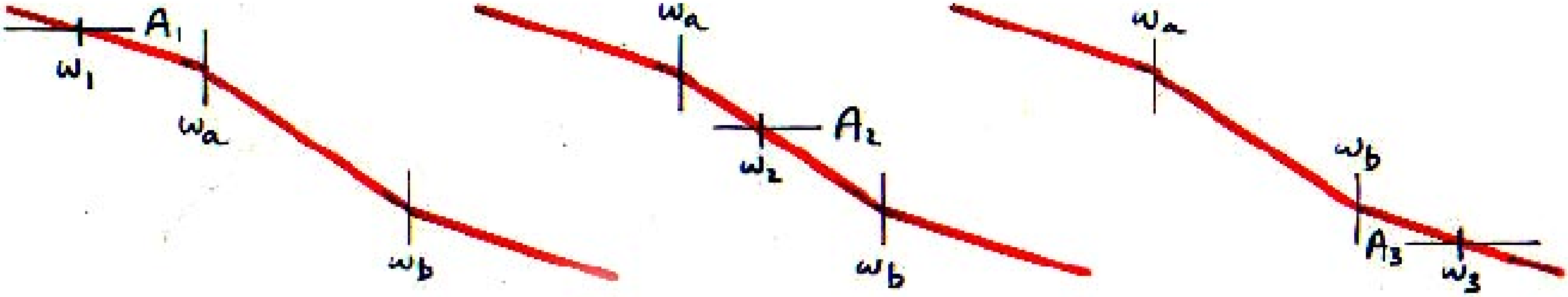
$$A = A_2 \frac{\left(1+\frac{\omega_3}{s}\right)\left(1+\frac{s}{\omega_6}\right)}{\left(1+\frac{\omega_4}{s}\right)\left(1+\frac{\omega_2}{s}\right)\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_5}\right)\left(1+\frac{s}{\omega_7}\right)}$$

$$A = A_3 \frac{\left(1+\frac{\omega_6}{s}\right)\left(1+\frac{\omega_3}{s}\right)}{\left(1+\frac{\omega_5}{s}\right)\left(1+\frac{\omega_4}{s}\right)\left(1+\frac{\omega_2}{s}\right)\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_7}\right)}$$

Exercise:

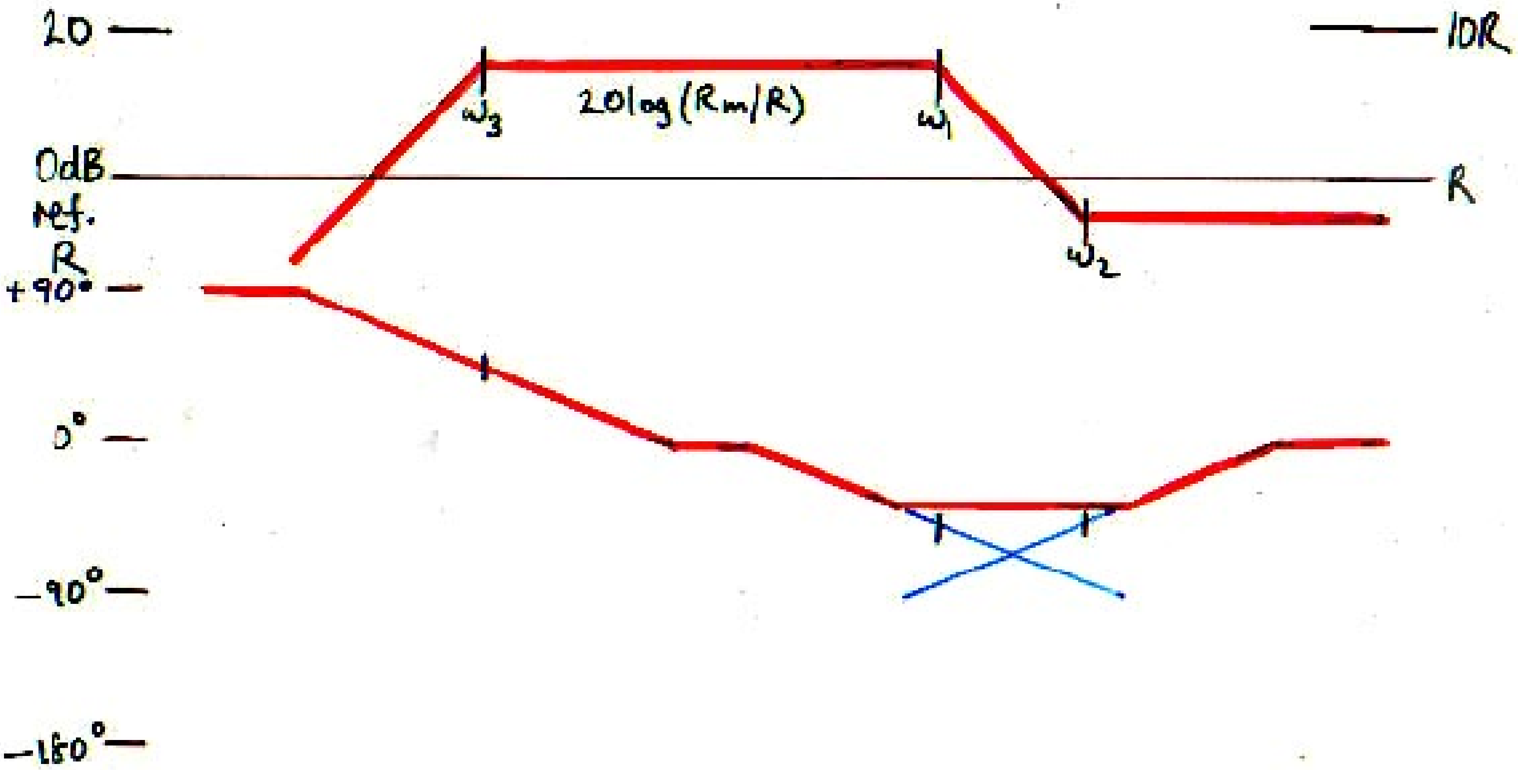
No flat gain

Identify the gain at any chosen frequency as "reference" gain

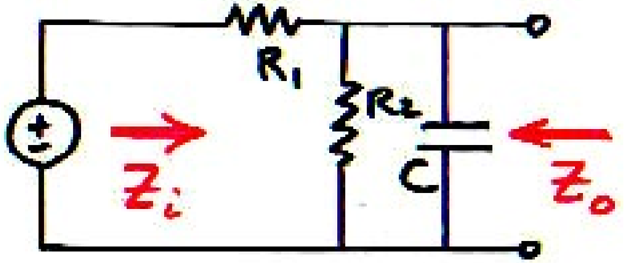


Impedance asymptotes

$$Z = R_m \frac{1 + \frac{s}{\omega_2}}{1 + \frac{s}{\omega_1}} \frac{1}{1 + \frac{\omega_3}{s}}$$



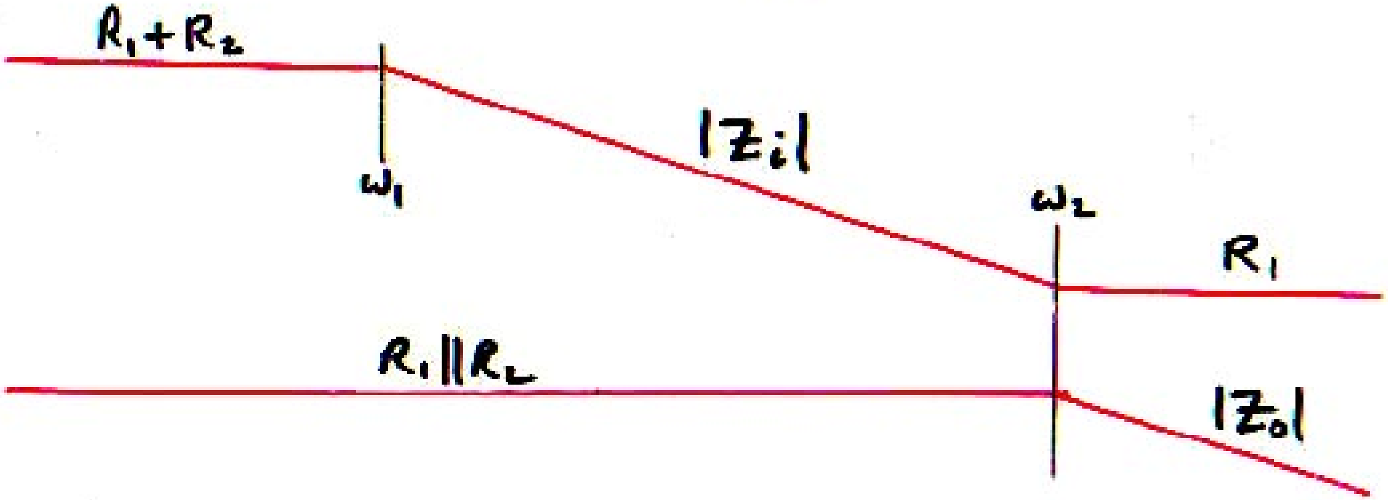
Input and output impedances



$\omega_1 \equiv \frac{1}{CR_2}$ $\qquad$ $\omega_2 \equiv \frac{1}{C(R_1\|R_2)}$

$$Z_i = R_1 + \frac{R_2}{1+sCR_2} = (R_1+R_2)\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = R_1\frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

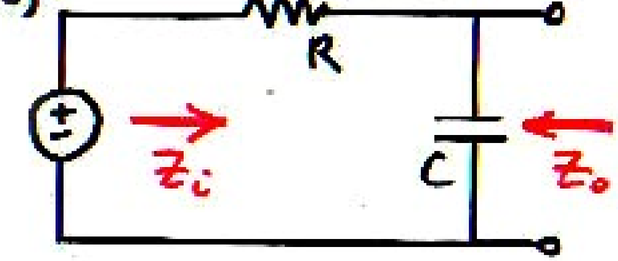
$$Z_o = R_1\|R_2\|\frac{1}{sC} = R_1\|R_2\frac{1}{1+\frac{s}{\omega_2}}$$



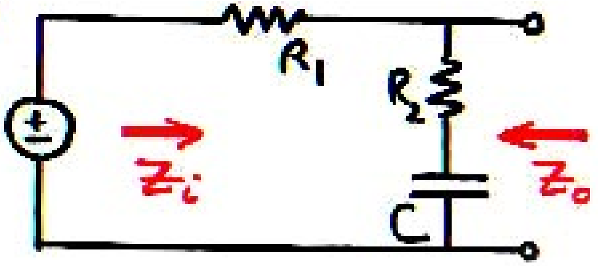
Exercise

Find the input and output impedances Z_i and Z_o in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

(a)

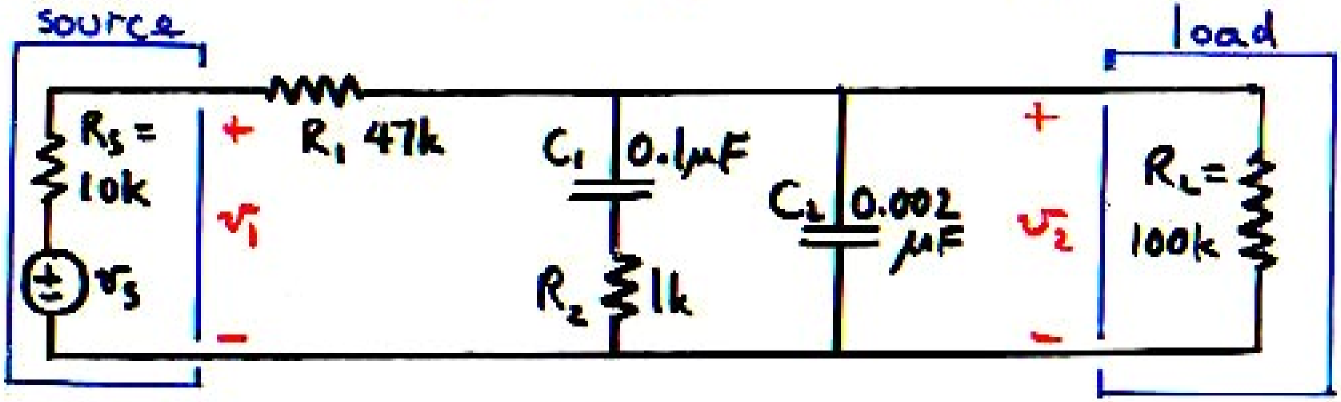


(b)



Example

Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod (1+s/\omega_x)}{\prod (1+s/\omega_y)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.

$$A = \frac{\dfrac{\left(\dfrac{R_L}{1+sC_2R_L}\right)\left(R_2+\dfrac{1}{sC_1}\right)}{\dfrac{R_L}{1+sC_2R_L}+R_2+\dfrac{1}{sC_1}}}{\dfrac{\left(\dfrac{R_L}{1+sC_2R_L}\right)\left(R_2+\dfrac{1}{sC_1}\right)}{\dfrac{R_L}{1+sC_2R_L}+R_2+\dfrac{1}{sC_1}}+R_1}$$

$$\text{a lot}\;\Big\downarrow\;\text{of algebra}$$

$$= \frac{R_L + sC_1R_2R_L}{[R_1+R_L]+s[C_1(R_1R_2+R_LR_2+R_1R_L)+C_2R_1R_L]+s^2[C_1C_2R_1R_2R_L]}$$

This is a high-entropy expression. To lower the entropy, write the polynomials in s with a leading term of unity:

$$A = \frac{R_L}{R_1+R_L}\;\frac{1+sC_1R_2}{1+s\left[C_1\underset{\underset{\displaystyle (R_2+R_1\|R_L)}{\downarrow}}{\left(\frac{R_1R_2+R_LR_2+R_1R_L}{R_1+R_L}\right)}+C_2\underset{\underset{\displaystyle (R_1\|R_L)}{\downarrow}}{\left(\frac{R_1R_L}{R_1+R_L}\right)}\right]+s^2\left[C_1C_2\underset{\underset{\displaystyle R_2(R_1\|R_L)}{\downarrow}}{\left(\frac{R_1R_2R_L}{R_1+R_L}\right)}\right]}$$

Now, recognize series/parallel resistance combinations:

$$A = \frac{R_L}{R_1+R_L}\;\frac{1+sC_1R_2}{1+s\left[C_1\underset{\Big\downarrow}{\left(\frac{R_1R_2+R_LR_2+R_1R_L}{R_1+R_L}\right)}+C_2\underset{\Big\downarrow}{\left(\frac{R_1R_L}{R_1+R_L}\right)}\right]+s^2\left[C_1C_2\underset{\Big\downarrow}{\left(\frac{R_1R_2R_L}{R_1+R_L}\right)}\right]}$$

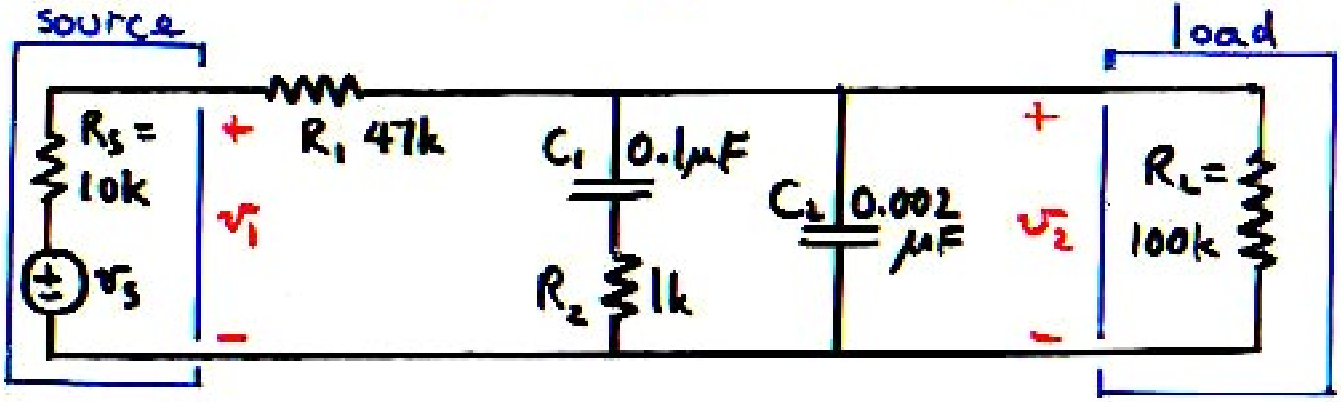
Now, recognize series/parallel resistance combinations: $(R_2 + R_1\|R_L)$ $\quad (R_1\|R_L)$ $\quad R_2(R_1\|R_L)$

The same result, <u>including</u> the series/parallel resistance grouping, could have been obtained with <u>less</u> algebra by elimination, first, of one of the loops of the original circuit.

Circuit with R_1 and R_L absorbed into a Thevenin equivalent:

Example

Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

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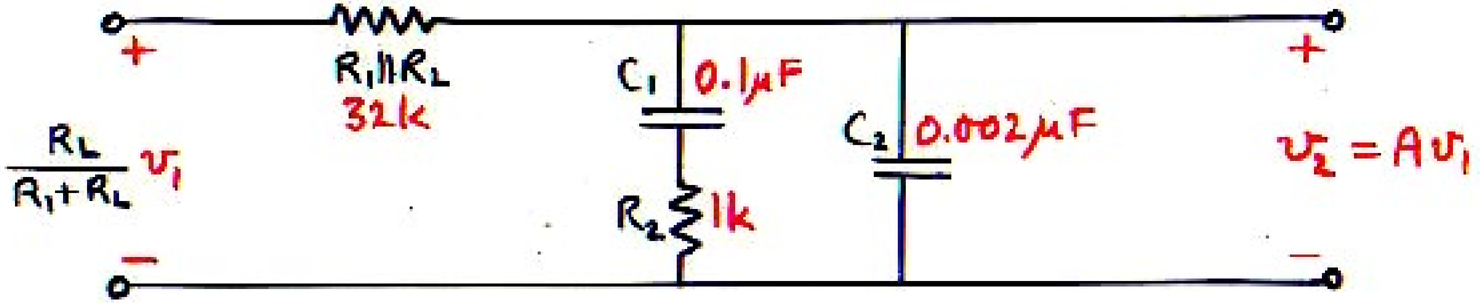
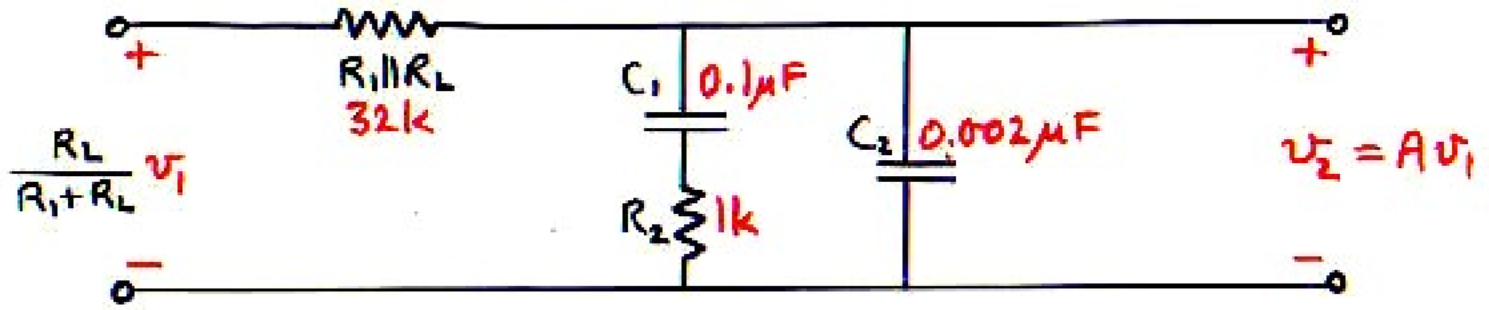


$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}+R_1\|R_L}\cdot\frac{R_L}{R_1+R_L}$$

less $\Big\downarrow$ algebra

$$= \frac{R_L}{R_1+R_L}\cdot\frac{1+sC_1R_2}{1+s\left[C_1(R_2+R_1\|R_L)+C_2(R_1\|R_L)\right]+s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}+R_1\|R_L}\cdot\frac{R_L}{R_1+R_L}$$

less $\Big\downarrow$ algebra

$$= \frac{R_L}{R_1+R_L}\cdot\frac{1+sC_1R_2}{1+s\left[C_1(R_2+R_1\|R_L)+\cancel{C_2(R_1\|R_L)}\right]+s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$

$\uparrow$

Use of numerical values to justify analytic approximation

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$\frac{1}{\omega_{1,3}} = \frac{C_1(\overset{\overset{3.3\times10^{-3}}{\downarrow}}{R_2}+R_1\|R_L) \pm \sqrt{C_1^2(\overset{\overset{10\times10^{-6}}{\downarrow}}{R_2}+R_1\|R_L)^2 - 4C_1\overset{\overset{0.026\times10^{-6}}{\downarrow}}{C_2}R_2(R_1\|R_L)}}{2}$$

This is useless for <u>design</u>, and in any case is inaccurate numerically.

Generalization: Use of Numerical Values to Justify Analytic Approximations

Use numbers to justify leaving out a term, but continue the analysis with the symbols.

This way, the analysis result can be used for design, because the numbers can be changed so that the answer has the desired value. (The approximation must be checked to ensure that it is not invalidated by the new numbers.)

Improved formulas for quadratic roots

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$= a(x-x_1)(x-x_2)$$

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Disadvantages of the conventional form

1. Complicated algebraic expressions in terms of element values:

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2+R_1\|R_L)\pm\sqrt{C_1^2(R_2+R_1\|R_L)^2-4C_1C_2R_2(R_1\|R_L)}}{2}$$

2. Computationally inaccurate when $4ac \ll b^2$:

$$\frac{1}{\omega_{1,3}} = 10^{-3}\,\frac{3.3\underset{\uparrow}{\pm}\sqrt{3.3^2-0.026}}{2}$$

small difference of large numbers, for one root

Disadvantages of the conventional form:

2. Small difference of large numbers, in some cases:

$$z^2 + bz + 1 = 0; \quad b = 45{,}000$$

$$z_1 = \frac{-b + \sqrt{b^2 - 4}}{2} \qquad z_2 = \frac{-b - \sqrt{b^2 - 4}}{2}$$

$$= \qquad\qquad =$$

Disadvantages of the conventional form:

2. Small difference of large numbers, in some cases:

$$z^2 + bz + 1 = 0; \quad b = 45{,}000$$

$$z_{1a} = \frac{-b + \sqrt{b^2 - 4}}{2} \qquad z_2 = \frac{-b - \sqrt{b^2 - 4}}{2}$$

$$=$$

$$=$$

$$z_{1b} = -\frac{b}{2}\left[1 - \sqrt{1 - 4/b^2}\right]$$

$$=$$

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$$z_{1a} = \frac{-b + \sqrt{b^2 - 4}}{2} \qquad z_2 = \frac{-b - \sqrt{b^2 - 4}}{2}$$

$$= -2.000{,}000{,}000 \times 10^{-5} \qquad = -44{,}999.999{,}98$$

$$z_{1b} = -\frac{b}{2}\left[1 - \sqrt{1 - 4/b^2}\right]$$

HPISC: works with 10 digits, last one rounded.

$$=$$

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$$z_{1b} = -\frac{b}{2}\left[1-\sqrt{1-4/b^2}\right]$$

$$= -2.250{,}000{,}000 \times 10^{-5}$$

HP15C: works with 10 digits, last one rounded.

Is z_{1a} or z_{1b} the correct answer? Or is neither correct?

Try a third algorithm: successive approximation

$$z^2 + bz + 1 = 0$$

$$bz_1 = -(1 + z_1^2)$$

$$z_1 = -\frac{1}{b}(1 + z_1^2)$$

Since b>>1, then $z_1 \ll 1$, so:

$$z_1 \approx -\frac{1}{b}\left(1 + \frac{1}{b^2}\right) = -\frac{1}{b} - \frac{1}{b^3}$$

$$= -2.222{,}222{,}22\dot{2} \times 10^{-5}$$

$$-0.000{,}000{,}001{,}097{,}393{,}690 \times 10^{-5}$$

So, the correct answer is

$z_1 = -2.222{,}222{,}22 \times 10^{-5}$ correct to 9 significant figures.

Note that:

$$\frac{z_{1a}}{z_1} - 1 = \frac{-2.000{,}000{,}000}{-2.222{,}222{,}22} - 1 = -10\%$$

$$\frac{z_{1b}}{z_1} - 1 = \frac{-2.250{,}000{,}000}{-2.222{,}222{,}22} - 1 = +1.25\%$$

b	Z_{1a} error	Z_{1b} error
42,000	5.00%	5.84%
43,000	7.50%	1.70%
44,000	10.0%	6.48%
45,000	-10.0%	1.25%
46,000	-8.00%	5.60%
47,000	-6.00%	-5.99%
48,000	-4.00%	3.68%
49,000	-2.00%	8.05%
50,000	0%	0%
51,000	2.00%	4.04%

Conclusions:

(a) The correct result was obtained with use of an algebraic approximation, whereas the incorrect results were obtained without an algebraic approximation.

(b) The numerical error introduced depends on the algebraic format. (Small difference of large numbers)

Hence, need a better algebraic format for the roots of a quadratic.

Better method:

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right] = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}, \qquad Q^2 \equiv \frac{ac}{b^2}$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]$$

$$= -\frac{b}{a}\,\frac{\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]} = -\frac{b}{a}\,\frac{\frac{1}{4}-\frac{1}{4}(1-4Q^2)}{F}$$

$$= -\frac{b}{a}\,\frac{Q^2}{F} = -\frac{c}{b}\,\frac{1}{F}$$

Better method:

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right] = -\frac{b}{a}F$$

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$$= -\frac{b}{a}\,\frac{\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]} = -\frac{b}{a}\,\frac{\frac{1}{4}-\frac{1}{4}(1-4Q^2)}{F}$$

$$= -\frac{b}{a}\,\frac{Q^2}{F} = -\frac{c}{b}\,\frac{1}{F}$$

Crucial step: Large numbers are subtracted exactly, leaving the small difference in analytic form.

Hence, both roots can be computed with equal accuracy:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad x_2 = -\frac{b}{a}F$$

Rewrite the two roots:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

x_2 is acceptable for all values; x_1 is unacceptable for $4ac \ll b^2$.
Rewrite x_2:

$$x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Now, instead of using the formula for x_1 directly, use the property of the quadratic that $x_1 x_2 = \frac{c}{a}$:

$$x_1 = \frac{c}{a}\frac{1}{x_2} = -\frac{c}{a}\,\frac{a}{b}\,\frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}}$$

Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\,\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Improved formulas for quadratic roots

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) \\ &= a(x - x_1)(x - x_2) \end{aligned} \qquad x_1 x_2 = \frac{c}{a}$$

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Rewrite the two roots:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad\qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

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Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\,\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad\qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

More elegant form:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

More elegant form:

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simple ratios of the original quadratic coefficients

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simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

This is exact for all values.

If $Q > 0.5$, F is complex $\Rightarrow$ complex roots

If $Q < 0.5$, F is real $\Rightarrow$ real roots

If $Q \ll 0.5$, $F \approx 1$

Note how simple the analytic roots, and therefore the quadratic factorization, become if $F \approx 1$.

More elegant form:

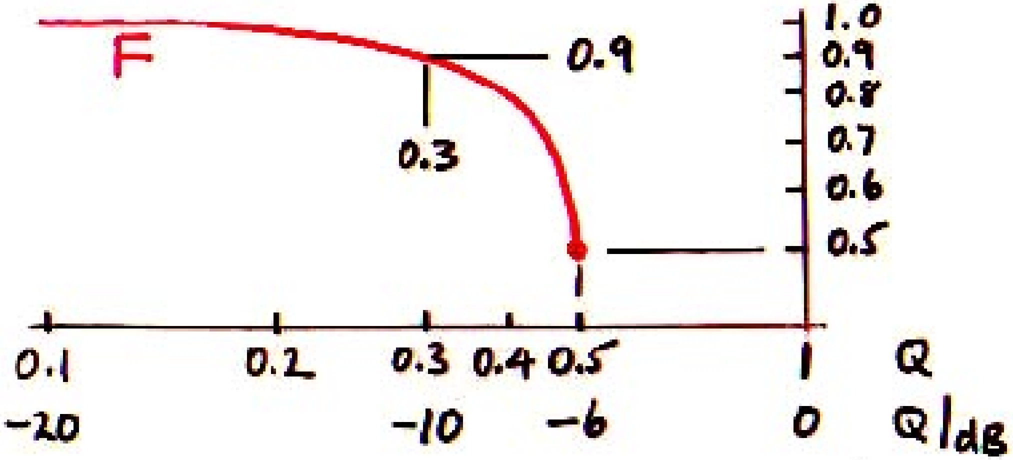
$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

$F \to 1$ very rapidly as Q drops below 0.5:



$F \approx 1$ with 10% error for $Q \le 0.3$

General result:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$= a\left(x + \frac{c}{b}\frac{1}{F}\right)\left(x + \frac{b}{a}F\right)$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}$$

$$x_1 = -\frac{c}{b}\frac{1}{F}$$

$$x_2 = -\frac{b}{a}F$$

General result:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$= a\left(x + \frac{c}{b}\frac{1}{F}\right)\left(x + \frac{b}{a}F\right) \overset{F=1}{\approx} a\left(x + \frac{c}{b}\right)\left(x + \frac{b}{a}\right)$$

Good approximation for real roots, $Q \equiv \sqrt{\frac{ac}{b^2}} \lesssim 0.5$: $F \approx 1$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}$$

$$x_1 = -\frac{c}{b}\frac{1}{F} \approx -\frac{c}{b}$$

$$x_2 = -\frac{b}{a}F \approx -\frac{b}{a}$$

Alternative format:

$$
\begin{aligned}
ax^2+bx+c &= c\left(1+\frac{b}{c}x+\frac{a}{c}x^2\right) \\
&= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right) \\
&= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right)
\end{aligned}
$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\overset{a_1}{\overset{\downarrow}{\frac{b}{c}}}x+\overset{a_2}{\overset{\downarrow}{\frac{a}{c}}}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Redefine coefficients:

$$1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= \left(1+a_1Fx\right)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{a_2}{a_1^2}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\overset{\overset{a_1}{\downarrow}}{\frac{b}{c}}x+\overset{\overset{a_2}{\downarrow}}{\frac{a}{c}}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right) \overset{F=1}{\approx} c\left(1+\frac{b}{c}x\right)\left(1+\frac{a}{b}x\right)$$

Good approximatoin
for real roots, $Q \leq 0.5$:
$F \approx 1$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Redefine coefficients:

$$1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= \left(1+a_1Fx\right)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right) \overset{F=1}{\approx} \left(1+a_1x\right)\left(1+\frac{a_2}{a_1}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{a_2}{a_1^2}$$

Generalization: Improved Formulas for Roots of a Quadratic

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2} \qquad\qquad Q^2 \equiv \frac{a_2}{a_1^{\,2}}$$

$$ax^2+bx+c = a(x-x_1)(x-x_2) \qquad\qquad 1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad\qquad x_1 = -\frac{1}{a_1F}$$

$$\frac{x_1}{x_2} = \frac{Q^2}{F^2}$$

$$x_2 = -\frac{b}{a}F \qquad\qquad x_2 = -\frac{a_1}{a_2}F$$

$$ax^2+bx+c = a\left(x+\frac{c}{b}\frac{1}{F}\right)\left(x+\frac{b}{a}F\right) \qquad\qquad 1+a_1x+a_2x^2 = (1+a_1Fx)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right)$$

For real roots, $Q \le 0.5$ and $F \approx 1$:

$$x_1 \approx -\frac{c}{b} \qquad\qquad x_1 \approx -\frac{1}{a_1}$$

$$\frac{x_1}{x_2} \approx Q^2$$

$$x_2 \approx -\frac{b}{a} \qquad\qquad x_2 \approx -\frac{a_1}{a_2}$$

$$\underset{\underset{\textcircled{1}}{\uparrow}}{a}x^2+\underset{\underset{\textcircled{2}}{\uparrow}}{b}x+\underset{\underset{\textcircled{3}}{\uparrow}}{c} \approx a\left(x+\underset{\underset{\textcircled{3}/\textcircled{2}}{\uparrow}}{\frac{c}{b}}\right)\left(x+\underset{\underset{\textcircled{2}/\textcircled{1}}{\uparrow}}{\frac{b}{a}}\right) \qquad\qquad \underset{\underset{\textcircled{1}}{\uparrow}}{1}+\underset{\underset{\textcircled{2}}{\uparrow}}{a_1}x+\underset{\underset{\textcircled{3}}{\uparrow}}{a_2}x^2 \approx (1+\underset{\underset{\textcircled{2}/\textcircled{1}}{\uparrow}}{a_1}x)\left(1+\underset{\underset{\textcircled{3}/\textcircled{2}}{\uparrow}}{\frac{a_2}{a_1}}x\right)$$

Advantages over the conventional formulas

1. Both roots can be computed with equal accuracy (avoids small difference of large numbers).
2. For real roots, to a very good approximation, there is no $\sqrt{\quad}$ anywhere in the results, and each root is a simple ratio of coefficients of the original quadratic.

Useful format of quadratic $1+a_1 s+a_2 s^2$

Define: $Q \equiv \frac{\sqrt{a_2}}{a_1}$

If $Q>0.5$ (F complex), roots are complex.

Leave in quadratic form:

$$1+a_1 s+a_2 s^2 = 1+\frac{a_1}{\sqrt{a_2}}\left(\sqrt{a_2}\, s\right)+\left(\sqrt{a_2}\, s\right)^2$$

$\frac{1}{Q}$ ↗

↖ normalized frequency

If $Q<0.5$ ($F \approx 1$), roots are real.

Factor into two real roots:

real corner frequencies

$$1+a_1 s+a_2 s^2 \approx \left(1+a_1 s\right)\left(1+\frac{a_2}{a_1} s\right)$$

$$= \left[1+\frac{a_1}{\sqrt{a_2}}\left(\sqrt{a_2}\, s\right)\right]\left[1+\frac{\sqrt{a_2}}{a_1}\left(\sqrt{a_2}\, s\right)\right]$$

$\frac{1}{Q}$ ↗ Q ↗ ↖ normalized frequency

Exercise:

Find, both analytically and numerically, the Q and hence the roots ω_1 and ω_3 of the quadratic:

$$1 + C_1[R_2 + R_1 \| R_L]s + [C_1 C_2 R_2 (R_1 \| R_L)]s^2$$

where $C_1 = 0.1\mu F$, $C_2 = 0.002\mu F$, $R_1 = 47k$, $R_2 = 1k$, $R_L = 100k$. Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.

Exercise:

Find, both analytically and numerically, the Q and hence the roots w_1 and w_3 of the quadratic:

$$1 + \underbrace{C_1[R_2 + R_1 \| R_L]}_{a_1} s + \underbrace{[C_1 C_2 R_2 (R_1 \| R_L)]}_{a_2} s^2$$

where $C_1 = 0.1\mu F$, $C_2 = 0.002\mu F$, $R_1 = 47k$, $R_2 = 1k$, $R_L = 100k$. Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.

$$Q^2 = \frac{a_2}{a_1^2} = \frac{C_1 C_2 R_2 (R_1 \| R_L)}{C_1^2 (R_2 + R_1 \| R_L)^2} = \frac{C_2}{C_1} \frac{R_2 \| \cancel{R_1 \| R_L}}{\cancel{R_2} + R_1 \| R_L} \approx \frac{C_2}{C_1} \frac{R_2}{R_1 \| R_L} = \frac{1}{50} \frac{1}{47 \| 100} = \frac{1}{1{,}600}$$

Hence, $F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \approx 1$, so the roots are real: $(F = 0.9994)$ $\quad Q = \frac{1}{40} << 0.5$

$$w_1 = \frac{1}{a_1} = \frac{1}{C_1 (R_2 + R_1 \| R_L)} \qquad f_1 = \frac{159}{0.1(1 + \underbrace{47 \| 100}_{32})} \text{ Hz} = 48 \text{ Hz}$$

$$w_3 = \frac{a_1}{a_2} \doteq \frac{C_1 (R_2 + R_1 \| R_L)}{C_1 C_2 R_2 (R_1 \| R_L)}$$

$$= \frac{1}{C_2 (R_2 \| R_1 \| R_L)} \qquad f_3 = \frac{159}{0.002(\underbrace{1 \| 32}_{0.97})} \text{ Hz} = 82 \text{ kHz}$$

Hence

$$A \approx \frac{R_L}{R_L+R_1} \frac{1+sC_1R_2}{[1+C_1(R_2+R_1 \| R_L)s][1+C_2(R_1 \| R_2 \| R_L)s]}$$

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$A_0 \equiv \frac{R_L}{R_L+R_1}$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1 \| R_L)}$$

$$\omega_2 \equiv \frac{1}{C_1R_2}$$

$$\omega_3 \equiv \frac{1}{C_2(R_1 \| R_2 \| R_L)}$$

Hence

$$A \approx \frac{R_L}{R_L+R_1} \frac{1+sC_1R_2}{[1+C_1(R_2+R_1 \| R_L)s][1+C_2(R_1 \| R_2 \| R_L)s]}$$

$$= A_0 \frac{(1+\frac{s}{\omega_2})}{(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_3})}$$

where

$$A_0 \equiv \frac{R_L}{R_L+R_1} = \frac{100}{100+47} = 0.68 \Rightarrow -3.4dB$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1 \| R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47 \| 100}_{32})} = 48Hz$$

$$\omega_2 \equiv \frac{1}{C_1R_2} \qquad f_2 = \frac{159}{0.1 \times 1} = 1.6kHz$$

$$\omega_3 \equiv \frac{1}{C_2(R_1 \| R_2 \| R_L)} \qquad f_3 = \frac{159}{0.002(47 \| 1 \| 100)} = 82kHz$$

The conventional quadratic formula for the two poles ω_1 and ω_3 is much higher entropy (gives much less useful information) than does the modified formula.

Conventional:

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2+R_1\|R_L) \pm \sqrt{C_1^2(R_2+R_1\|R_L)^2 - 4C_1C_2R_2(R_1\|R_L)}}{2}$$

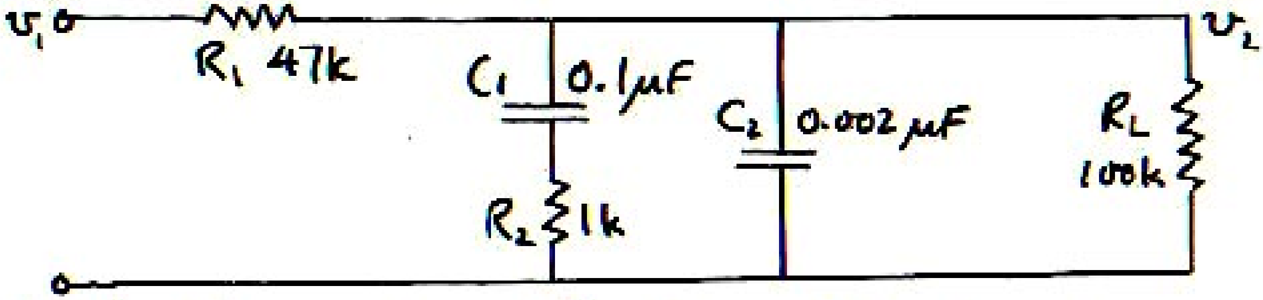
Modified:

$$\omega_1 = \frac{1}{C_1(R_2+R_1\|R_L)} \qquad \omega_3 = \frac{1}{C_2(R_1\|R_2\|R_L)}$$

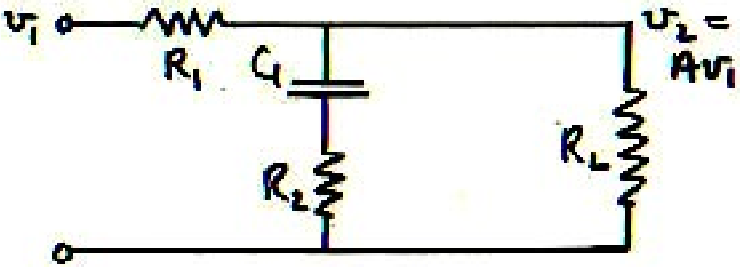
Note, in particular, (when the two roots are real and well-separated) that the modified formula is much lower entropy and not only gives both roots with equal numerical accuracy, but also exposes the fact that C_1 affects only ω_1 and C_2 affects only ω_3 — which is useful information for design purposes.

A still better solution

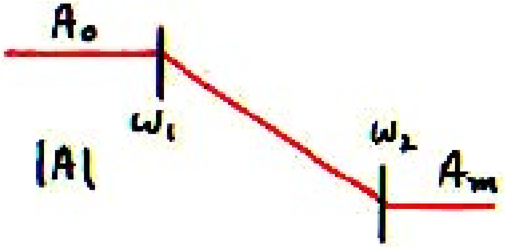
Look at the original circuit and consider response as frequency increases:



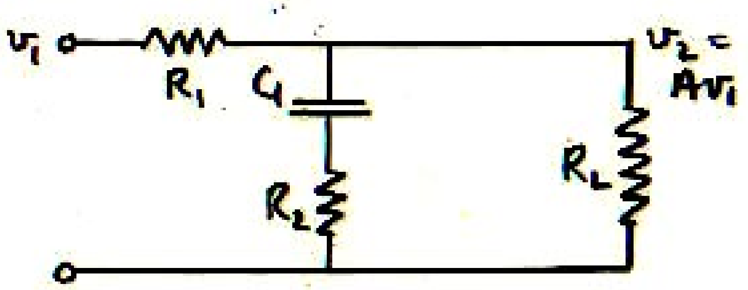
At low frequencies, both capacitances are open, so have flat response. As frequency increases, the reactance of C_1, the larger capacitance, comes down causing a pole. When the reactance of C_1 drops below R_2, the response flattens causing a zero. However, at this frequency the reactance of C_2 is still 50 times higher than R_2, so C_2 has negligible effect.

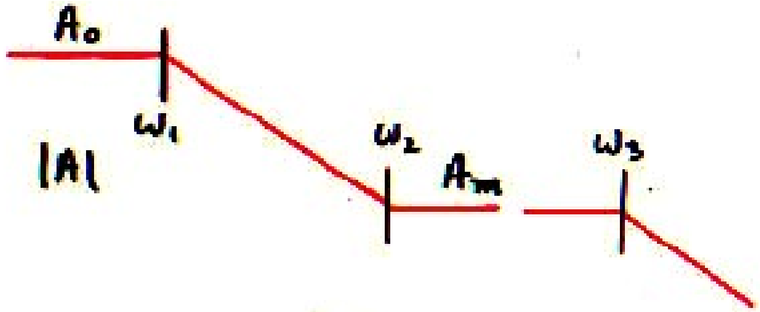


v_1
R_1
C_1
R_2
R_L
$v_2 = Av_1$

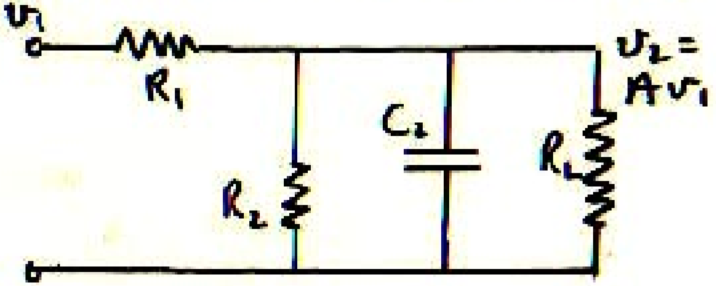


A_0
ω_1
$|A|$
ω_2
A_m



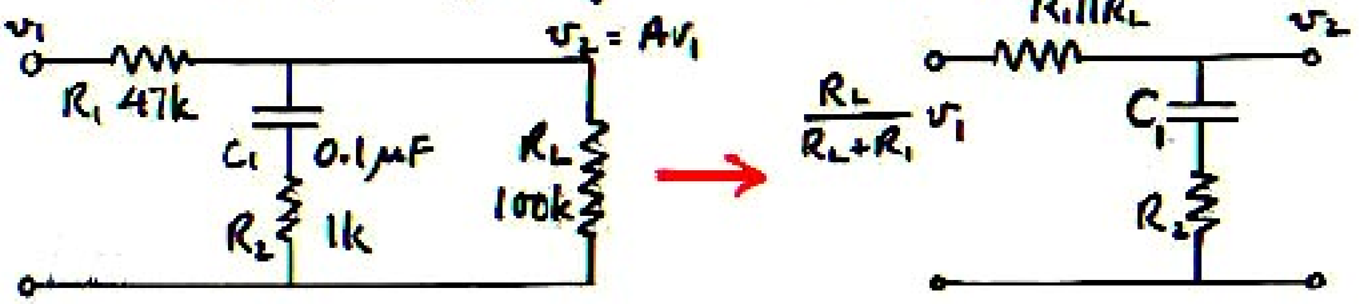


At still higher frequencies, the reactance of C_2 drops below R_2, causing a second pole



Hence, the solution can be obtained in two parts, each containing only one reactance (one pole).

Low-frequency range (C_2 open)



$$A = \frac{R_L}{R_L+R_1}\,\frac{R_2+\frac{1}{sC_1}}{R_2+\frac{1}{sC_1}+R_1\|R_L} = A_0\,\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_m\frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

where

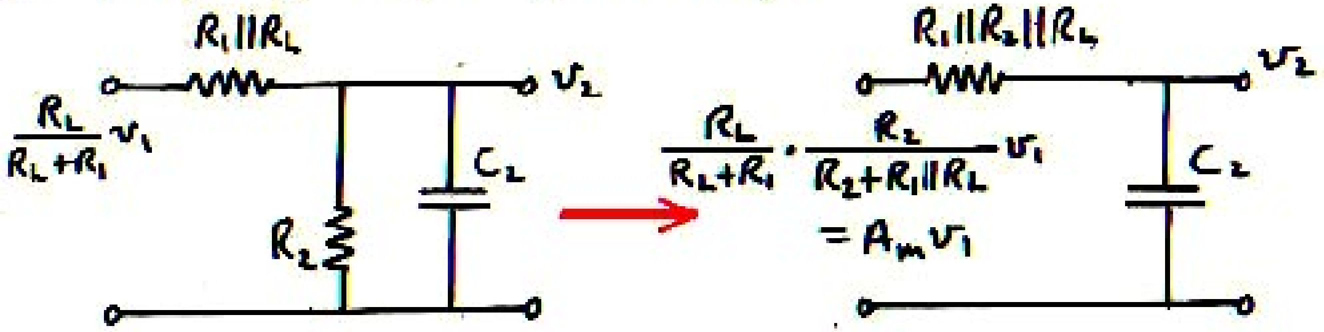
$$A_0 \equiv \frac{R_L}{R_L+R_1} = \frac{100}{100+47} = 0.68 \Rightarrow -3.4\text{dB}$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})} = 48\text{Hz}$$

$$\omega_2 \equiv \frac{1}{C_1R_2} \qquad f_2 = \frac{159}{0.1\times 1} = 1.6\text{kHz}$$

$$A_m \equiv A_0\frac{\omega_1}{\omega_2} = \frac{R_L}{R_L+R_1}\cdot\frac{R_2}{R_2+R_1\|R_L} = 0.68\,\frac{0.048}{1.6} = 0.02 \Rightarrow -34\text{dB}$$

High-frequency range (C_1 short)

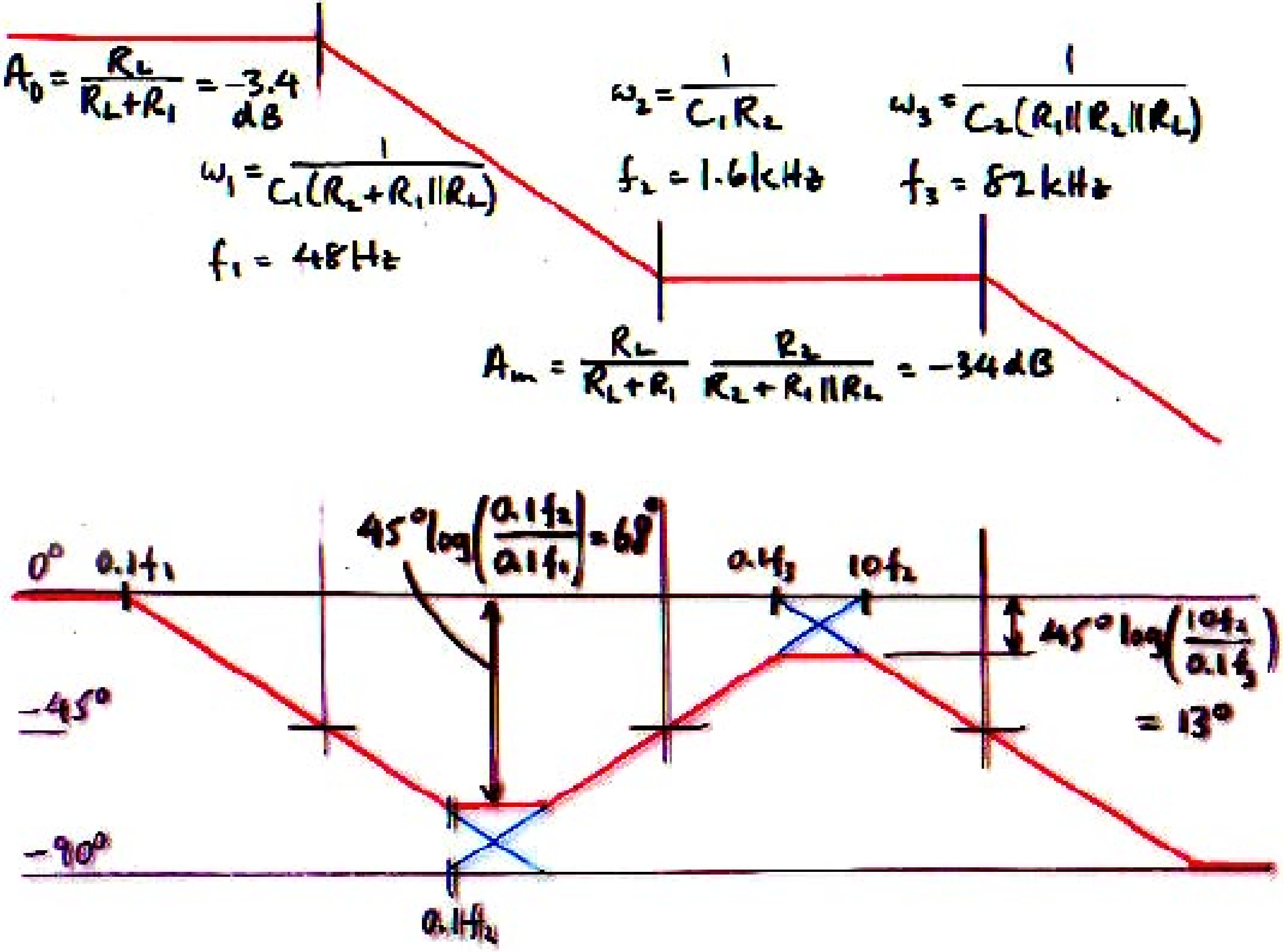


$$A = A_m \frac{1}{1+\frac{s}{\omega_3}} \quad \text{where} \quad \omega_3 \equiv \frac{1}{C_2 (R_1 \| R_2 \| R_L)}$$

$$f_3 = \frac{159}{0.002 (47 \| 1 \| 100)} = 82\,\text{kHz}$$

Hence, overall response is

$$A = A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)} = A_m \frac{\left(1+\frac{\omega_2}{s}\right)}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_3}\right)}$$


$A_0 = \frac{R_L}{R_L+R_1} = -3.4$ dB
$\omega_1 = \frac{1}{C_1(R_2+R_1\|R_L)}$
$f_1 = 48$ Hz
$\omega_2 = \frac{1}{C_1 R_2}$
$f_2 = 1.6$ kHz
$\omega_3 = \frac{1}{C_2(R_1\|R_2\|R_L)}$
$f_3 = 82$ kHz
$A_m = \frac{R_L}{R_L+R_1} \frac{R_2}{R_2+R_1\|R_L} = -34$ dB
$45° \log\left(\frac{0.1f_2}{0.1f_1}\right) = 68°$
0°
$0.1f_1$
$0.1f_3$
$10f_2$
$45° \log\left(\frac{10f_2}{0.1f_3}\right)$
$= 13°$
$-45°$
$-90°$
$0.1f_2$

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2 \overset{\overset{3.3\times10^{-3}}{\downarrow}}{+} R_1\|R_L) \pm \sqrt{C_1^2(\overset{\overset{10\times10^{-6}}{\downarrow}}{R_2}+R_1\|R_L)^2 - 4C_1\overset{\overset{0.026\times10^{-6}}{\downarrow}}{C_2}R_2(R_1\|R_L)}}{2}$$

This is useless for <u>design</u>, and in any case is inaccurate numerically.

Generalization: Presentation of Results

Sketch magnitude and phase by straight-line asymptotes, and label salient features (flat gains, corner frequencies, Q's, etc.) with both analytic expressions and numerical values.

This is a compact summary so that both the analytic and numerical results can be interpreted at a glance, which is especially useful for reports, design reviews, etc. so that managers can easily and quickly see and understand the results obtained by others.

For design, the element values that must be changed to give different numerical results can easily be seen.