6

NULL DOUBLE INJECTION (NDI)

AND THE

EXTRA ELEMENT THEOREM (EET)

Null Double Injection

Consider a system with a single injected (input) signal u_{i1}, and two dependent (output) signals u_{o1}, u_{o2}:

Consider a second injected signal u_{i2} which alone would give:

If the system is linear, in general each output is a linear sum of the outputs due to each input:

$$\circ\xrightarrow{u_{i1}} \qquad\qquad \xrightarrow{u_{o1} = A_1 u_{i1} + A_2 u_{i2}}\circ$$

$$\circ\xrightarrow{u_{i2}} \qquad\qquad \xrightarrow{u_{o2} = B_1 u_{i1} + B_2 u_{i2}}\circ$$

Note definition of "gains:"

$$\left.\frac{u_{o1}}{u_{i1}}\right|_{u_{i2}=0} = A_1 \qquad \left.\frac{u_{o1}}{u_{i2}}\right|_{u_{i1}=0} = A_2$$

$$\left.\frac{u_{o2}}{u_{i1}}\right|_{u_{i2}=0} = B_1 \qquad \left.\frac{u_{o2}}{u_{i2}}\right|_{u_{i1}=0} = B_2$$

By adjustment of u_{i2} relative to u_{i1} (in magnitude and phase), either output can be nulled:

For example, if u_{o1} is nulled:

$$0 = A_1 u_{i1} + A_2 u_{i2}\Big|_{u_{o1}=0}$$

$$u_{o2} = B_1 u_{i1} + B_2 u_{i2}\Big|_{u_{o1}=0}$$

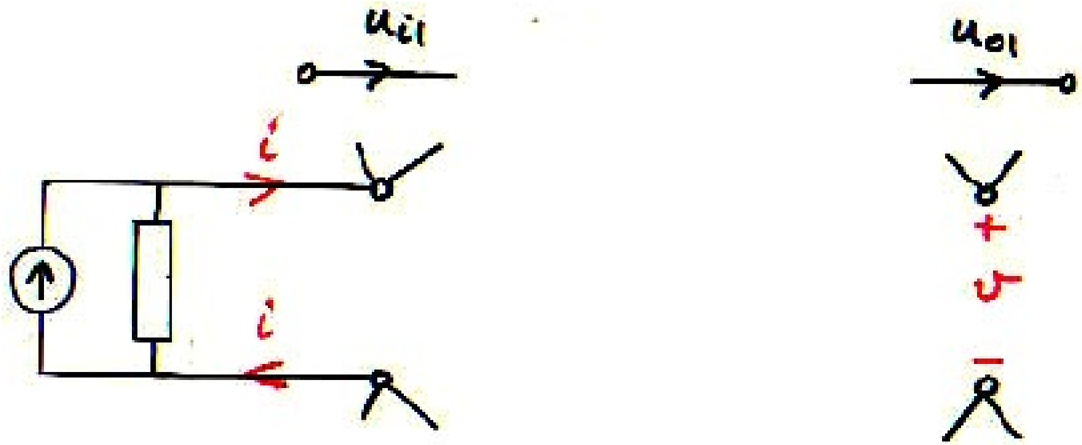
By elimination of u_{i1}:

$$\left.\frac{u_{o2}}{u_{i2}}\right|_{u_{o1}=0} = \frac{A_1 B_2 - A_2 B_1}{A_1}$$

Notice the difference from

$$\left.\frac{u_{o2}}{u_{i2}}\right|_{u_{i1}=0} = B_2$$

Particular example: $u_{i2} = i,\ u_{o2} = v$:



$$u_{o1} = A_1 u_{i1} + A_2 i$$

$$v = B_1 u_{i1} + B_2 i$$

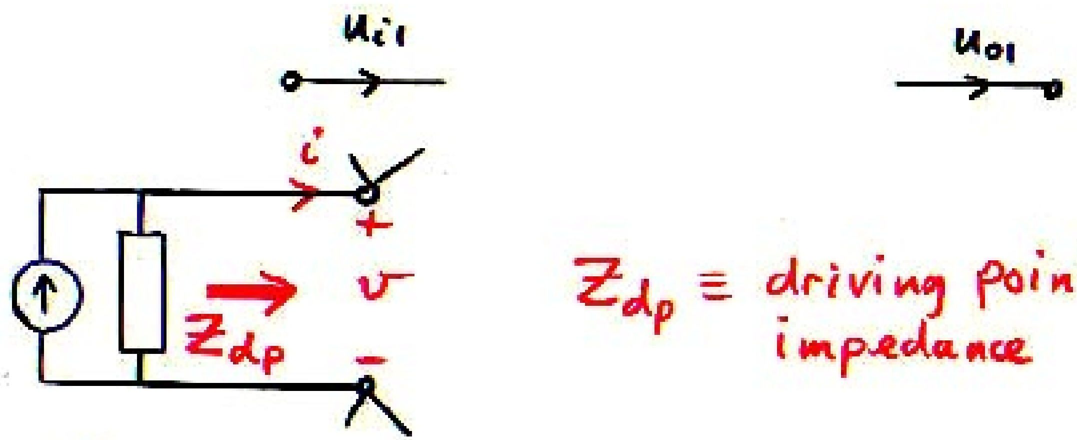
So

$$\left.\frac{v}{i}\right|_{u_{i1}=0} = B_2$$

$$\left.\frac{v}{i}\right|_{u_{o1}=0} = \frac{A_1 B_2 - A_2 B_1}{A_1}$$

Special case: the Extra Element Theorem

The second output v is across the second input terminals:



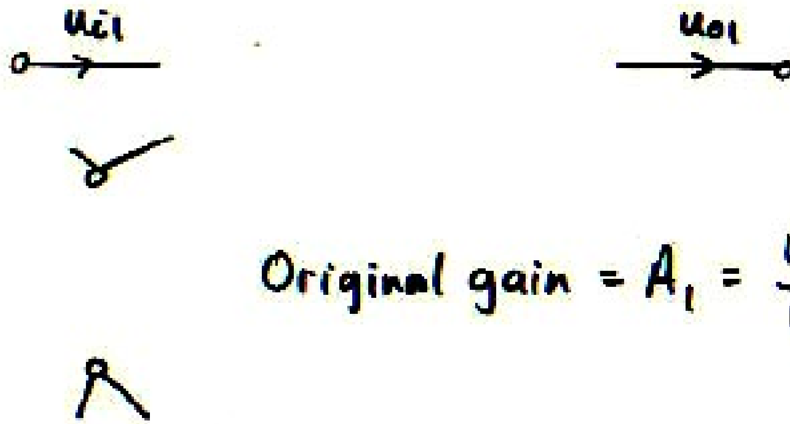
$Z_{dp} \equiv$ driving point impedance

Then

$$\frac{v}{i}\bigg|_{u_{i1}=0} = Z_{dp}\Big|_{u_{i1}=0} \equiv Z_d = B_2$$

$$\frac{v}{i}\bigg|_{u_{o1}=0} = Z_{dp}\Big|_{u_{o1}=0} \equiv Z_n = \frac{A_1 B_2 - A_2 B_1}{A_1}$$

Consider the original circuit without $u_{i2} = i$:



Original gain $= A_1 = \left.\frac{u_{o1}}{u_{i1}}\right|_{u_{i2}=0}$

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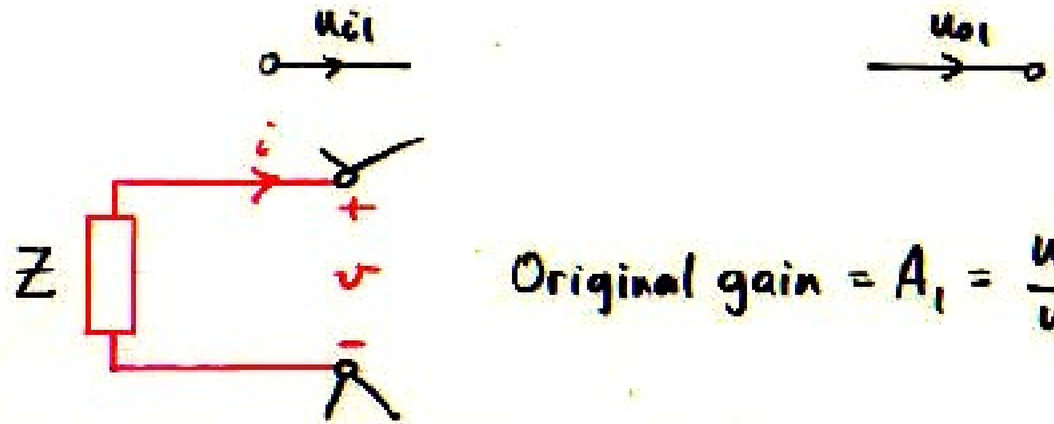

Original gain $= A_1 = \left.\dfrac{u_{o1}}{u_{i1}}\right|_{u_{i2}=0}$

Now consider the circuit with an extra element Z:

$$i = -\frac{v}{Z}$$

So

$$u_{o1} = A_1 u_{i1} - \frac{A_2}{Z} v$$

$$v = B_1 u_{i1} - \frac{B_2}{Z} v$$

Eliminate v:

$$u_{o1} = A_1 \frac{1 + \frac{1}{Z}\frac{A_1 B_2 - A_2 B_1}{A_1}}{1 + \frac{1}{Z} B_2} u_{i1}$$

Hence

$$u_{o1} = A_1 \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} u_{i1}$$

or

$$\text{gain}\Big|_{Z} = \text{gain}\Big|_{Z=\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}}$$

This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.

Hence

$$u_{o1} = A_1 \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} u_{i1}$$

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This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.

The Theorem also proves that any transfer function (e.g. gain) of a linear system is a bilinear function of any single element (e.g. Z).

forming Δ_{11} the terms by which z is multiplied must be the minor Δ_{11jj} obtained by omitting both the first and jth rows and columns. If we let Δ^0 and Δ_{11}^0 represent, respectively, Δ and Δ_{11} when $z = 0$, therefore, we have

$$Z = \frac{\Delta^0 + z\Delta_{jj}}{\Delta_{11}^0 + z\Delta_{11jj}}. \tag{1-11}$$

Since Δ_{jj} and Δ_{11jj} are evidently independent of z they can equally well be written as Δ_{jj}^0 and Δ_{11jj}^0. This will occasionally be done in later analysis in order to facilitate further transformations.

The relation between Z_T and z can be found in similar fashion. It is given by

$$Z_T = \frac{\Delta^0 + z\Delta_{jj}}{\Delta_{12}^0 + z\Delta_{12jj}}. \tag{1-12}$$

If z represents a unilateral coupling term, instead of a bilateral element, the expansion is essentially the same. Thus, if we suppose that z is a part of Z_{ij} in the original determinant, we readily find

$$Z = \frac{\Delta^0 + z\Delta_{ij}}{\Delta_{11}^0 + z\Delta_{11ij}} \tag{1-13}$$

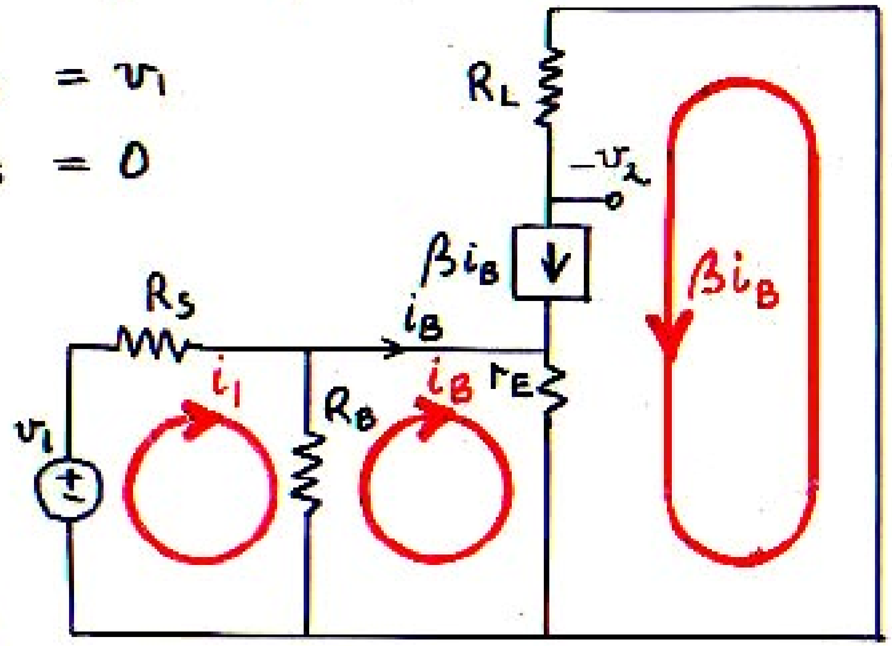
and

$$Z_T = \frac{\Delta^0 + z\Delta_{ij}}{\Delta_{12}^0 + z\Delta_{12ij}}. \tag{1-14}$$

The "brute-force" method: loop analysis

$$(R_s + R_B)\, i_1 \qquad - R_B \cdot i_B = v_1$$

$$-R_B\, i_1 + [R_B + (1+\beta) r_E]\, i_B = 0$$



$$i_B = \frac{\begin{vmatrix} R_s + R_B & v_1 \\ -R_B & 0 \end{vmatrix}}{\begin{vmatrix} R_s + R_B & -R_B \\ -R_B & R_B + (1+\beta) r_E \end{vmatrix}}$$

$$= \frac{R_B v_1}{(R_s + R_B)[R_B + (1+\beta) r_E] - R_B^2}$$

$$= \frac{R_B v_1}{R_s R_B + (1+\beta) r_E R_s + R_B^2 + (1+\beta) r_E R_B - R_B^2}$$

Finally, $\quad v_2 = R_L \beta i_B$

which leads to:

$$A_m \equiv \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta) r_E R_s + (1+\beta) r_E R_B + R_s R_B}$$

Implementation:

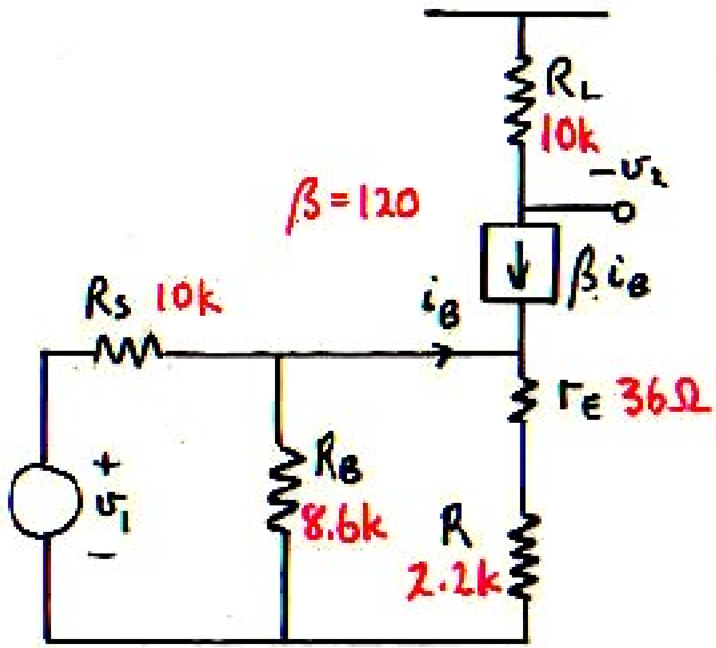
All that is needed is to calculate the driving point impedance across the terminals to which the extra element is to be added, under two conditions:

$$Z_d \equiv Z_{dp}\Big|_{u_{i1}=0} \quad \text{(original input zero)}$$

$$Z_n \equiv Z_{dp}\Big|_{u_{o1}=0} \quad \text{(original output nulled)}$$

Example: The previously designed CE amplifier

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \to 1/sC_2$ is desired.

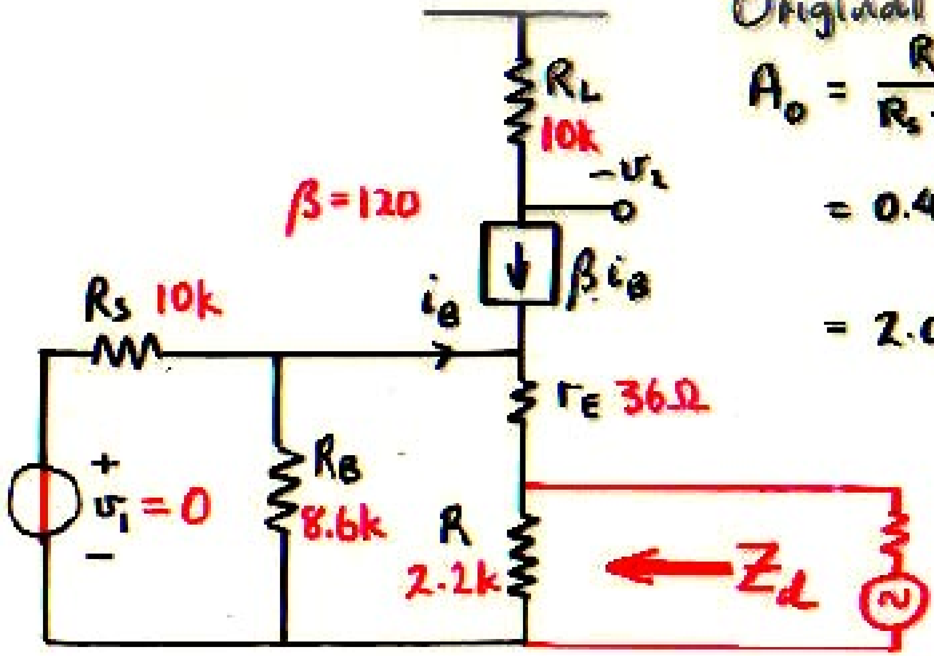


Original gain:

$$A_0 = \frac{R_B}{R_S + R_B} \, \frac{\alpha R_L}{R + r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 0.46 \, \frac{10}{2.2 + 0.036 + 0.039}$$

$$= 2.0 \Rightarrow 6\,\text{dB}$$



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$$A_0 = \frac{R_B}{R_s + R_B} \; \frac{\alpha R_L}{R + r_e + (R_s \| R_B)/(1+\beta)}$$

$$= 0.46 \; \frac{10}{2.2 + 0.036 + 0.039}$$

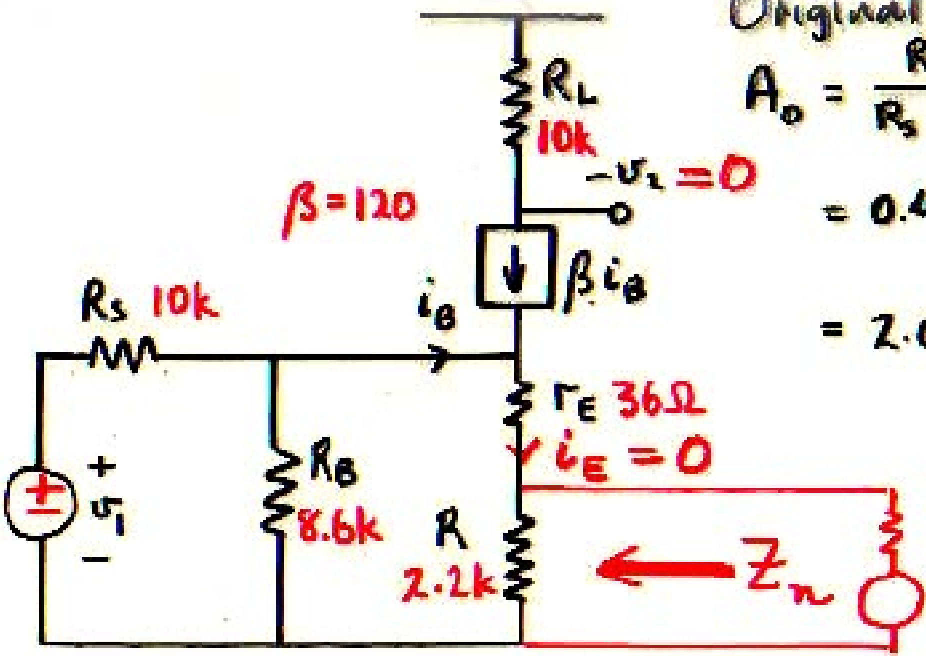
$$= 2.0 \Rightarrow 6\,dB$$

Step 1. Calculate Z_d by shorting $u_{i1} = v_1$, and applying a second injected signal across R:

$$Z_d = R_d = R \| [r_E + (R_S \| R_B)/(1+\beta)]$$

$$= 2.2 \| [0.036 + \underbrace{(10 \| 8.6)/120}_{0.039}]$$

$$= 75\,\Omega$$



Original gain:

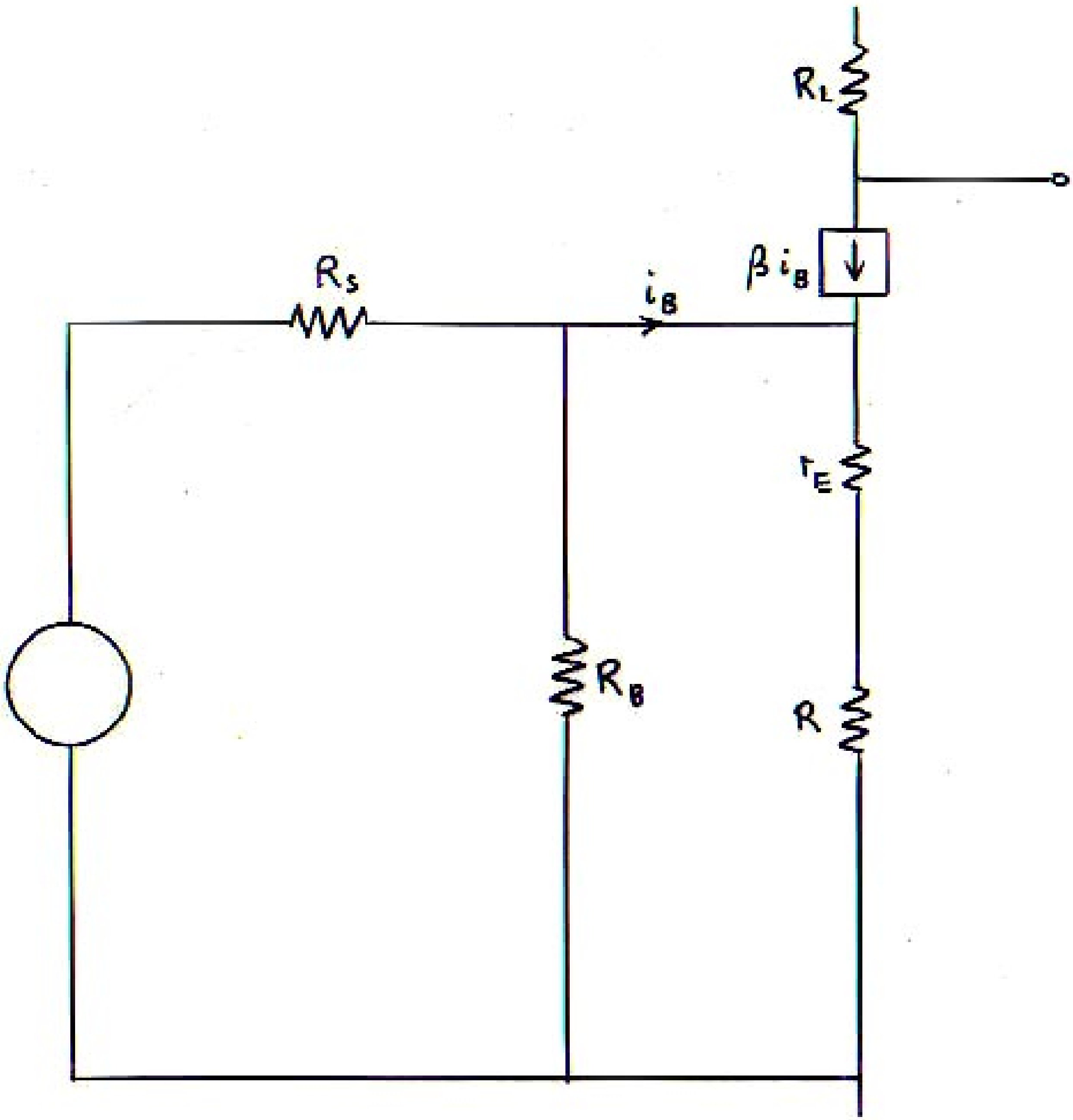
$$A_o = \frac{R_B}{R_s + R_B} \; \frac{\alpha R_L}{R + r_e + (R_s \| R_B)/(1+\beta)}$$

$$= 0.46 \; \frac{10}{2.2 + 0.036 + 0.039}$$

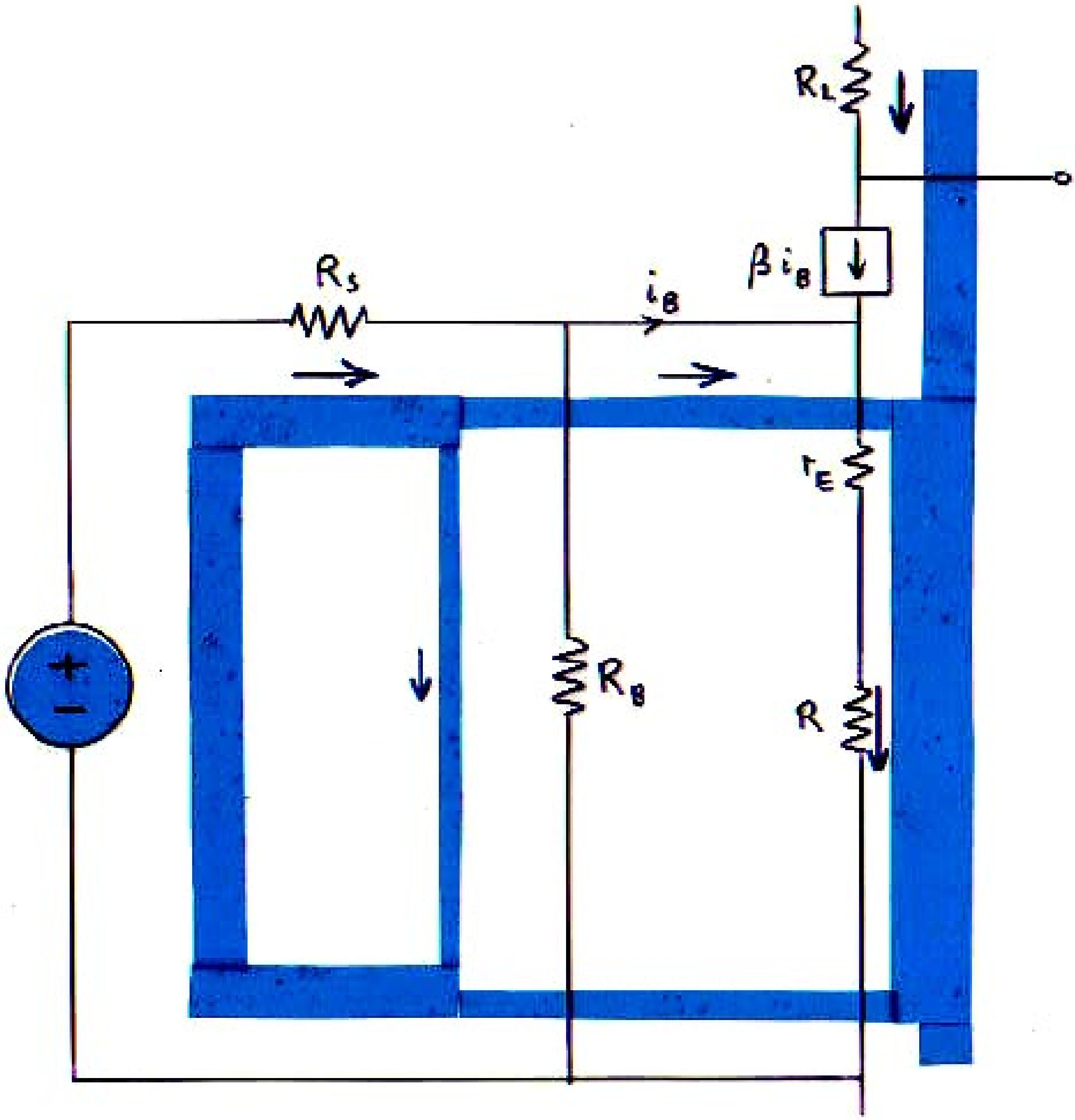
$$= 2.0 \Rightarrow 6\,\text{dB}$$

Step 2. Calculate Z_n by applying a second injected signal across R, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$. Then, since $v_2 = 0$, $i_E = 0$, hence:

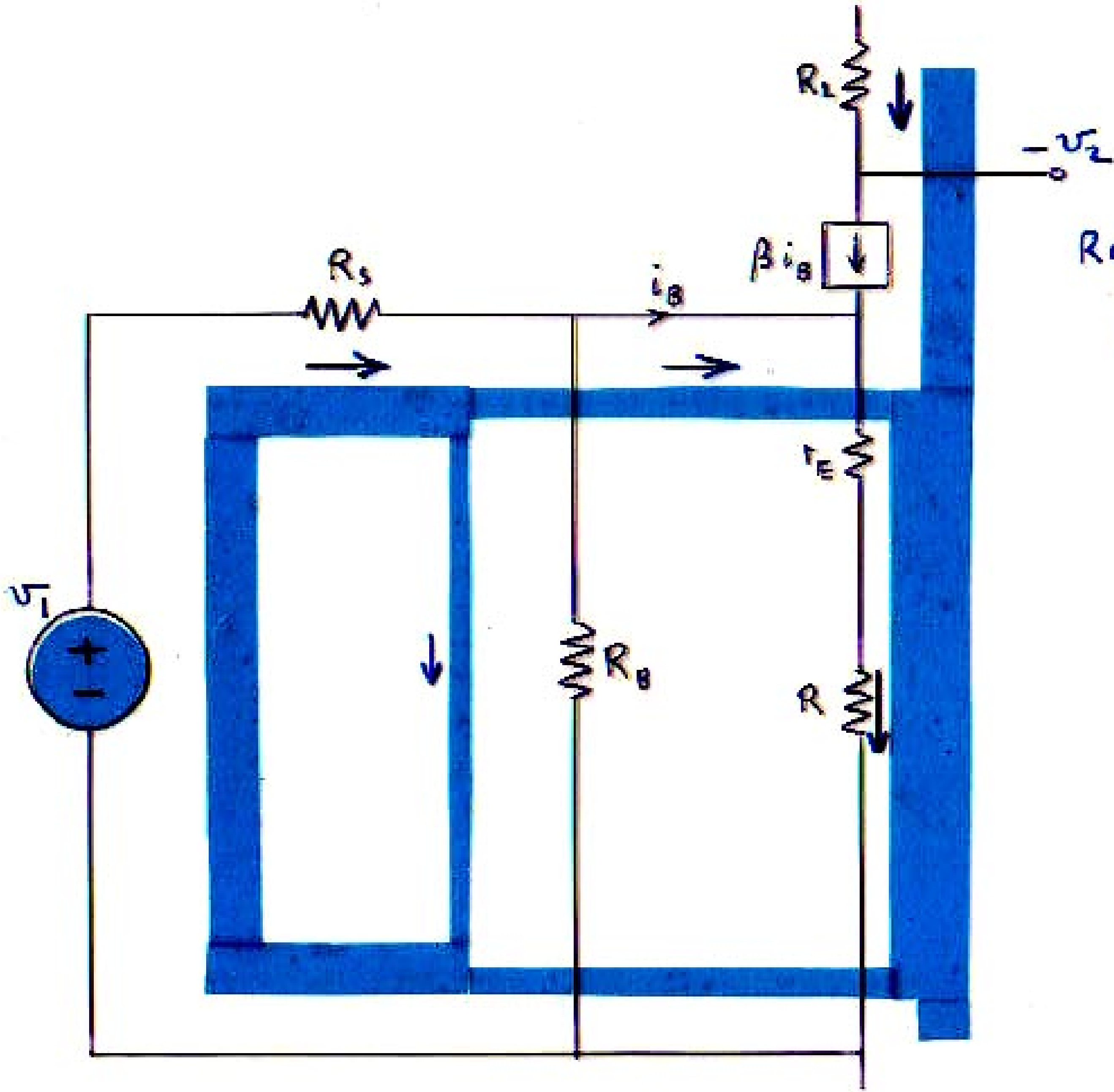
$$Z_n = R_n = R = 2.2\text{k}$$



R_L
βi_B
R_s
i_B
r_E
R_B
R

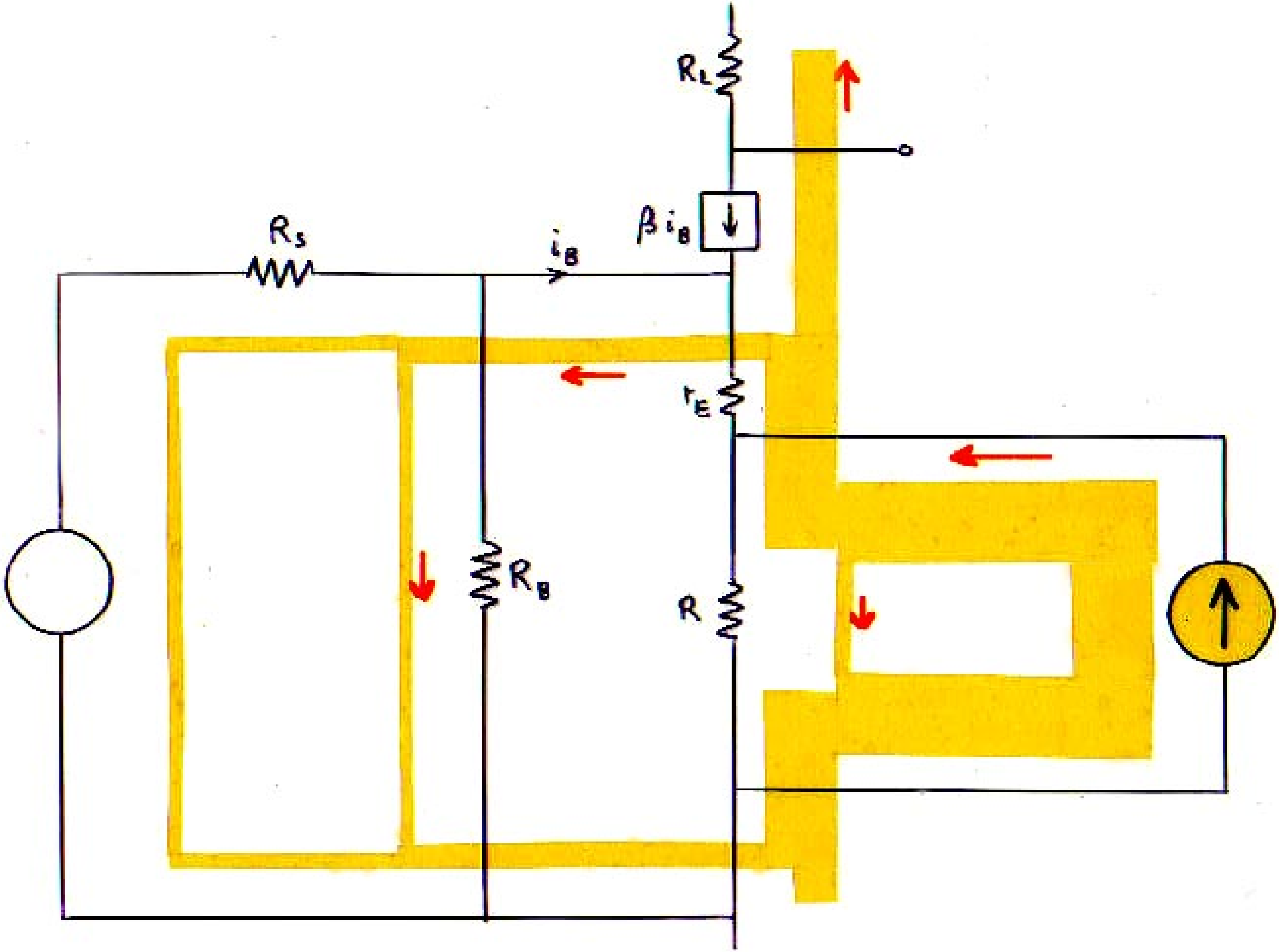


R_L
βi_B
R_S
i_B
r_E
R_B
R
+
−

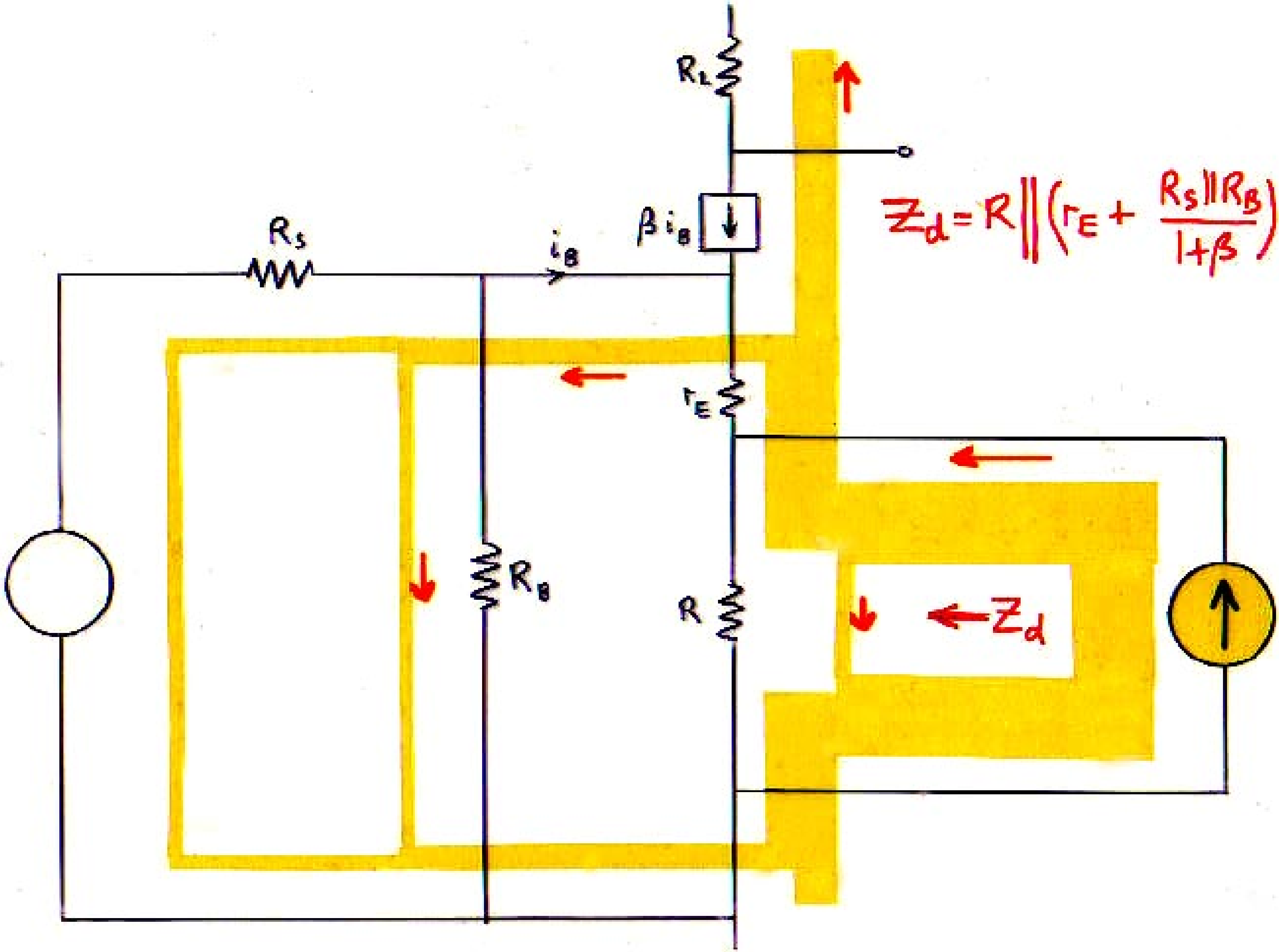


Reference gain:

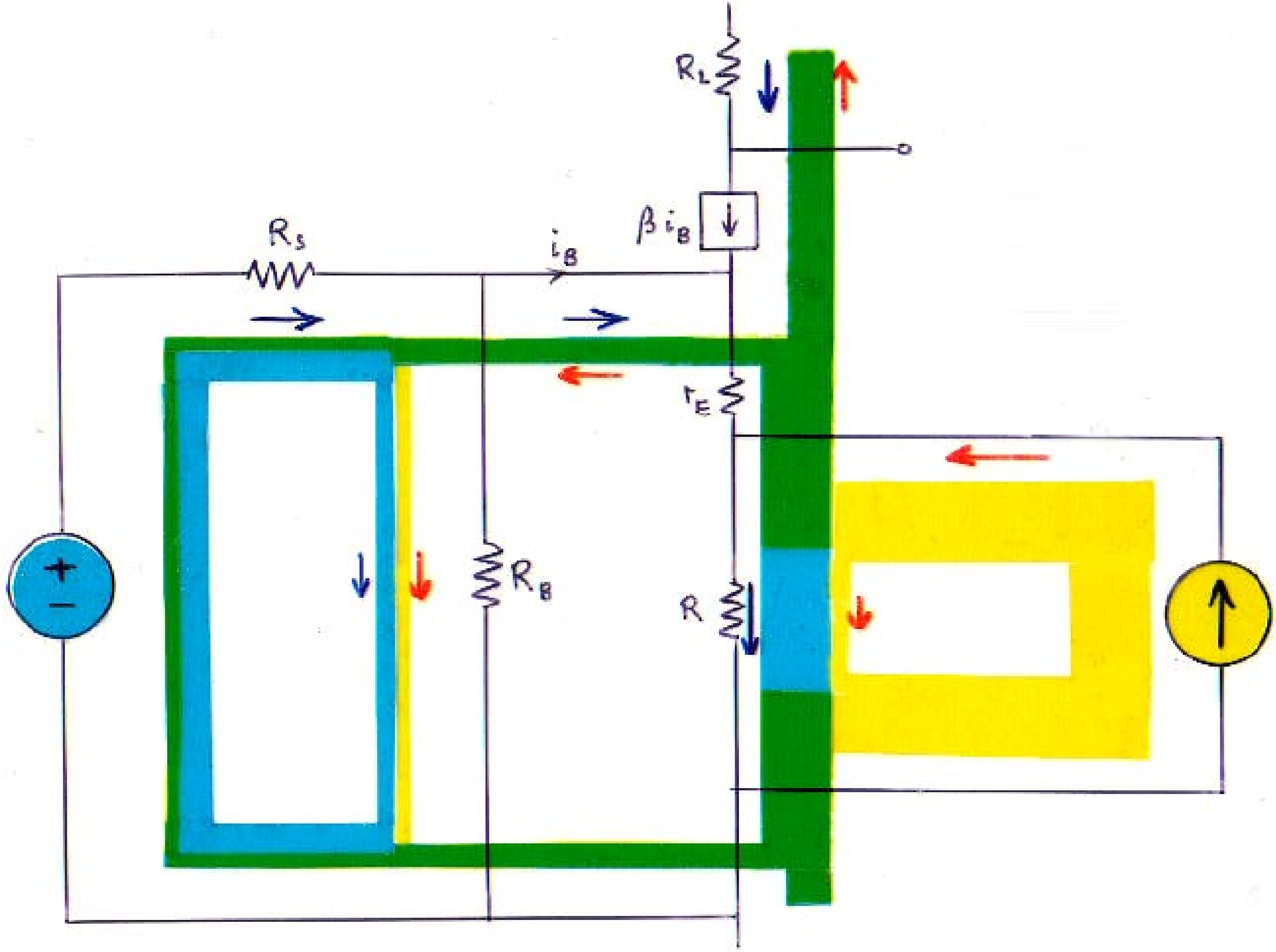
$$A_o = \frac{v_2}{v_1}$$



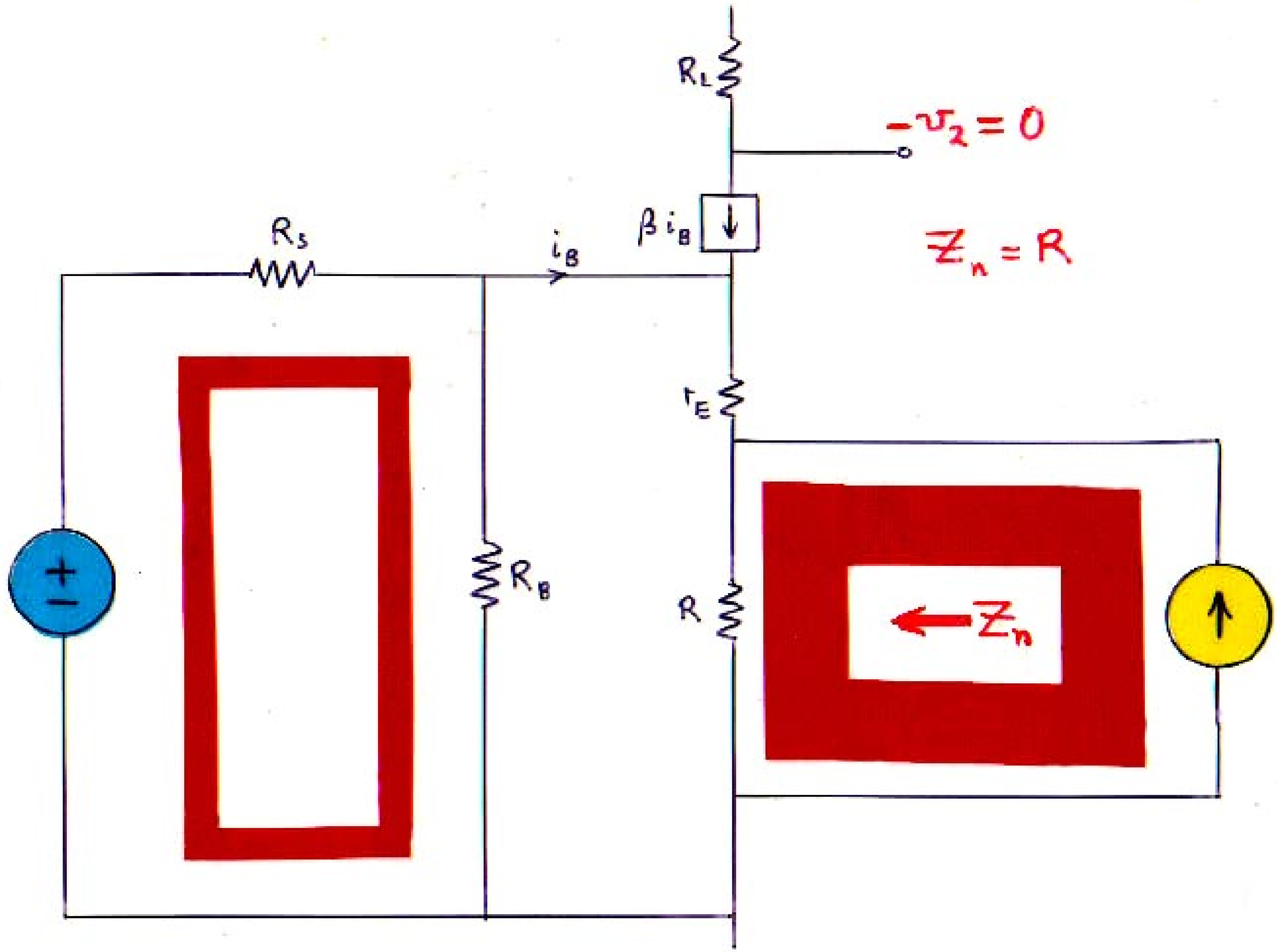
R_L
βi_B
R_S
i_B
r_E
R_B
R



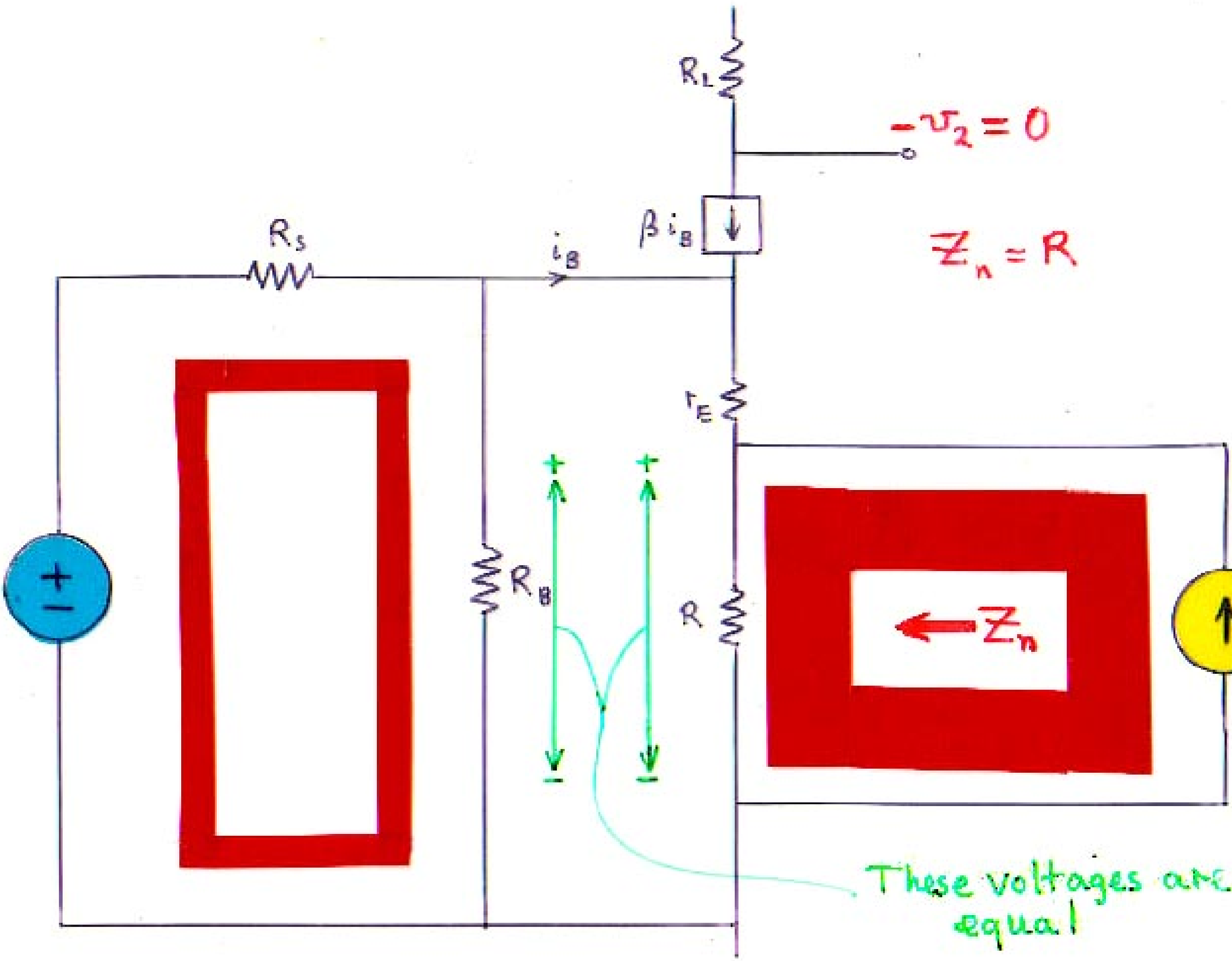
R_L
βi_B
R_S
i_B
r_E
R_B
R
Z_d
$Z_d = R \,\|\, \left(r_E + \frac{R_S \| R_B}{1+\beta} \right)$



R_L
βi_B
R_S
i_B
r_E
R_B
R
+
−



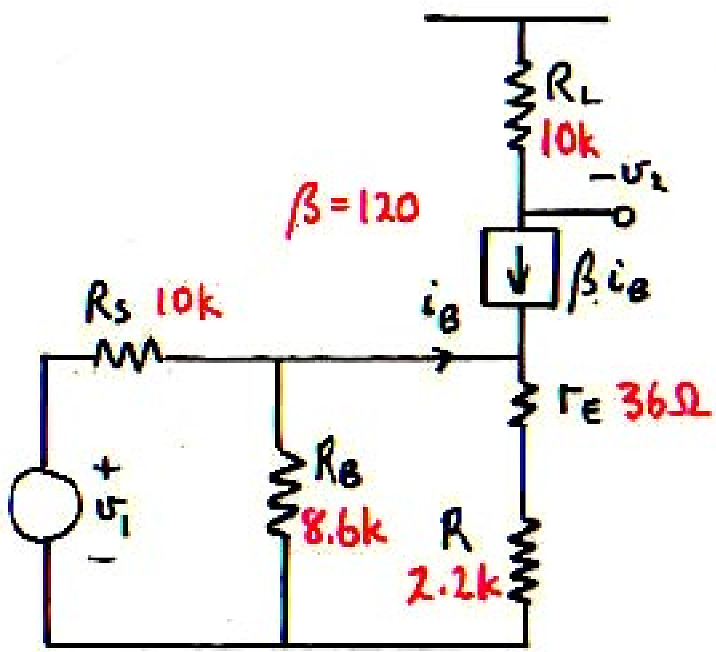
R_L
$v_2 = 0$
βi_B
$Z_n = R$
R_S
i_B
r_E
R_B
R
Z_n



R_L
$v_2 = 0$
βi_B
R_s
i_B
$Z_n = R$
r_E
R_B
R
Z_n
These voltages are equal

Example: The previously designed CE amplifier

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \to 1/sC_2$ is desired.



Original gain:

$$A_0 = \frac{R_B}{R_S + R_B} \; \frac{\alpha R_L}{R + r_E + (R_S \| R_B)/(1+\beta)}$$

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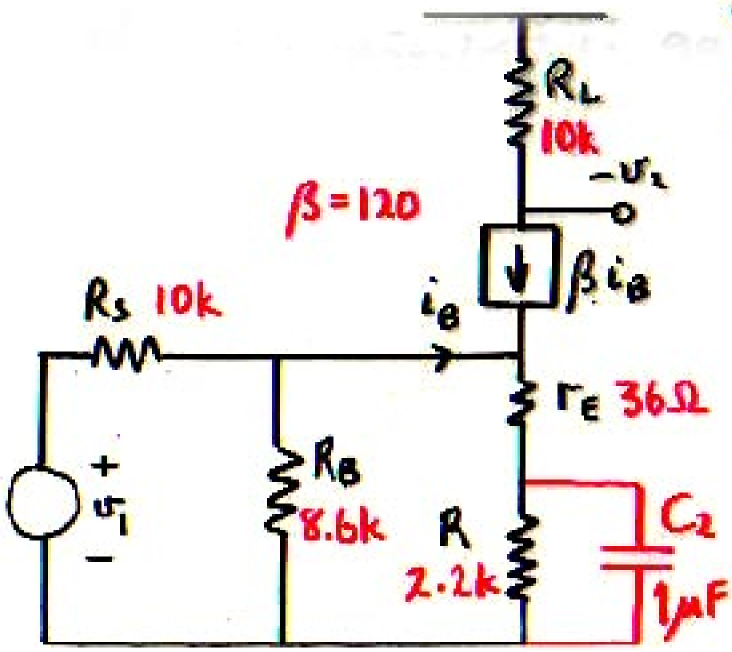
$$= 2.0 \Rightarrow 6\,\text{dB}$$



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Hence, corrected gain in presence of $C_2 = 1\mu F$ bypass capacitance is:

$$A = A_0 \frac{1 + \frac{R_n}{Z}}{1 + \frac{R_d}{Z}} = A_0 \frac{1 + sC_2R_n}{1 + sC_2R_d} = A_0 \frac{1 + \frac{s}{\omega_1}}{1 + \frac{s}{\omega_2}} = A_m \frac{1 + \frac{\omega_1}{s}}{1 + \frac{\omega_2}{s}}$$

where

$$\omega_1 \equiv \frac{1}{C_2R_n} \quad f_1 = \frac{159}{1 \times 2.2} = 72\,Hz \quad \omega_2 \equiv \frac{1}{C_2R_d} = \frac{159}{1 \times 0.075} = 2.1\,kHz$$

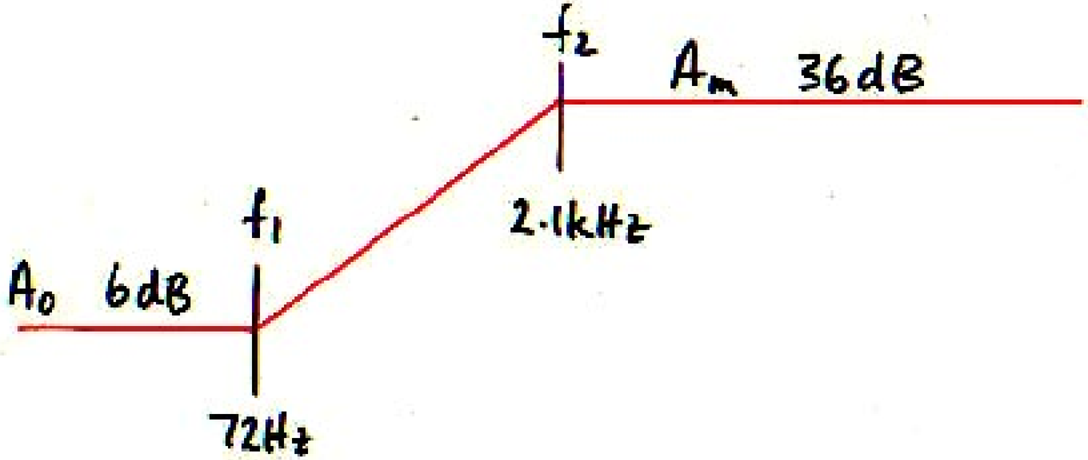
$$A_m = A_0 \frac{\omega_2}{\omega_1} = A_0 \frac{R_n}{R_d}$$

$$= \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{R + r_E + (R_S \| R_B)/(1+\beta)} \frac{R[R + r_E + (R_S \| R_B)/(1+\beta)]}{R[r_E + (R_S \| R_B)/(1+\beta)]}$$

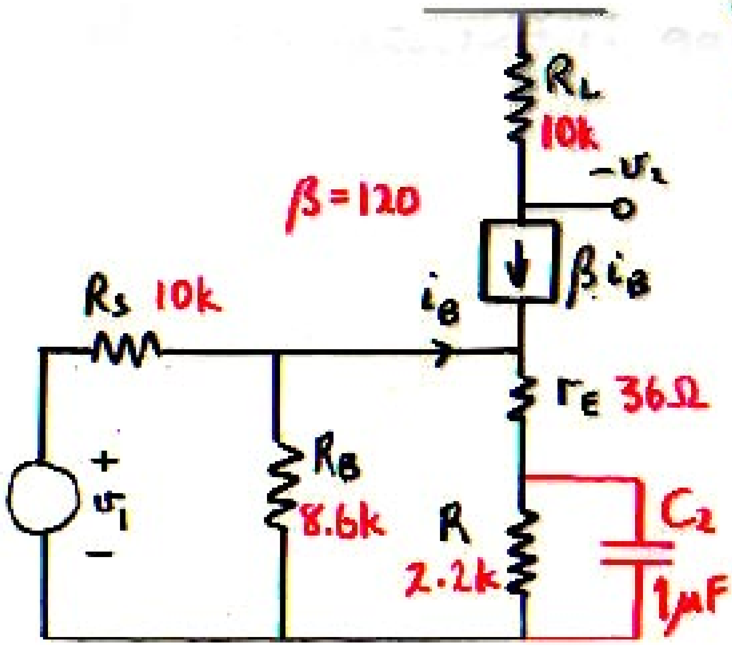
$$= \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)} = 62 \Rightarrow 36\,dB$$

NOTE: <u>Nulling</u> a voltage is <u>not</u> the same as <u>shorting</u> it!

NOTE: the null double injection <u>calculation</u> is easier than the single injection <u>calculation</u>!



f_2
A_m 36dB
f_1
2.1kHz
A_0 6dB
72Hz



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$$= \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{R + r_E + (R_S \| R_B)/(1+\beta)} \frac{R[R + r_E + (R_S \| R_B)/(1+\beta)]}{R[r_E + (R_S \| R_B)/(1+\beta)]}$$

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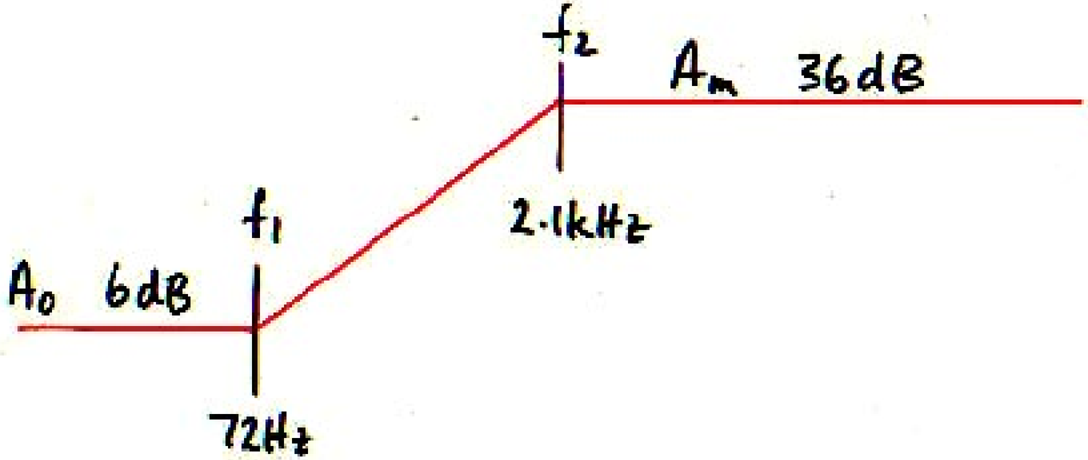
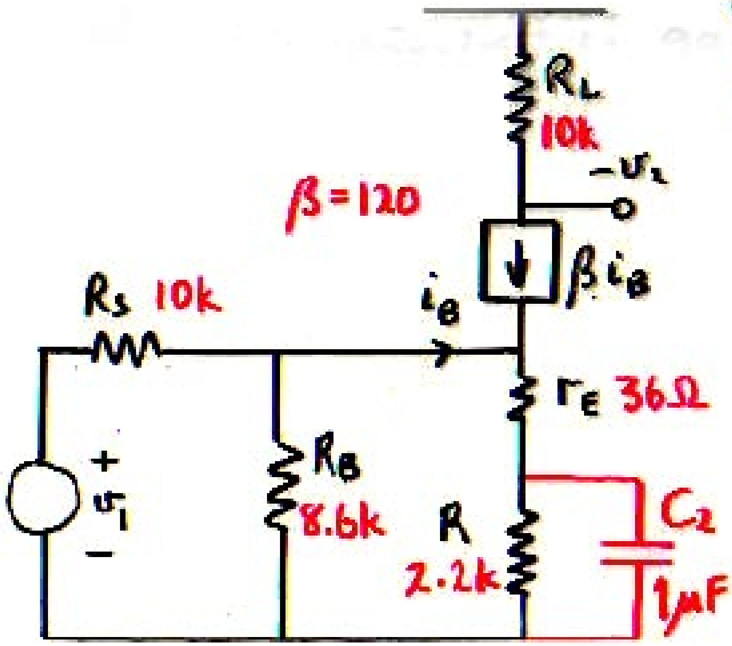


f2
Am 36dB
f1
2.1kHz
A0 6dB
72Hz



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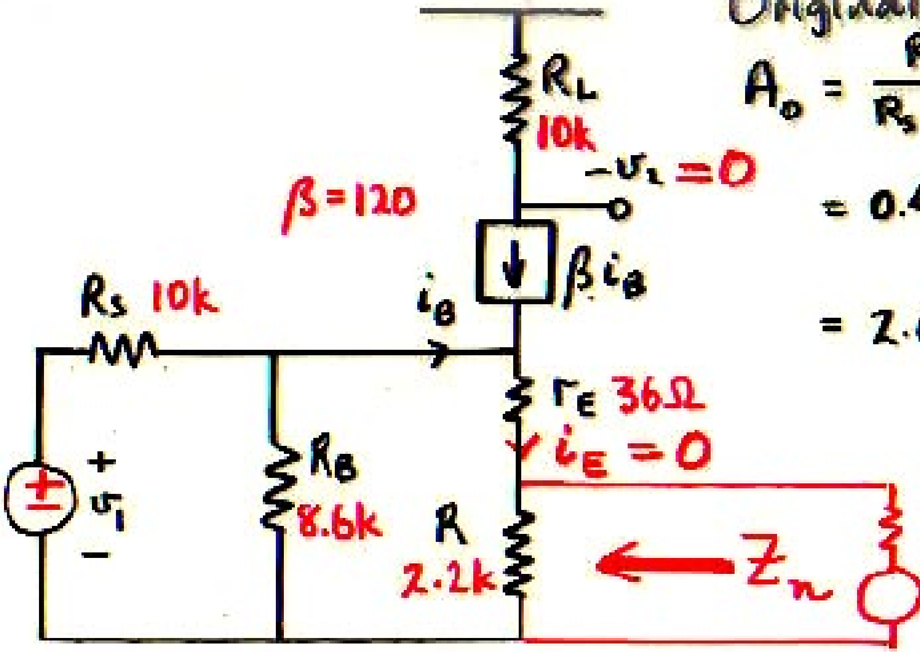

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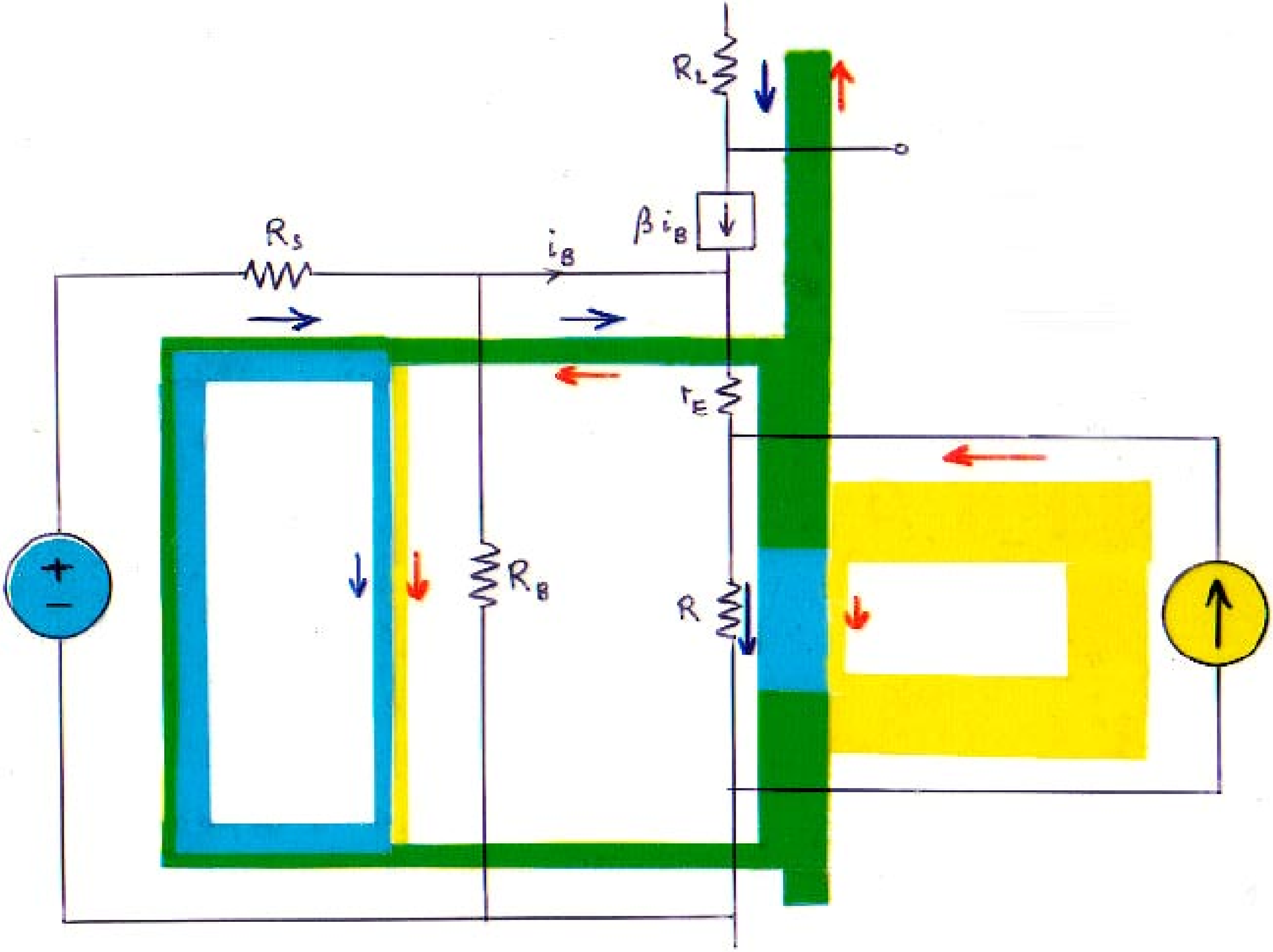
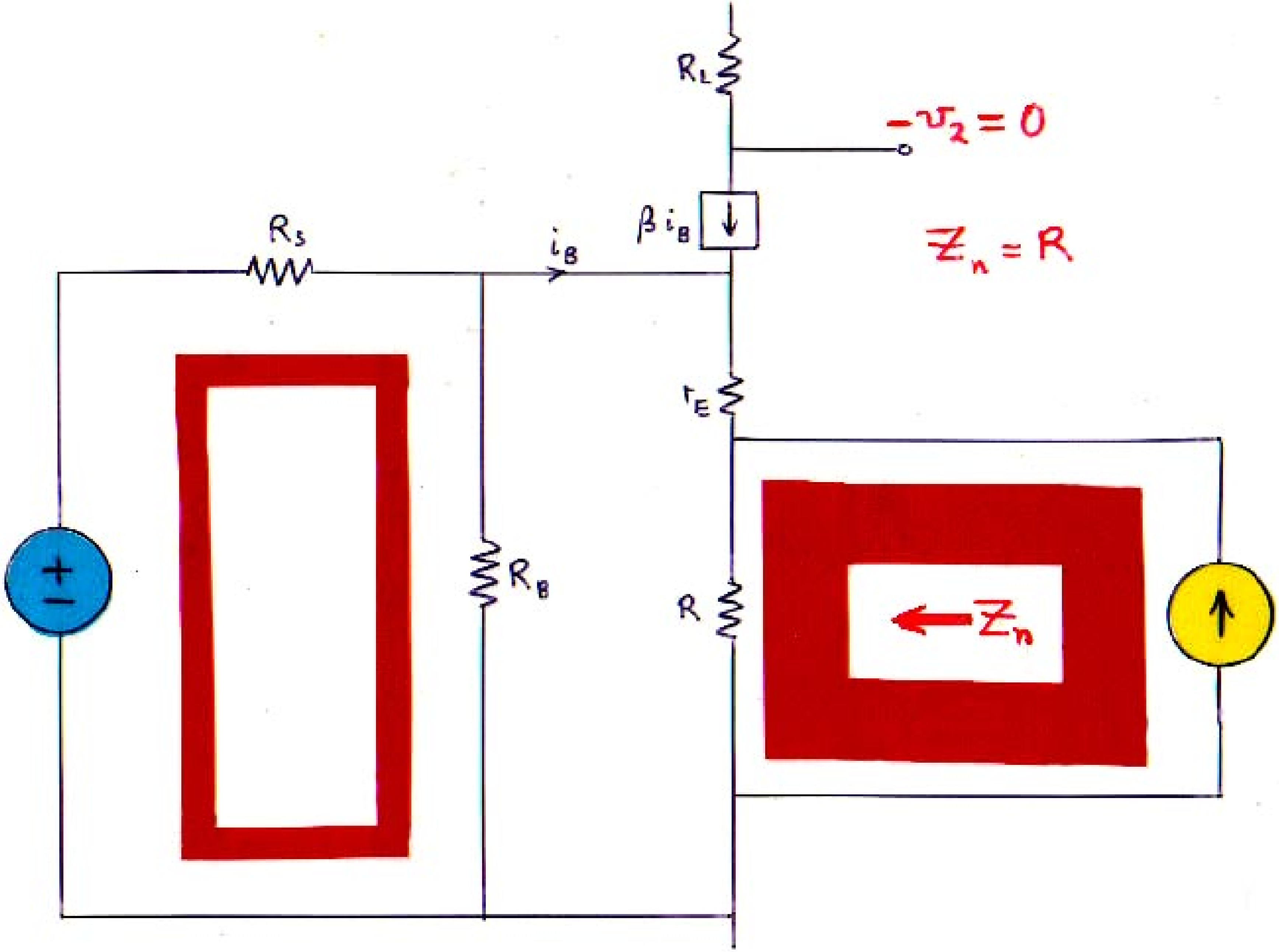
Step 2. Calculate Z_n by applying a second injected signal across R, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$. Then, since $v_2 = 0$, $i_E = 0$, hence:

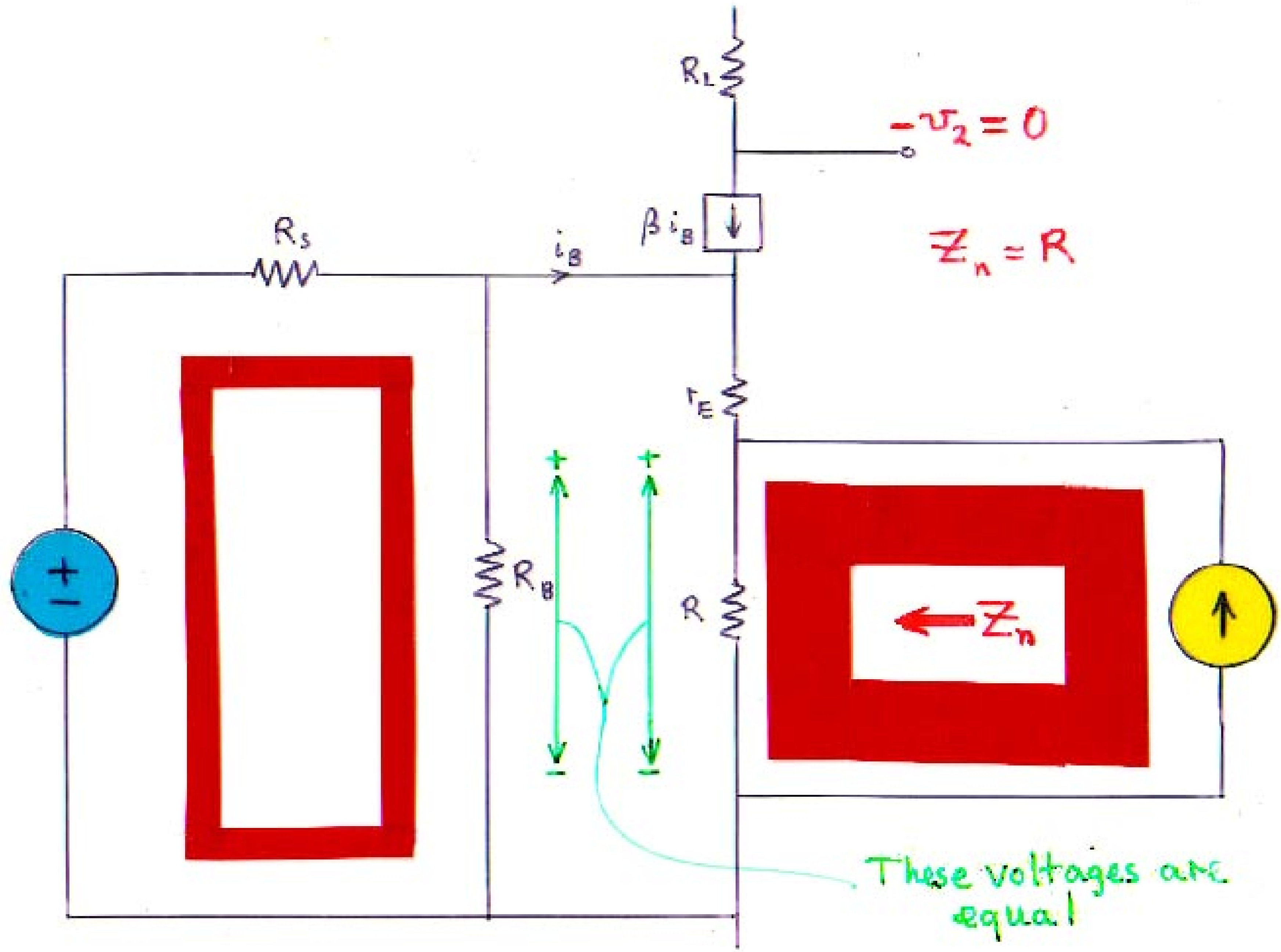
$$Z_n = R_n = R = 2.2k$$



R_L
βi_B
R_S
i_B
r_E
R_B
R
+
−



R_L
$-v_2 = 0$
βi_B
R_S
i_B
$Z_n = R$
r_E
R_B
R
Z_n



R_L
$-v_2 = 0$
βi_B
R_s
i_B
$Z_n = R$
r_E
R_B
R
Z_n
These voltages are equal

The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element.

There is a corresponding form to find the correction factor resulting from an extra series element:

reference gain

$$\text{gain}\Big|_{Z} = \text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\Big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

$$= \frac{Z_n}{Z_d}\,\text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

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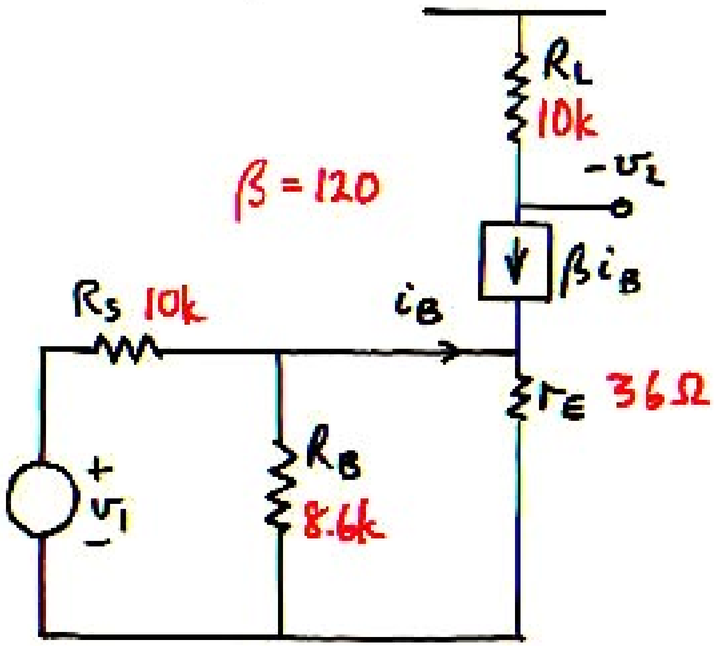
$$\text{gain}\big|_{Z} = \text{gain}\big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

reference gain

$$= \text{gain}\big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}} = \frac{Z_n}{Z_d}\,\text{gain}\big|_{Z=\infty} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

This must be the gain when $Z=0$

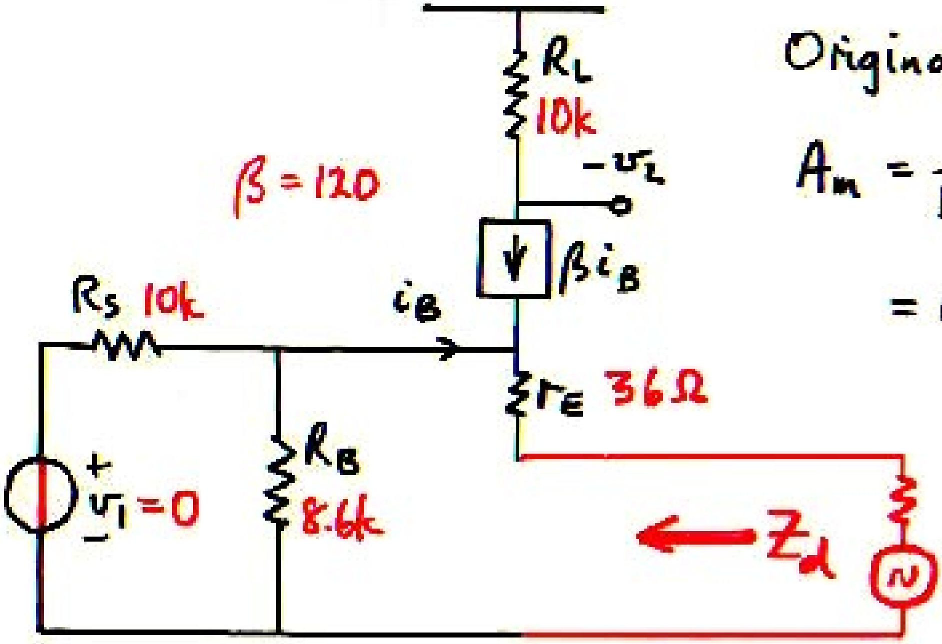
Example: An alternative to the method of the previous example is to find the correction factor to the midband gain A_m resulting from addition of the series "extra element" $Z \to R \| 1/sC$.



Original gain:

$$A_m = \frac{R_B}{R_s + R_B} \frac{\alpha R_L}{r_E + (R_s \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\,\text{dB}$$



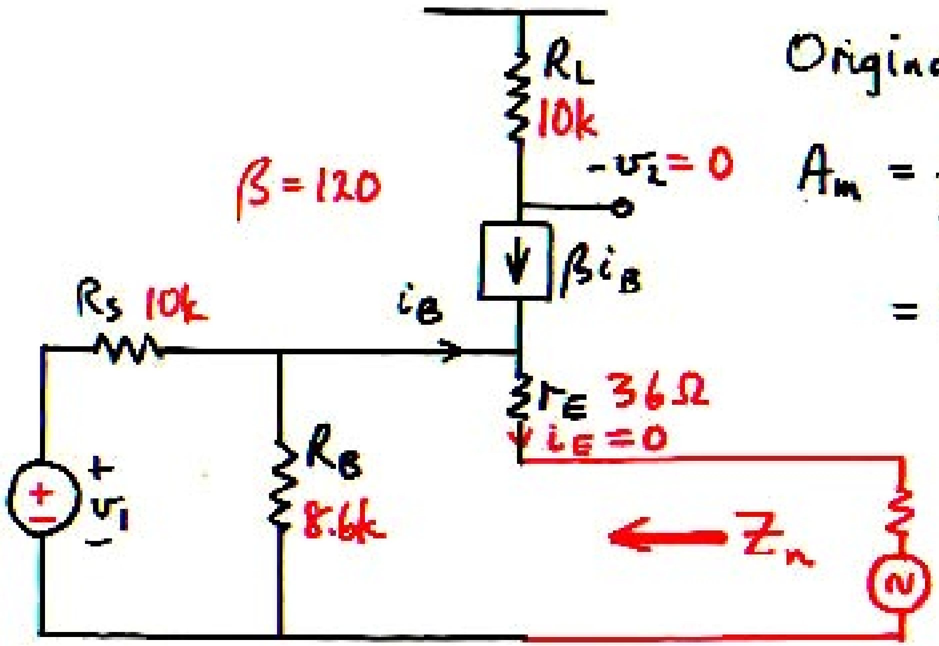
Original gain:

$$A_m = \frac{R_B}{R_S+R_B}\,\frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\text{dB}$$

Step 1. Calculate Z_d by shorting $u_{i1} = v_i$ and applying a second injected signal in series with r_E:

$$Z_d \equiv R_d' = r_E + (R_S \| R_B)/(1+\beta)$$



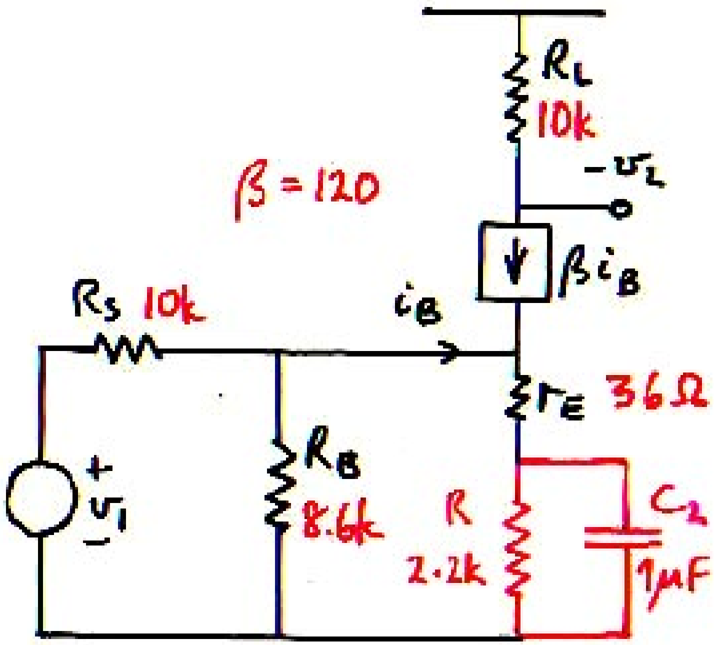
Original gain:

$$A_m = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\,\text{dB}$$

Step 2. Calculate Z_n by applying a second injected signal in series with r_E, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$. Then, since $v_2 = 0$, $i_E = 0$, hence

$$Z_n \equiv R_n' = \infty$$



Original gain:

$$A_m = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\text{dB}$$

Hence, corrected gain in presence of $R \| 1/sC_2$ is:

$$A = A_m \frac{1 + \frac{Z}{R_n'}}{1 + \frac{Z}{R_d'}} = A_m \frac{1}{1 + \frac{R}{R_d'} \frac{1}{1 + sC_2 R}} = A_m \frac{1 + 1/sC_2 R}{1 + 1/sC_2 (R \| R_d')}$$

However, $R \| R_d' = R_d$, so

$$A = A_m \frac{1 + 1/sC_2 R}{1 + 1/sC_2 R_d} \longrightarrow \text{same result as before}$$

Generalization: Extra Element Theorem — #1

There are two forms of Extra Element Theorem:

1.

$$\text{gain}\big|_{Z} = \text{gain}\big|_{Z=\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}}$$

Provides a correction factor for an extra element added in shunt across a node pair.

2.

$$\text{gain}\big|_{Z} = \text{gain}\big|_{Z=0} \frac{1 + \frac{Z}{Z_n}}{1 + \frac{Z}{Z_d}}$$

Provides a correction factor for an extra element added in series with a branch.

The "extra element" Z can be any two-terminal combination of impedances.

Note that in all cases the null double injection calculation is easier than the single injection calculation.

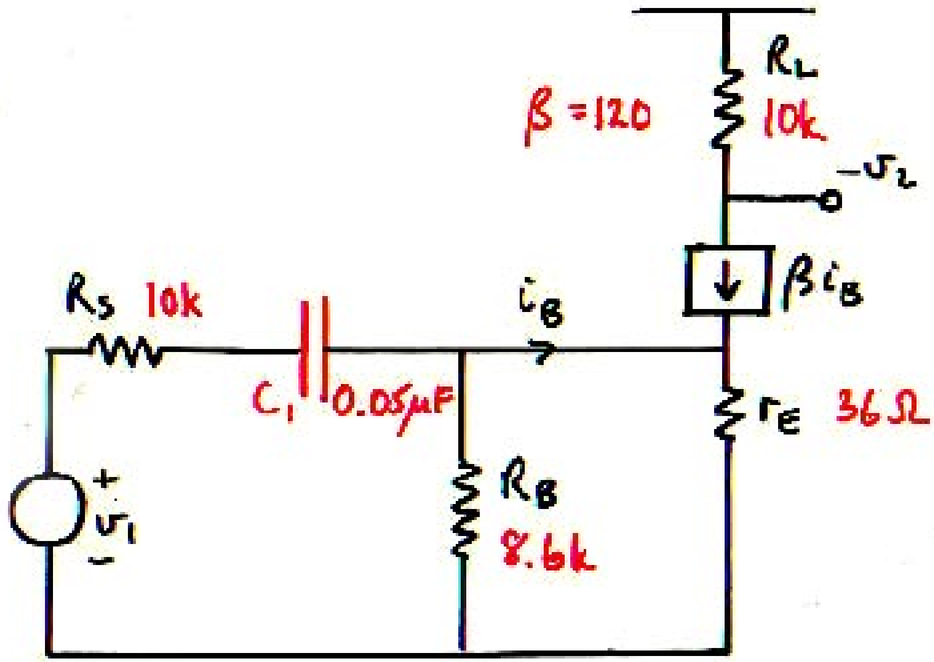
This results from use of the null condition (which makes several other quantities zero);

and

because the relation between u_{i1} and u_{i2} to produce the null is never needed — only the null itself is used.

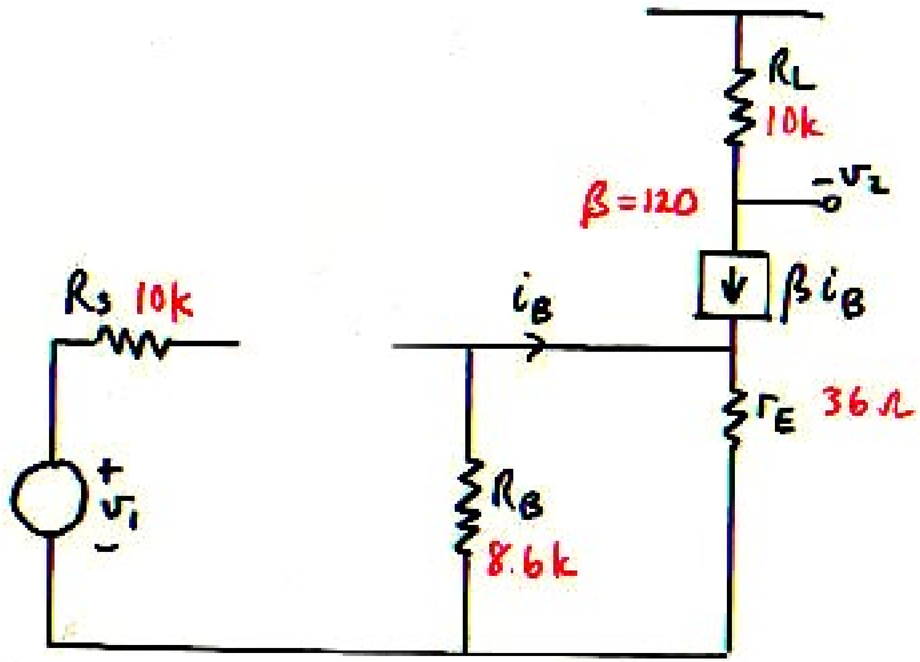
Exercise

In the CE amplifier stage, find the correction factor to the midband gain A_m resulting from inclusion of the coupling capacitance $C_1 = 0.05\mu F$:



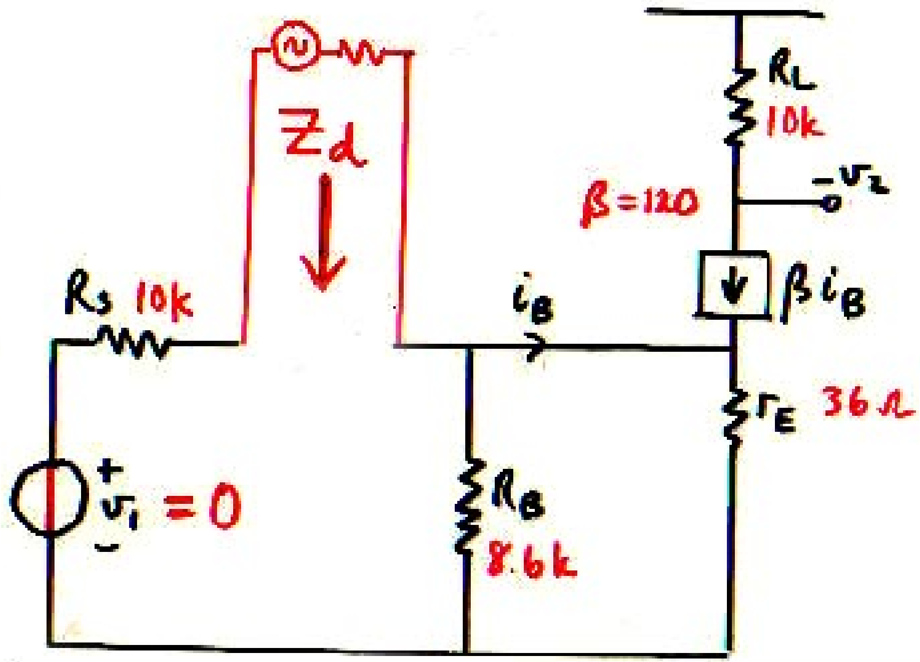
$A_m = 36dB$

Exercise Solution



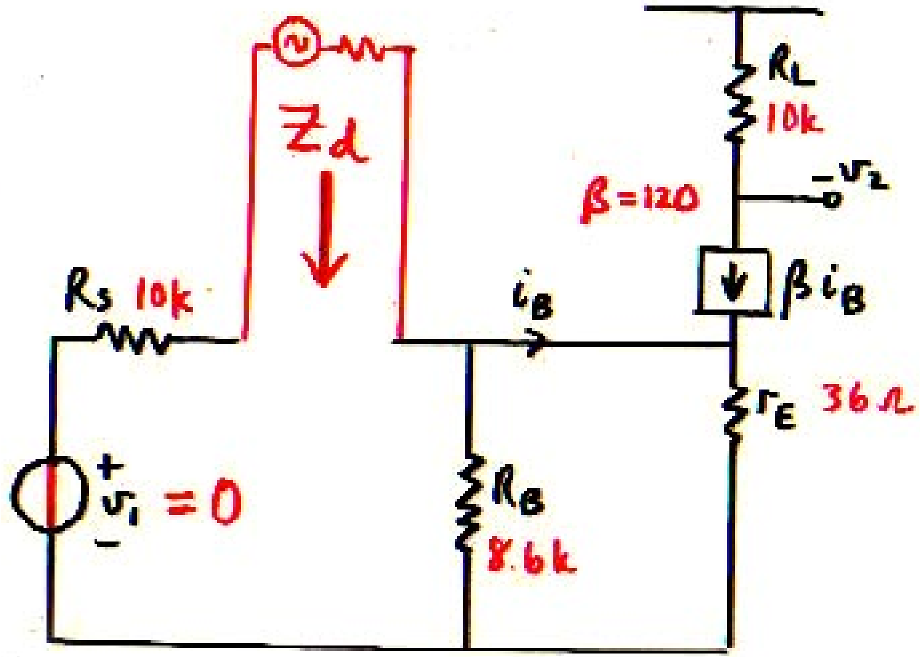
$A_m = 36\text{dB}$

Exercise Solution



$A_m = 36\,dB$

Exercise Solution

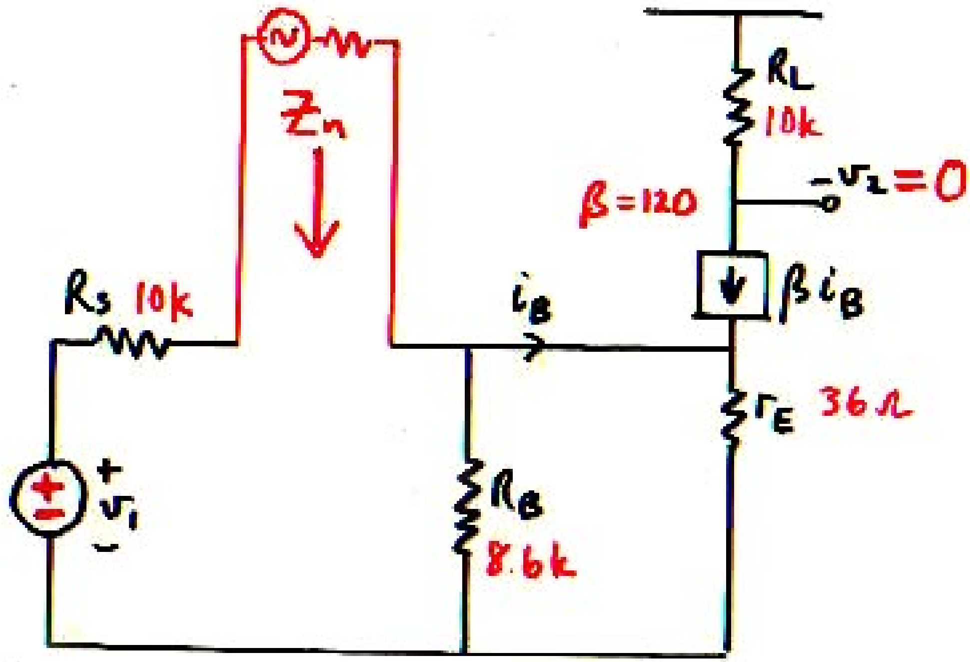


$A_m = 36\,dB$

Step 1.

$$
\begin{aligned}
Z_d = R_d &= R_s + R_B \| (1+\beta) r_E \\
&= 10 + 8.6 \| (120 \times 0.036) \\
&= 10 + 8.6 \| 4.3 \\
&= 13\,k
\end{aligned}
$$

Exercise Solution



$A_m = 36\text{dB}$