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1. MOTIVATION AND BACKGROUND

The Design-Oriented Analysis (D-OA) Paradigm
Premise:
Not much of what you learned in school has turned out to be much use
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You don't solve equations simultaneously; instead:
You solve them sequentially
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An Engineer's story
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An Engineer's story
Falling off a cliff
Most of us "fall off a cliff" when we begin our first job

**Why Most of Us Need Technical Therapy...**
**First Manifestation of Technical Disability:**
Problem:
New graduate engineers are unable to translate the principles and methods they have learned to the real world. What can be done?
The Design Process
Design iteration loops
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Design iteration loops
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Design iteration loops
The design process consists of a succession of iteration loops:
"How to present the results" is important:

1. If you are a design engineer writing a report or appearing before a design review committee;

2. If you are a Test, Reliability, or System Integration Engineer dealing with someone else's design.

The D-OA approach is valuable for all these engineers.
Realization:
Design is the Reverse of Analysis,
Realization:

Design is the Reverse of Analysis, because:

The Starting Point of the Design Problem
(the Specification)
Realization:
  Design is the Reverse of Analysis, because:
The Starting Point of the Design Problem (the Specification) is the Answer to the Analysis Problem
Conventional problem-solving approach:

1. Put everything into the model and simplify later.
2. Postpone approximation as long as possible, and don't even dare to make an approximation unless you can justify it on the spot.
3. The "answer" is acceptable in whatever form it emerges from the algebra.
4. The more work you do, the more valuable the result.
5. Every problem is a brand-new problem, and requires a brand-new strategy to solve it.

This is a recipe for failure!
Syndromes of Technical Disability:
Algebraic diarrhoea, which leads to
Algebraic paralysis
Fear of approximation
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The negative results of the conventional paradigm are often masked while the student is in school.
Why does the conventional approach fail?

Mathematicians tell us:

\[ \text{# of equations must} = \text{# of unknowns} \]

Engineers face:

\[ \text{# of equations} < \ll \text{# of unknowns} \]

but have to solve the problem, anyway.
How can we overcome the negative results of the conventional approach?

1. Divide and Conquer:

   It's easier to solve many simpler problems than one large one.
2. We must make the equations we do have *work harder* by expressing them in "Low Entropy" form.

A *High Entropy Expression* is one in which the arrangement of terms and element symbols conveys no information other than that obtained by substitution of numbers.
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*A High Entropy Expression* is one in which the arrangement of terms and element symbols conveys no information other than that obtained by substitution of numbers.

*A Low Entropy Expression* is one in which the terms and element symbols are ordered and grouped so that their physical origin and relative importance are apparent. Only in this way can one *change* the values in an informed manner in order to *change* the analysis answer (that is, to make it meet the Specification).
Low Entropy Expressions are essential in order to navigate the Design Iteration Loop:
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Low Entropy Expressions are essential in order to navigate the Design Iteration Loop:

- The analysis result must be "inverted" and used "backwards" to select a better set of values.
- Model, with perhaps some tentative values.

To "invert" the analysis result, it must be in "Low Entropy" form.
Design-Oriented Analysis (D-OA) keeps the entropy low at every step along the way to a low entropy result.
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The way to do this is:
Avoid solving simultaneous equations.
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There may be many such paths (algorithms), each of which gives a different Low Entropy Expression.
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There may be many such paths (algorithms), each of which gives a different Low Entropy Expression.

Avoid multiplying out the series/parallel combinations.
How can we overcome the negative results of the conventional approach?
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3. Recognize that we *don't want* an exact answer: it would be too complicated to use, even if we could get it.
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3. Recognize that we don't want an exact answer: it would be too complicated to use, even if we could get it.

   Therefore, substitute for the missing equations with: inequalities, approximations, assumptions, and tradeoffs.
D-OA problem-solving approach: D-OA Rules

1. Put only enough into the model to get the answer you need.

2. Make all the approximations you can, as soon as you can, justified or not. Plow through the problem leaving behind you a wake of assumptions and approximations. You can't lose by trying.

3. Figure out in advance as many of the quantities as you can that you want to have in the answer, and put them into the statement of the problem as soon as possible – even into the circuit model.

4. The less work you do, the more valuable the result. You control the algebra. You make the algebra come out in low entropy form by applying strategic mental energy before and during the math.

5. Every problem in not unique. There are problem solving strategies that apply to almost all engineering problems.

This is a recipe for success!
The benefits of applying the D-OA Rules are:

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1. You can *fend off* algebraic paralysis.

2. Approximations are *good* things, not an admission of defeat.

3. Algebra is *malleable*; you have *choices*. You are *empowered* to exercise control: the math is your *slave*, not your master.

**D-OA is the only kind of analysis worth doing!**
1. Background & Motivation

- Presenting Results
  - Ch 3
  - Ch 4
  - Ch 5

- Combining Results
  - Ch 6

- Extending Results:
  - Input/Output Impedance Theorem
    - I/O IT
    - Ch 7
  - Null Double Injection
    - NDI
    - Ch 8
  - Dissection Theorem
    - DT
    - Ch 9
2. LOW ENTROPY EXPRESSIONS

The Key to D-OA
Conventional analysis

Gain of CE amplifier

Midband means frequencies at which reactive effects are negligible.
The "brute-force" method: loop analysis

\[
(R_S + R_B) i_1 - R_B i_B = v_1 \\
-R_B i_1 + [R_B + (1+\beta)R_E] i_B = 0 \\

i_B = \frac{\begin{bmatrix} R_S + R_B & v_1 \\ -R_B & 0 \end{bmatrix} \begin{bmatrix} R_S + R_B \\ -R_B \end{bmatrix}}{R_S + R_B + (1+\beta)R_E R_S + R_B^2 + (1+\beta)R_E R_B - R_B^2} \\

v_1 = \frac{R_B v_1}{R_S R_B + (1+\beta)R_E R_S + R_B^2 + (1+\beta)R_E R_B - R_B^2} \\

Finally, \ v_2 = R_L \beta i_B \\
\text{which leads to:} \\
\alpha_m = \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta)R_E R_S + (1+\beta)R_E R_B + R_B R_L} \\

This is a high entropy expression
Apply mental energy to:

\[
A_m = \frac{\beta R_E R_L}{(1+\beta) R_E R_S + (1+\beta) R_E R_B + R_S R_B}
\]

\[
= \frac{\beta R_E R_L}{(1+\beta) R_E (R_S + R_B) + R_S R_B}
\]

\[
= \frac{R_E}{R_S + R_B} \cdot \frac{\beta R_L}{(1+\beta) R_E + R_S R_B}
\]

\[
= \frac{R_E}{R_S + R_B} \cdot \frac{\alpha R_L}{R_E + (R_S R_B)/(1+\beta)}
\]

(1) (2)
The Low Entropy result exposes the following additional information, not apparent from the High Entropy version:
(a) The \( R_b/(R_s+R_b) \) factor is identified as a voltage divider;
(b) Resistances appear in series/parallel combinations, so it is clear which ones are dominant;
(c) The relative values of the two terms labeled (1) and (2) determine the sensitivity of the gain \( A \) to variations of \( \beta \).

The additional information makes possible a much better informed choice of element values.
Disadvantages of the “brute-force” method:
1. No direct physical interpretation of the result.
2. Obscures relationships as to how element values affect the result.
3. Difficult to use for design: given \(A_m\) (the specification), how do you choose element values?
4. Purely algebraic derivation increases likelihood of mistakes.
Advantages of the Low-Entropy form of the result:
1. Direct physical interpretation of the result.
2. Clarifies relationships as to how element values affect the result.
3. Easy to use for design: given $A_m$ (the specification), how do you choose element values?
4. ?
It is easier to keep the Entropy low from the start of the analysis than it is to lower the Entropy once it has increased.
The "brute-force" method: loop analysis

\[
\begin{align*}
(R_S + R_E) i_1 - R_E i_B &= v_1 \\
-R_B i_1 + [R_E + (1+\beta) R_E] i_B &= 0
\end{align*}
\]

\[
i_B = \frac{R_E v_1}{(R_S + R_E) [R_E + (1+\beta) R_E] - R_E^2}
\]

\[
= \frac{R_E v_1}{R_S R_E + (1+\beta) R_E R_S + R_E^2 + (1+\beta) R_E R_E - R_E^2}
\]

Finally, \( v_L = R_L \beta i_B \)

which leads to:

\[
A_{in} = \frac{v_L}{v_1} = \frac{\beta R_E R_L}{(1+\beta) R_E R_S + (1+\beta) R_E R_E + R_E R_E}
\]
Reflection of impedances

Better method #1:

\[ i_1 = \frac{v_1}{R_S + R_B \| (1+\beta)R_E} \]
\[ v_2 = \beta R_L i_B \]
\[ A_m = \frac{v_2}{v_1} = \frac{R_B}{R_B + (1+\beta)R_E} \cdot \frac{\beta R_L}{R_S + R_B \| (1+\beta)R_E} \]

"reflect" emitter impedance to the base.
The "brute-force" method: loop analysis

\[
(R_s + R_b) i_1 - R_b i_B = v_i \\
-R_b i_1 + [R_b + (1+\beta) r_e] i_B = 0
\]

\[
i_B = \frac{[R_s + R_b] v_i}{(R_s + R_b) [R_b + (1+\beta) r_e] - R_b^2}
\]

Finally, \( v_z = R_L \beta i_B \)

which leads to:

\[
A_m = \frac{v_z}{v_i} = \frac{\beta R_b R_L}{(1+\beta) r_e R_s + (1+\beta) r_e R_b + R_s R_b}
\]
Doing the algebra on the circuit diagram

Better method: Use Thevenin's Theorem at the start

\[ i_E = \frac{V_E}{r_E + (R_s || R_b) / (1 + \beta)} \]

\[ \frac{V_2}{V_s} = \frac{RL}{r_E + (R_s || R_b) / (1 + \beta)} = \alpha \frac{\text{total collector load}}{\text{total emitter load including reflected base impedance}} \]

\[ A_m = \frac{V_2}{V_1} = \frac{R_b}{R_s + R_b} \cdot \frac{\alpha R_L}{r_E + (R_s || R_b) / (1 + \beta)} \]
Don't leave out the penultimate line, because this is where the relative importance of the various element contributions is exposed!
The results for $A_m$ by the two methods are of course the same, but the element contributions are grouped differently. Any grouping contains more useful information about the relative contributions of the various elements than does the multiplied-out result obtained by the "brute-force" solution of simultaneous loop or node equations.
**Generalization:** Current and Voltage Dividers

**Current divider**

\[
\frac{i_2}{i_{\text{tot}}} = \frac{z_1}{z_1 + z_2}
\]

- current in one branch
- total current
- opposite branch impedance
- sum of branch impedances

This is the dual of the:

**Voltage divider**

\[
\frac{v_2}{v_{\text{tot}}} = \frac{z_2}{z_1 + z_2}
\]

- voltage at tap
- total voltage
- tap impedance to ground
- sum of impedances to ground
Generalization: Loop and Node Removal

Every time Thévenin’s theorem is used, one loop is removed from the circuit:

Every time Norton’s theorem is used, one node is removed from the circuit:
Generalization: **Loop and node removal by Thevenin and Norton reduction.**

By successive use of the Thevenin and Norton theorems, a multi-loop, multi-node circuit can be reduced to a simple form from which the analytical results can be written by inspection.

This is an example of the powerful technique of doing the algebra on the circuit diagram.
Another example:

This is how element groupings arise naturally, by circuit reduction through successive loop and node removal.
Generalization: Advantages of Doing the Algebra on the Circuit Diagram

1. Simultaneous solution of multiple loop or node equations is replaced by sequential, simple, semigraphical steps.

2. The element values in the successively reduced models automatically appear in usefully grouped combinations (to facilitate tradeoffs).

3. Less likelihood of making algebraic mistakes.

4. Because the physical origins of all terms in the analytic results remain explicit, the results are in optimum form for design: element values can be chosen so that the results meet the specifications.
BOTTOM LINE:

AVOID solving simultaneous equations.

Instead, follow the signal path from input to output by Thevenin/Norton reduction, voltage/current dividers, and reflection of impedances.

This automatically generates Low Entropy Expressions; AVOID multiplying out the series/parallel expressions.

There may be many such paths (algorithms), each of which gives a different Low Entropy Expression.
3. NORMAL AND INVERTED POLES AND ZEROS

How to choose the gain at any frequency as the Reference Gain
"Flat gain"

\[ \frac{v_2}{v_1} = \frac{R_e \frac{1}{sC}}{R_1 + R_e \frac{1}{sC}} = \frac{R_e}{R_1 + \frac{R_e}{1+sC}} \]

The hard way:

\[ \frac{v_2}{v_1} = \frac{R_e}{R_1 + R_e + sC R_e R_e} \]
This format is commonly considered to be "the answer."

However, it is much better to extract the constant term from both the numerator and denominator polynomials in $s$: 

\[
\frac{V_o}{V_i} = \frac{R_E}{R_1 + R_2 + sCR_1R_E}
\]
This normalizes the polynomials, and exposes a zero-frequency gain and a corner frequency.
This is a special case of the general result as a ratio of polynomials in complex frequency $s$:

$$A = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \ldots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots}$$

Extraction of the constant term from numerator and denominator defines the zero-frequency reference gain $A_{\text{ref}}$ and normalizes the polynomials:

$$A = A_{\text{ref}} \frac{1 + \frac{b_1}{b_0} s + \frac{b_2}{b_0} s^2 + \frac{b_3}{b_0} s^3 + \ldots}{1 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} s^2 + \frac{a_3}{a_0} s^3 + \ldots}$$
Factorization of the polynomials defines the poles and zeros, and hence the final (preferred) "factored pole-zero" form:

\[
A = A_{\text{ref}} \frac{\left(1 + \frac{s}{\omega_{z1}}\right) \left(1 + \frac{s}{\omega_{z2}}\right) \left(1 + \frac{s}{\omega_{z3}}\right) \ldots}{\left(1 + \frac{s}{\omega_{p1}}\right) \left(1 + \frac{s}{\omega_{p2}}\right) \left(1 + \frac{s}{\omega_{p3}}\right) \ldots}
\]

The reference gain and the poles and zeros should, of course, be low entropy expressions in terms of the circuit elements.
Return to the example:

"Flat gain"

\[ \frac{v_o}{v_i} = \frac{R_e}{R_1 + R_2 \left| \frac{1}{sC} \right|} = \frac{R_e}{1 + sCR_e} \]

The hard way:

\[ \frac{v_o}{v_i} = \frac{R_e}{R_1 + R_2 + sCR_e} \]

\[ = \frac{R_e}{R_1 + R_2 + sCR_e} \cdot \frac{1}{1 + sC(R_1 \| R_2)} \]
"Flat gain"

\[
\frac{v_o}{v_i} = \frac{R_e \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1 + sC R_e}}{R_1 + \frac{R_2}{1 + sC R_e}} = \frac{R_e}{R_1 + R_2 + sC R_e R_L}
\]

The hard way:

\[
\frac{v_o}{v_i} = \frac{R_e}{R_1 + R_2 + sC R_e R_L}
\]

The easy way:

\[
\frac{v_o}{v_i} = \frac{R_e}{R_1 + R_2 + sC (R_1 \| R_2)}
\]

\[
\frac{v_o}{v_i} = \frac{R_e}{1 + sC (R_1 \| R_2)}
\]
"Flat gain"

\[
\frac{v_2}{v_1} = \frac{R_e \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{R_e}{1 + sCR_2} \frac{R_e}{R_1 + \frac{R_e}{1 + sCR_2}}
\]

The hard way:

\[
\frac{v_2}{v_1} = \frac{R_e}{R_1 + R_2 + sCR_1R_2} = \frac{R_e \cdot 1}{R_1 + R_2 + sC(R_1R_2)}
\]

Result:

\[
\frac{v_2}{v_1} = A = A_1 \frac{1}{1 + \frac{3}{\omega_1}} \text{ where } A_1 = \frac{R_e}{R_1 + R_2}, \quad \omega_1 = \frac{1}{C(R_1R_2)}
\]
Single-pole response:

\[ A = A_1 \frac{1}{1 + \frac{s}{\omega_i}} \]

flat gain \[ l \] normal pole

\[ |A| = A_1 \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_i})^2}} \]

\[ \angle A = -\tan^{-1}(\frac{\omega}{\omega_i}) \]
Single-pole response:

\[ A = A_1 \frac{1}{1 + \frac{s}{s_1}} \]

Flat gain \[ \frac{1}{1 + \frac{s}{s_1}} \] normal pole

\[ |A| = A_1 \sqrt{1 + \left(\frac{\frac{f}{f_1}}{\frac{f}{f_1}}\right)^2} \]

-20 dB/dec concave downward

\[ \tan^{-1}\left(\frac{f}{f_1}\right) \]

Phase lag

-45°/dec

0°

-90°

-45°

\[ \frac{A}{A} \]

\[ 1 \]

\[ \frac{1}{10} \]

\[ w_1 \]

\[ 2w_1 \]

\[ \frac{w_1}{2} \]
Single-zero response:

\[ A = A_0 \left( 1 + \frac{s}{\omega_n} \right) \]

- Flat gain \( A_0 \)
- Normal zero \( \omega_n \)

\[ |A| = A_0 \sqrt{1 + \left( \frac{\omega}{\omega_n} \right)^2} \]

- 1 dB
- 3 dB
- 1 dB
- Concave upward
- +20 dB/dec

\[ \omega_n \]

- 0 dB

\[ +90^\circ \]

- +45^\circ \]

- -5.7^\circ \]

\[ +45^\circ \]

- +45^\circ \]

\[ \theta = \tan^{-1} \left( \frac{\omega}{\omega_n} \right) \]

- +45^\circ/dec
- Phase lead

\[ \omega_n \]
Generalization: Property of Magnitude and Phase Graphs

A corner can be "seen" from further away on the phase graph than on the magnitude graph.

OR:

The phase gives a more accurate value of a nearby corner frequency than does the magnitude.
Normal and Inverted poles and zeros

\[ |A| \uparrow \]

\[ A = A_m \]
Normal and Inverted poles and zeros

\[ |A| \]

\[ A = A_m \]
Normal and Inverted poles and zeros

\[ A = A_m \frac{(1 + \frac{2}{\omega_c})(1 + \frac{2}{\omega_b})}{(1 + \frac{2}{\omega_n})} \]
Normal and Inverted poles and zeros

\[ A = A_m \]
Normal and Inverted poles and zeros

\[ A = A_m \frac{(1 + \frac{s}{w_b})}{(1 + \frac{s}{w_1})(1 + \frac{s}{w_e})} \]

Inversion of pole-zero factors \(\leftrightarrow\) vertical inversion of magnitude graph
Normal and Inverted poles and zeros

$A = A_m$
Normal and Inverted poles and zeros

\[ A_I = A_{Im} \frac{(1 + \frac{w_c}{s})}{(1 + \frac{w_b}{s})(1 + \frac{w_b}{s})} \]

"inverted" poles and zeros

Inversion of frequency terms $\iff$ horizontal reversal of magnitude graph
Normal and Inverted poles and zeros

\[ A = A_m \]

\[ |A| \uparrow \]

\[ A_m \]
Normal and Inverted poles and zeros

\[ A = A_m \frac{(1 + \frac{w_c}{s})(1 + \frac{w_b}{s})}{1 + \frac{w_e}{s}} \]
Normal and Inverted poles and zeros

\[ A_l = A_{lm} \frac{(1 + \frac{w_{lc}}{s})}{(1 + \frac{w_{lb}}{s})(1 + \frac{w_{le}}{s})} \]

Inversion of frequency terms $\iff$ horizontal reversal of magnitude graph
Normal and Inverted poles and zeros

\[ A = A_m \frac{\left(1 + \frac{w_a}{s}\right)}{\left(1 + \frac{w_a}{s}\right)\left(1 + \frac{w_b}{s}\right)} \quad \text{"normal" poles and zeros} \]

\[ \text{"inverted" poles and zeros} \quad \frac{\left(1 + \frac{w_b}{s}\right)}{\left(1 + \frac{w_b}{s}\right)\left(1 + \frac{w_c}{s}\right)} \left(1 + \frac{w_c}{s}\right) \]

\[
\text{Inversion of frequency terms } \Longleftrightarrow \text{ horizontal reversal of magnitude graph}
\]
Relationships to conventional forms:

\[ A = A_m \frac{(1 + \frac{s}{w_c})}{s} = A_m \frac{s}{w_c} \frac{s}{w_c} = \frac{(s + 1)}{(s + \frac{1}{w_c})} \]

\[ = A_m \frac{w_c s}{w_c w_b} \frac{(1 + s/w_c)}{1 + \frac{s}{w_c}} = \frac{s}{w_c} \frac{(1 + \frac{s}{w_c})}{(1 + \frac{s}{w_c})(1 + \frac{s}{w_b})} \]

Conventional form
(normal poles and zeros)

Where is \( w_a \) on the graph? Where is \( A_m \) in the formula?

\( w_x \) is not a useful parameter.
Relationships to conventional forms:

\[ A = A_m \frac{(1 + \frac{w_c}{s})}{(1 + \frac{w_a}{s})(1 + \frac{w_b}{s})} \]

\[ = \frac{A_m w_c s}{w_a w_b} \frac{(1 + \frac{s}{w_c})}{(1 + \frac{s}{w_a})(1 + \frac{s}{w_b})} = \frac{s}{w_x} \frac{(1 + \frac{s}{w_c})}{(1 + \frac{s}{w_a})(1 + \frac{s}{w_b})} \]

\[ \text{conventional form (normal poles and zeros)} \]

Where is \( w_x \) on the graph? Where is \( A_m \) in the formula?

\( w_x \) is not a useful parameter.
\[
A = A_m \frac{(1 + \frac{s}{w_3})}{(1 + \frac{s}{w_1})(1 + \frac{s}{w_2})} = A_m \frac{1}{w_3} \frac{(w_2 + s)}{(w_1 + s)(w_2 + s)}
\]

\[
= A_m \frac{w_1 w_2}{w_3} \frac{(s + w_3)}{(s + w_1)(s + w_2)} = w_y \frac{(s + w_3)}{(s + w_1)(s + w_2)}
\]

*conventional form*

Where is \(w_y\) on the graph? Where is \(A_m\) in the formula? \(w_y\) is not a useful parameter.
\[ A = A_m \frac{(1 + \frac{s}{w_3})}{(1 + \frac{s}{w_1})(1 + \frac{s}{w_2})} = A_m \frac{1}{w_3} \frac{\left( \frac{w_3 + s}{w_1 + s} \right)}{w_2} \frac{\left( \frac{w_3 + s}{w_1 + s} \right)}{w_2} \]

\[ = \frac{A_m w_1 w_2}{w_3} \frac{(s+w_3)}{(s+w_1)(s+w_2)} = w_y \frac{(s+w_3)}{(s+w_1)(s+w_2)} \]

Where is \( w_y \) on the graph? Where is \( A_m \) in the formula? \( w_y \) is not a useful parameter.
If there is no "flat gain", use a reference value:

\[ A = A_3 \frac{1}{\omega_3} = A_3 \frac{\omega_3}{5} \]

\[ A = A_4 \frac{5}{\omega_4} \]

\[ 0^\circ \]
\[ -45^\circ \]
\[ -90^\circ \]
Exercise 3.1
Write factored pole-zero forms from asymptotes
Exercise 3.1 - Solution

Express the gains in factored pole-zero form

\[ A = A_1 \frac{1}{(1+\frac{w_1}{s})(1+\frac{s}{w_2})} \]
Exercise 3.1 - Solution

Express the gains in factored pole-zero form

\[ A = A_1 \frac{1}{\left(1 + \frac{\omega_1}{\omega}\right)\left(1 + \frac{\omega}{\omega_2}\right)} \]

\[ A = A_1 \frac{1 + \frac{\omega_1}{\omega}}{1 + \frac{\omega}{\omega_2}} \]
Exercise 3.1 - Solution

Express the gains in factored pole-zero form

\[ A = A_1 \frac{1}{(1 + \frac{\omega_1}{s})(1 + \frac{\omega_2}{s})} \]
\[ A = A_1 \frac{1 + \frac{\omega_1}{s}}{1 + \frac{s}{\omega_1}} \]
\[ A = A_1 \left( \frac{\omega_1}{s} \right) \frac{1 + \frac{\omega_2}{s}}{1 + \frac{s}{\omega_2}} \]
Exercise 3.1 - Solution

Express the gains in factored pole-zero form

\[
A = A_1 \frac{1}{(1 + \frac{\omega_1}{s})(1 + \frac{s}{\omega_2})} \\
A = A_1 \frac{1 + \frac{\omega_1}{s}}{1 + \frac{s}{\omega_2}} \\
A = A_1 \left(\frac{\omega_1}{s}\right) \frac{1 + \frac{s}{\omega_2}}{1 + \frac{s}{\omega_1}} \\
A = A_1 \left(\frac{\omega_1}{s}\right)^2 \frac{1 + \frac{s}{\omega_2}}{1 + \frac{s}{\omega_1}}
\]
Lag-lead network

\[ \frac{v_2}{v_1} = A = \frac{R_2}{R_1 + R_2} \cdot \frac{\frac{1}{sC} + R_3}{\frac{1}{sC} + R_2 + R_1R_2} \]

\[ A = A_1 \frac{1 + \frac{3}{w_2}}{1 + \frac{3}{w_1}} \quad \text{where} \quad A_1 = \frac{R_2}{R_1 + R_2} \quad w_1 = \frac{1}{C(R_3 + R_1R_2)} \quad w_2 = \frac{1}{C R_3} \]

\[ 0^\circ \quad -90^\circ \]

\[ A_1 \quad 1A_1 \quad w_1 \quad 10w_1 \quad 0.1w_2 \quad \]

\[ A_2 \]
In this case, there are two flat gains. As derived, the low-frequency flat gain \( A_1 \) appears as coefficient, together with normal pole and zero:

\[
A = A_1 \frac{1 + \frac{\omega}{\omega_1}}{1 + \frac{s}{\omega_1}}
\]

Equally well, directly from the \( |A| \) asymptotes, the result could be written with the high-frequency flat gain \( A_2 \) as coefficient, together with inverted zero and pole:

\[
A = A_2 \frac{1 + \frac{\omega}{\omega_1}}{1 + \frac{s}{\omega_1}}
\]

What is the relation between \( A_1 \) and \( A_2 \)? One form of the result can be derived from the other algebraically:

\[
A = \left( A_1 \right) \frac{1 + \frac{\omega}{\omega_2}}{1 + \frac{s}{\omega_1}} = A_1 \frac{\frac{\omega}{\omega_2} + 1}{\frac{s}{\omega_1} + 1} = A_1 \frac{\frac{\omega}{\omega_2} + 1}{\frac{s}{\omega_1} + 1} = A_1 \frac{\omega_2}{\omega_1} \frac{1 + \frac{s}{\omega_1}}{1 + \frac{s}{\omega_1}}
\]

This is \( A_1 \big|_{s \to 0} \)
In this case, there are two flat gains. As derived, the low-frequency flat gain $A_1$ appears as coefficient, together with normal pole and zero:

$$A = A_1 \frac{1 + \frac{s}{w_2}}{1 + \frac{s}{w_1}}$$

Equally well, directly from the $|A|$ asymptotes, the result could be written with the high-frequency flat gain $A_2$ as coefficient, together with inverted zero and pole:

$$A = A_2 \frac{1 + \frac{w_2}{s}}{1 + \frac{w_1}{s}}$$

What is the relation between $A_1$ and $A_2$? One form of the result can be derived from the other algebraically:

$$A = \left( A_1 \right) \frac{1 + \frac{s}{w_2}}{1 + \frac{s}{w_1}} = A_1 \frac{\frac{s}{w_2}}{\frac{s}{w_1}} \frac{\frac{w_2}{s} + 1}{\frac{w_1}{s} + 1} = A_1 \frac{\frac{w_1}{w_2}}{\frac{w_1}{w_2} + 1}$$

This is $A_1|_{s \to 0}$

This is $A_1|_{s \to \infty}$, so must be $A_2$. 

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Result:
\[ \frac{A_2}{A_1} = \frac{w_1}{w_2} \]

For the lag-lead network:
\[ A_2 = A_1 \frac{w_1}{w_2} = \frac{R_2}{R_1 + R_2} \cdot \frac{CR_3}{\epsilon(R_3 + R_1R_2)} \]
which is obvious from the reduced model.

**Generalization: Gain-Bandwidth Trade-Off**
For a single-slope (± 20dB/dec)

Ratio of flat gains = Ratio of corner frequencies that separate them

This is a form of gain-bandwidth trade-off.
More than one flat gain

\[ A = A_1 \frac{1 + \frac{w_2}{w_1}}{1 + \frac{w_1}{w_2}} = A_1 \frac{w_2}{w_1} \frac{1 + \frac{w_2}{w_1}}{1 + \frac{w_1}{w_2}} = A_2 \frac{1 + \frac{w_1}{w_2}}{1 + \frac{w_1}{w_2}} \]

Hence: "gain-bandwidth tradeoff:"

\[ \frac{A_2}{A_1} = \frac{w_2}{w_1} \]

Either flat gain can be used as "reference" gain.
Lag-lead network

\[ \frac{V_2}{V_1} = A = \frac{R_2}{R_1 + R_2} \cdot \frac{1}{\frac{1}{j\omega C} + R_3} \]

\[ A = A_1 \frac{1 + \frac{\omega}{\omega_1}}{1 + \frac{1}{\omega_1}} \]

where \( A_1 = \frac{R_2}{R_1 + R_2} \)

\[ \omega_1 = \frac{1}{C(R_3 + R_1 R_2)} \]

\[ \omega_2 = \frac{1}{CR_3} \]
If \( w_2 > 100w_1 \), phase asymptotes do not overlap and the phase lag reaches 90° before returning to zero.

If \( w_2 < 100w_1 \), the phase asymptotes do overlap, and the phase lag reaches a maximum, less than 90°, which is a function of the ratio of the flat gains.
Find the maximum phase lag $\phi_{\text{max}}$ as a function of the gain ratio $k \equiv A_1/A_2 = \omega_2/\omega_1$.

$$
\phi_{\text{max}} = -45^\circ \log \frac{0.1 \omega_2}{0.1 \omega_1} = -45^\circ \log k \quad (k < 100)
$$
Exercise 3.2
Write factored pole-zero forms for different Reference Gains, and write $A_2$ and $A_3$ in terms of $A_1$.

Exercise:
No flat gain
Identify the gain at any chosen frequency as "reference" gain.
Exercise 3.2 - Solution

Exercice:
No flat gain
Identify the gain at any chosen frequency as "reference" gain

\[ A_1 = A_i \frac{\omega_1}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \]
\[ A_2 = A_2 \left( \frac{\omega_2}{s} \right)^2 \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \]
\[ A_3 = A_3 \frac{\omega_3}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \]
Exercise 3.2 - Solution

Exercise:
No flat gain
Identify the gain at any chosen frequency as "reference" gain

\[ A = A_1 \frac{\omega_1}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \]

\[ A = A_2 \left( \frac{\omega_2}{s} \right) \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \]

\[ A = A_3 \frac{\omega_3}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \]

Exercise: Express \( A_2 \) and \( A_3 \) in terms of \( A_1 \).
Exercise 3.2 - Solution

Exercise:
No flat gain
Identify the gain at any chosen frequency as "reference" gain

\[ A = A_1 \frac{w_1}{w_a} \frac{1 + \frac{s}{w_b}}{1 + \frac{s}{w_a}} \]
\[ A = A_2 \left( \frac{w_2}{s} \right)^2 \frac{1 + \frac{s}{w_b}}{1 + \frac{s}{w_a}} \]
\[ A = A_3 \frac{w_3}{s} \frac{1 + \frac{s}{w_b}}{1 + \frac{s}{w_a}} \]

Exercise: Express \( A_2 \) and \( A_3 \) in terms of \( A_1 \).

\[ A_2 = \left( A_1 \frac{w_1}{w_a} \right) \left( \frac{w_a}{w_2} \right)^2 \]

\[ = A_1 \frac{w_1 w_a}{w_2^2} \]
Exercise 3.2 - Solution

Exercise:
No flat gain
Identify the gain at any chosen frequency as “reference” gain

\[ A = \frac{A_1 w_1}{w_a} \frac{1 + \frac{s}{w_b}}{1 + \frac{s}{w_a}} \]

\[ A = \frac{A_2 (\frac{w_b}{s})^2 + \frac{s}{w_b}}{1 + \frac{s}{w_a}} \]

\[ A = \frac{A_3 w_3}{s} \frac{1 + \frac{s}{w_b}}{1 + \frac{s}{w_a}} \]

Exercise: Express \( A_2 \) and \( A_3 \) in terms of \( A_1 \).

\[ A_2 = \left( \frac{A_1 w_1}{w_a} \right) \left( \frac{w_a}{w_b} \right)^2 \]

\[ A_3 = \left( \frac{A_1 w_1}{w_a} \right) \left( \frac{w_a}{w_b} \right) \frac{s}{w_3} \]

\[ = A_1 \frac{w_1 w_a}{w_2^2} \]

\[ = A_1 \frac{w_1 w_a}{w_3 w_b} \]
Any flat gain can be used as "reference" gain \( A_{\text{ref}} \). With respect to \( A_{\text{ref}} \), poles and zeros above \( A_{\text{ref}} \) are normal, those below \( A_{\text{ref}} \) are inverted.

\[ A = A_1 \frac{(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_6})}{(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_1})(1 + \frac{s}{\omega_6})(1 + \frac{s}{\omega_5})(1 + \frac{s}{\omega_7})} \]

\[ A = A_2 \frac{(1 + \frac{s}{\omega_4})}{(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_5})(1 + \frac{s}{\omega_7})} \]

\[ A = A_3 \frac{(1 + \frac{s}{\omega_5})(1 + \frac{s}{\omega_5})}{(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_3})(1 + \frac{s}{\omega_5})(1 + \frac{s}{\omega_7})} \]
If you don't use inverted poles and zeros, you are stuck with the zero-frequency gain as the reference gain.

The principal benefit of using inverted poles and zeros is that you can choose the gain at any frequency as the reference gain.
Impedance asymptotes

\[ Z = R_m \frac{1 + \frac{\omega}{\omega_1}}{1 + \frac{\omega}{\omega_2}} \frac{1}{1 + \frac{\omega}{\omega_3}} \]
Input and output impedances

\[ Z_i = R_i + \frac{R_2}{1 + \frac{1}{SCR_2}} = (R_i + R_2) \left( 1 + \frac{\frac{1}{S}}{1 + \frac{\frac{1}{S}}{\omega_2}} \right) = R_i \left( 1 + \frac{\omega_2}{1 + \frac{\omega_2}{\omega_1}} \right) \]

\[ Z_o = R_i \| R_2 \| \frac{1}{S} = R_i \| R_2 \left( 1 + \frac{\frac{1}{S}}{\omega_2} \right) \]
Exercise 3.3
Write input and output impedances $Z_i$ and $Z_o$ in factored pole-zero forms.

Exercise
Find the input and output impedances $Z_i$ and $Z_o$ in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

(a) \[ \text{Network A} \]
(b) \[ \text{Network B} \]
Exercise 3.3 - Solution

Exercise

Find the input and output impedances $Z_i$ and $Z_o$ in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

(a) \[ Z_i \rightarrow C \rightarrow Z_o \]

(b) \[ Z_i \rightarrow C \rightarrow Z_o \]
Exercise 3.3 - Solution

Exercise

Find the input and output impedances $Z_i$ and $Z_o$ in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

\[(a)\]

\[Z_i = R + \frac{1}{sC} = R(1 + \frac{\omega_1}{s}) \quad \omega_1 = \frac{1}{CR}\]

\[Z_o = R \parallel \frac{1}{sC} = R \frac{1}{1 + \frac{s}{\omega_1}}\]

\[(b)\]

\[Z_i \quad Z_o\]
Exercise 3.3 - Solution

\[ Z_i = R + \frac{1}{sC} \]
\[ = R \left( 1 + \frac{w_i}{s} \right) \quad \omega_i = \frac{1}{CR} \]

\[ Z_0 = R \left| 1 + \frac{1}{sC} \right| \]
\[ = R \frac{1}{1 + \frac{1}{w_i}} \]
Exercise 3.3 - Solution

Exercise.

Find the input and output impedances $Z_i$ and $Z_o$ in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

(a) $Z_i = R_i + R_x + \frac{1}{sC}$

$= (R_i + R_x)(1 + \frac{\omega_3}{s})$

$\omega_3 = \frac{1}{sC(R_i + R_x)}$

(b) $Z_o = R \parallel (R_x + \frac{1}{sC})$

$= (R \parallel R_x)(1 + \frac{\omega_4}{s})$

$\omega_4 = \frac{1}{sC R_x}$
Exercise 3.3 - Solution

\[ Z_i = R_i + R_2 + \frac{1}{\frac{sC}{sC}} \]
\[ = (R_i + R_2)(1 + \frac{\omega_3}{\frac{sC}{sC}}) \]
\[ \omega_3 = \frac{C(R_i + R_2)}{sC} \]

\[ Z_o = R_1 || (R_2 + \frac{1}{\frac{sC}{sC}}) \]
\[ = (R_1 || R_2) \frac{1 + \omega_3}{1 + \frac{\omega_3}{\frac{sC}{sC}}} \]
\[ \omega_4 = \frac{1}{CR_2} \]
4. AN IMPROVED FORMULA FOR QUADRATIC ROOTS

The Conventional Formula suffers from two congenital defects
Example

Analyze the following circuit for the gain response \( \frac{V_2}{V_1} \), using the given values to justify appropriate analytic approximations.

Express the result in the factored pole-zero form

\[
\frac{V_2}{V_1} = A = A_0 \frac{\prod (1+s/w_p)}{\prod (1+s/w_z)}
\]

Sketch \(|A|\) and \(\angle A\) showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.
\[
A = \frac{\left(\frac{R_L}{1+sC_1R_L}\right)(R_2+sC_1)}{\frac{R_L}{1+sC_1R_L} + R_2 + \frac{1}{sC_1}} + R_1
\]

\[
\frac{R_L}{1+sC_2R_L} + \frac{R_2 + \frac{1}{sC_1}}{1+sC_2R_L + R_2 + \frac{1}{sC_1}}
\]

\[\downarrow\text{a lot of algebra}\]

\[
A = \frac{R_L + sC_1R_2R_L}{[R_1 + R_L] + 5[C_1(R_1R_2 + R_LR_2 + R_1R_L) + C_1R_1R_L] + 5^2[C_2R_1R_2R_L]}
\]

This is a high-entropy expression. To lower the entropy, write the polynomials in $s$ with a leading term of unity.
Now, recognize series/parallel resistance combinations:

\[ A = \frac{R_L}{R_1 + R_L} \frac{1 + sC_1R_L}{1 + s\left[C_1\left(\frac{R_1R_2 + R_1R_L + R_2R_L}{R_1 + R_L}\right) + C_2\left(\frac{R_2R_L}{R_1 + R_L}\right)\right] + s^2\left[C_1C_2\left(\frac{R_1R_2R_L}{R_1 + R_L}\right)\right]} \]
\[ A = \frac{R_L}{R_1 + R_L} \left( \frac{1 + sC_1R_2}{1 + s\left[ C_1\left( R_1R_2 + R_1R_2 + R_1R_L \right) + C_2\left( \frac{R_1R_2}{R_1 + R_L} \right) \right] + s^2 \left[ C_1C_2\left( \frac{R_1R_2R_L}{R_1 + R_L} \right) \right]} \right) \]

Now, recognize series/parallel resistance combinations:

\[ (R_2 + R_1||R_L) \quad (R_1||R_L) \quad R_2(R_1||R_L) \]

The same result, including the series/parallel resistance grouping, could have been obtained with less algebra by elimination, first, of one of the loops of the original circuit.

Circuit with \( R_1 \) and \( R_L \) absorbed into a Thvenin equivalent:
Example

Analyze the following circuit for the gain response $\frac{v_2}{v_1}$, using the given values to justify appropriate analytic approximations.

Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} = A = A_0 \frac{\prod (1+s/w_1)}{\prod (1+s/w_2)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.
4. Improved Quadratic Roots

\[ A = \frac{\frac{1}{sC_2} \left( R_x + \frac{1}{sC_1} \right)}{R_x + \frac{1}{s} \left( \frac{1}{C_x} + \frac{1}{C_1} \right)} + R \times R_L \]

\[ = \frac{R_L}{R_L + R} \cdot \frac{1 + 5C_1 R_x}{1 + s \left[ C_1 (R_x + R \times R_L) + C_2 (R \times R_L) \right] + s^2 \left[ C_1 C_2 R_x (R \times R_L) \right]} \]
Improved Quadratic Roots

\[
A = \frac{\frac{1}{sC_2} \left( R_2 + \frac{1}{sC_1} \right)}{R_z + \frac{1}{sC_1} + \frac{1}{sC_2}} + R_{L_1} + R_{L_2}
\]

\[
A = \frac{R_L}{R_1 + R_L} \cdot \frac{1 + sC_1 R_2}{1 + s \left[ C_1 \left( R_2 + R_{L_1} R_{L_2} \right) + C_2 \left( R_{L_1} R_{L_2} \right) \right] + s^2 \left[ C_1 C_2 R_2 (R_{L_1} R_{L_2}) \right]}
\]

Use of numerical values to justify analytic approximation
Generalization: Use of Numerical Values to Justify Analytic Approximations

Use numbers to justify leaving out a term, but continue the analysis with the symbols.

This way, the analysis result can be used for design, because the numbers can be changed so that the answer has the desired value. (The approximation must be checked to ensure that it is not invalidated by the new numbers.)

You can't lose by trying!
\[ A = \frac{R_L}{R_1 + R_L} \frac{1 + s C_1 R_2}{1 + s [C_1 (R_2 + (R_1 || R_L))] + s^2 [C_1 C_2 R_2 (R_1 || R_L)]} \]

\[ = A_0 \left( \frac{1 + \frac{s}{\omega_b}}{(1 + \frac{s}{\omega_1})(1 + \frac{s}{\omega_3})} \right) \]

where

\[ \frac{1}{\omega_{13}} = \frac{C_1 (R_2 + R_{1||R_L}) \pm \sqrt{C_1^2 (R_2 + R_{1||R_L})^2 - 4 C_1 C_2 R_2 (R_1 || R_L)^2}}{2} \]

This is useless for design, and in any case is inaccurate numerically.
Improved formulas for quadratic roots

\[ ax^2 + bx + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) \]

\[ = a(x - x_1)(x - x_2) \]

\[ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Disadvantages of the conventional form

1. Complicated algebraic expressions in terms of element values:

\[ \frac{1}{\omega_{1,3}} = \frac{C_1(R_2 + R_{11}R_1) \pm \sqrt{C_1^2(R_2 + R_{11}R_1)^2 - 4C_2R_2(R_{11}R_1)}}{2} \]

2. Computationally inaccurate when \( 4ac \ll b^2 \):

\[ \frac{1}{\omega_{1,3}} = 10^{-3} \frac{3.3 \pm \sqrt{3.3^2 - 0.026}}{2} \]

small difference of large numbers, for one root

These are congenital defects!
Better method:
\[ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}\right] = -\frac{b}{a} F \]

where
\[ F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}, \quad Q^2 \equiv \frac{ac}{b^2} \]

\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4Q^2}\right] \]
\[ = -\frac{b}{a}\left[\frac{\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4Q^2}}{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}}\right] = -\frac{b}{a}\left(\frac{1 - \sqrt{1 - 4Q^2}}{\sqrt{1 - 4Q^2}}\right) = -\frac{b}{a}\frac{1 - \sqrt{1 - 4Q^2}}{F} \]
\[ = -\frac{b}{a} \frac{Q^2}{F} = -\frac{c}{b} \frac{1}{F} \]
Better method:

\[ x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \right] = -\frac{b}{a} F \]

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2}, \quad Q^2 = \frac{ac}{b^2} \]

\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{a} \left[ \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4Q^2} \right] \]

\[ = -\frac{b}{a} \frac{\left[ \frac{1}{2} - \frac{1}{2} \sqrt{1 - 4Q^2} \right] \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \right]}{\left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \right]} = -\frac{b}{a} \frac{\frac{1}{4} - \frac{1}{4} (1 - 4Q^2)}{F} \]

\[ = -\frac{b}{a} \frac{Q^2}{F} = -\frac{c}{b} \frac{1}{F} \]

**Crucial step:** Large numbers are subtracted exactly, leaving the small difference in analytic form.

Hence, both roots can be computed with equal accuracy:

\[ x_1 = -\frac{c}{b} \frac{1}{F}, \quad x_2 = -\frac{b}{a} F \]
Rewrite the two roots:
\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

\[ x_2 \] is acceptable for all values; \( x_1 \) is unacceptable for \( 4ac < b^2 \).

Rewrite \( x_2 \):
\[ x_2 = \frac{c}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \]

Now, instead of using the formula for \( x_1 \) directly, use the property of the quadratic that \( x_1 x_2 = \frac{c}{a} \):
\[ x_1 =\frac{c}{a} x_2 = \frac{c}{a} \frac{a}{b} \frac{1}{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \]

Thus, the improved formulas for the quadratic roots are:
\[ x_1 = -\frac{c}{b} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \quad x_2 = -\frac{b}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \]
Rewrite the two roots:
\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

\( x_2 \) is acceptable for all values; \( x_1 \) is unacceptable for \( 4ac < b^2 \).

Rewrite \( x_2 \):
\[ x_2 = \frac{-b}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \]

Now, instead of using the formula for \( x_1 \) directly, use the property of the quadratic that \( x_1 x_2 = \frac{c}{a} \):
\[ x_1 = \frac{c}{a} x_2 = -\frac{c}{a} \frac{a}{b} \frac{1}{2 + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \]

Thus, the improved formulas for the quadratic roots are:
\[ x_1 = -\frac{c}{b} \left[ \frac{1}{2 + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \right] \quad x_2 = -\frac{b}{a} \left[ \frac{1}{2 + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \right] \]

\[ ax^2 + bx + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) = a(x - x_1)(x - x_2) \]
Rewrite the two roots:

\[ x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

\( x_2 \) is acceptable for all values; \( x_1 \) is unacceptable for \( 4ac < b^2 \).

Rewrite \( x_2 \):

\[ x_2 = -\frac{b}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \]

Now, instead of using the formula for \( x_1 \) directly, use the property of the quadratic that \( x_1 x_2 = \frac{c}{a} \):

\[ x_1 = \frac{c}{a} \frac{1}{x_2} = -\frac{c}{a} \frac{a}{b} \frac{1}{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \]

Thus, the improved formulas for the quadratic roots are:

\[ x_1 = -\frac{c}{b} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \quad x_2 = -\frac{b}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \]
\[ ax^2 + bx + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) \]
\[ = a(x - x_1)(x - x_2) \]

More elegant form:
\[ x_1 = -\frac{c}{b} \quad \frac{x_1}{x_2} = \frac{Q}{F} \quad x_2 = -\frac{b}{a} \]

where
\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \quad \text{in which } Q^2 = \frac{ac}{b^2} \]
\[ ax^2 + bx + c = a \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) = a(x - x_1)(x - x_2) \]

More elegant form:
\[ x_1 = -\frac{c}{b} \quad x_1 = \frac{Q}{\bar{P}} \quad x_2 = -\frac{b}{a} - \frac{Q}{\bar{P}} \]

Simple ratios of the original quadratic coefficients

where
\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 = \frac{ac}{b^2} \]

root ratio
\[ ax^2 + bx + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) = a(x - x_1)(x - x_2) \]

More elegant form:

\[ x_1 = -\frac{c}{b} + \frac{1}{2} F \]
\[ x_2 = \frac{Q^2}{F^2} \]
\[ x_1 = -\frac{b}{a} F \]

Where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \text{ in which } Q^2 = \frac{ac}{b^2} \]

This is exact for all values.

If \( Q > 0.5 \), \( F \) is complex \( \Rightarrow \) complex roots

If \( Q < 0.5 \), \( F \) is real \( \Rightarrow \) real roots

If \( Q < 0.5 \), \( F \approx 1 \)

Note how simple the analytic roots, and therefore the quadratic factorization, become if \( F \approx 1 \).
\[ a x^2 + b x + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) = a (x - x_1)(x - x_2) \]

More elegant form:

\[ x_1 = -\frac{c}{b} \quad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \quad x_2 = -\frac{b}{a} F \]

Root ratio

Simple ratios of the original quadratic coefficients

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 = \frac{ac}{b^2} \]

\[ F \rightarrow 1 \text{ very rapidly as } Q \text{ drops below 0.5} : \]

\[ F \approx 1 \text{ with 10% error for } Q \leq 0.3 \]

Remember this graph!
One more time!

\[ a x^2 + bx + c = a \left( x^2 + \frac{b}{a} x + \frac{c}{a} \right) = a(x - x_1)(x - x_2) \]

Bad!

\[ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Good!

\[ x_1 = -\frac{c}{b} \left[ \frac{1}{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \right] \quad x_2 = -\frac{b}{a} \left[ \frac{1}{\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}} \right] \]
One more time!

\[ ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = a(x-x_1)(x-x_2) \]

Bad!

\[ x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Good!

\[ x_1 = -\frac{c}{b}\left[\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}\right], \quad x_2 = -\frac{b}{a}\left[\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}}\right] \]

Further information:

TT DVD Ch 3
Paper

http://www.RDMiddlebrook.com

4. Improved Quadratic Roots
Math in action!
Apply mathematical tools to real-world problems...
see inside for details.

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

IMPORTANT: Dated Materials Enclosed
Math in action!
Apply mathematical tools to real-world problems...
see inside for details.

\[ x_1 = -\frac{b}{a} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \]
\[ x_2 = -\frac{c}{b} \left/ \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \right] \right. \]
General result:
\[ ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) \]
\[ = a(x - r_1)(x - r_2) \]
\[ = a(x + \frac{c - 1}{b})(x + \frac{b + 1}{a}) \]

where
\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4ac}{b^2}} \]
\[ x_1 = -\frac{c - 1}{b} F \]
\[ x_2 = -\frac{b + 1}{a} F \]
General result:
\[ ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a(x - x_1)(x - x_2) = a(x + \frac{c}{bF})(x + \frac{b}{aF}) \]

where
\[ F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}} \]

Good approximation for real roots, \( Q = \frac{ac}{b^2} \leq 0.5; \)
\[ F \approx 1 \]
\[ F = 1 \approx a(x + \frac{c}{b})(x + \frac{b}{a}) \]

\[ x_1 = -\frac{c}{bF} \approx -\frac{c}{b} \]
\[ x_2 = -\frac{b}{aF} \approx -\frac{b}{a} \]
Alternative format:

\[ ax^2 + bx + c = c \left( 1 + \frac{b}{c}x + \frac{a}{c}x^2 \right) \]
\[ = c \left( 1 - \frac{a}{x_1} \right) \left( 1 - \frac{a}{x_2} \right) \]
\[ = c \left( 1 + \frac{b}{c}F\bar{x} \right) \left( 1 + \frac{a}{b}F\bar{x} \right) \]

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{\Omega^2}{Q^2}} \]
\[ \Omega^2 = \frac{ac}{b^2} \]
Alternative format:

\[ a x^2 + b x + c = c \left(1 + \frac{b}{c} x + \frac{a}{c} x^2 \right) \]
\[ = c \left(1 - \frac{a}{x_1} \right) \left(1 - \frac{a}{x_2} \right) \]
\[ = c \left(1 + \frac{b}{c} F x \right) \left(1 + \frac{a}{b} F x \right) \]

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 Q^2} \]
\[ Q^2 = \frac{ac}{b^2} \]

Redefine coefficients:

\[ 1 + a_1 x + a_2 x^2 = \left(1 - \frac{a_2}{x_1} \right) \left(1 - \frac{a_2}{x_2} \right) \]
\[ = \left(1 + a_1 F x \right) \left(1 + \frac{a_2}{a_1} \frac{1}{F} x \right) \]

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4 Q^2} \]
\[ Q^2 = \frac{a_2}{a_1^2} \]
Alternative format:

\[ ax^2 + bx + c = c \left(1 + \frac{b}{c} x + \frac{a}{c} x^2\right) \]

\[ = c \left(1 - \frac{x}{x_1} \right) \left(1 - \frac{x}{x_2} \right) \]

\[ = c \left(1 + \frac{b}{c} Fx \right) \left(1 + \frac{a}{b} \frac{1}{F} x \right) \]

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \]

\[ Q^2 = \frac{a_1 x}{b} \]

Good approximation for real roots, \( Q \leq 0.5 \): \( F \approx 1 \)

Redefine coefficients:

\[ 1 + a_1 x + a_2 x^2 = \left(1 - \frac{x}{x_1} \right) \left(1 - \frac{x}{x_2} \right) \]

\[ = \left(1 + a_1 Fx \right) \left(1 + \frac{a_2}{a_1} \frac{1}{F} x \right) \]

where

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \]

\[ Q^2 = \frac{a_2}{a_1} \]
Generalization: Improved Formulas for Roots of a Quadratic

\[ F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \]

\[ Q^2 = \frac{ac}{b^2} \]

\[ ax^2 + bx + c = a(x-x_1)(x-x_2) \]

\[ x_1 = -\frac{c}{b} \frac{1}{F} \]

\[ x_2 = -\frac{b}{a} F \]

\[ \frac{x_1}{x_2} = \frac{Q^2}{F^2} \]

\[ 1 + a_1 x + a_2 x^2 = (1 - \frac{x}{x_1})(1 - \frac{x}{x_2}) \]

\[ x_1 = -\frac{1}{a_1 F} \]

\[ x_2 = -\frac{a_1}{a_2} F \]

\[ ax^2 + bx + c = a\left(x + \frac{c}{b} \frac{1}{F}\right)\left(x + \frac{b}{a} F\right) \]

\[ 1 + a_1 x + a_2 x^2 = (1 + a_1 F x) \left(1 + \frac{a_2}{a_1} F x\right) \]

\[ \text{For real roots, } Q \leq 0.5 \text{ and } F \approx 1: \]

\[ x_1 \approx -\frac{c}{b} \]

\[ x_2 \approx -\frac{b}{a} \]

\[ \frac{x_1}{x_2} \approx Q^2 \]

\[ x_1 \approx -\frac{1}{a_1} \]

\[ x_2 \approx -\frac{a_1}{a_2} \]

\[ ax^2 + bx + c \approx a\left(x + \frac{c}{b} \frac{1}{F}\right)\left(x + \frac{b}{a} F\right) \]

\[ 1 + a_1 x + a_2 x^2 \approx (1 + a_1 F x) \left(1 + \frac{a_2}{a_1} F x\right) \]
Advantages over the conventional formulas

1. Both roots can be computed with equal accuracy (avoids small difference of large numbers).

2. For real roots, to a very good approximation, there is no \( \sqrt{ } \) anywhere in the results, and each root is a simple ratio of coefficients of the original quadratic.
Useful format of quadratic \( 1 + a_1 s + a_2 s^2 \)

Define: \( Q = \frac{\sqrt{a_2}}{a_1} \)

If \( Q > 0.5 \) (F complex), roots are complex.
Leave in quadratic form:
\[
1 + a_1 s + a_2 s^2 = 1 + \frac{a_1}{\sqrt{a_2}} (\sqrt{a_2} s) + (\sqrt{a_2} s)^2
\]

If \( Q < 0.5 \) (F \approx 1), roots are real.
Factor into two real roots:
\[
1 + a_1 s + a_2 s^2 \approx (1 + a_1 s) (1 + \frac{a_2}{a_1} s)
\]

\[
= \left[1 + \frac{a_1}{\sqrt{a_2}} (\sqrt{a_2} s)\right]\left[1 + \frac{\sqrt{a_2}}{a_1} (\sqrt{a_2} s)\right]
\]

\( \frac{1}{Q} \) \( Q \) normalized frequency
Return to the circuit example:

Analyze the following circuit for the gain response $v_2/v_1$, using the given values to justify appropriate analytic approximations:

Express the result in the factored pole-zero form:

$$\frac{v_2}{v_1} = A = A_0 \frac{\Pi (1+s/\omega_1)}{\Pi (1+s/\omega_2)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.
\[ A = \frac{\frac{1}{sC_2} \left( R_2 + \frac{1}{sC_1} \right)}{R_2 + \frac{1}{sC_1} + \frac{1}{sC_2}} + \frac{R_L}{R_1 + R_L} \]

\[ \Rightarrow \quad \text{less algebra} \]

\[ = \frac{R_L}{R_1 + R_L} \cdot \frac{1 + sC_1 R_2}{1 + s[C_1 (R_2 + R_L) + C_2 (R_L R_L)] + 5[C_1 C_2 R_2 (R_L R_L)]} \]

*Use of numerical values to justify analytic approximation*
Find both analytically and numerically, the Q and hence the roots \( w_1 \) and \( w_3 \) of the quadratic:

\[
1 + C_1 [R_2 + R_{11} R_L] s + [C_1 C_2 R_2 (R_{11} R_L)] s^2
\]

where \( C_1 = 0.5 \mu F \), \( C_2 = 0.002 \mu F \), \( R_1 = 47 k \Omega \), \( R_2 = 1 k \Omega \), \( R_L = 100 k \Omega \).

Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.
Find both analytically and numerically, the \( Q \) and hence the roots \( \omega_1 \) and \( \omega_2 \) of the quadratic:

\[
1 + C_1 \left[ \frac{R_2 + R_{11} R_L}{a_1} \right] s + \left[ C_1 C_2 R_2 \left( \frac{R_{11} R_L}{a_2} \right) \right] s^2
\]

where \( C_1 = 0.1 \mu F \), \( C_2 = 0.002 \mu F \), \( R_1 = 47 \Omega \), \( R_2 = 1 \Omega \), \( R_L = 100 \Omega \).

Express the analytic results in terms of series/parallel element combinations, and express the numerical results in \( \text{Hz} \) or \( \text{kHz} \).

\[
Q^2 = \frac{a_2}{a_1} = \frac{C_1 C_2 R_2 (R_{11} R_L)}{C_1^2 (R_1 + R_{11} R_L)^2} = \frac{C_2}{C_1 R_1} \frac{R_2 R_{11} R_L}{R_1 + R_{11} R_L} \approx \frac{C_2}{C_1} \frac{R_2}{R_{11} R_L} = \frac{1}{50} \frac{1}{47 \Omega/100} = \frac{1}{1,000}
\]

Hence, \( F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \approx 1 \), so the roots are real: \( F = 0.9994 \), \( Q = \frac{1}{40} \ll 0.5 \)

\[
\omega_1 = \frac{1}{a_1} = \frac{1}{C_1 (R_2 + R_{11} R_L)} \quad \frac{f_1}{0.1(1 + 47 \Omega/100)} \text{Hz} = 48 \text{kHz}
\]

\[
\omega_2 = \frac{a_1}{a_2} = \frac{C_1 (R_1 + R_{11} R_L)}{C_1 C_2 R_2 (R_{11} R_L)} \quad \frac{f_2}{0.002 (\frac{100}{0.97})} \text{Hz} = 82 \text{kHz}
\]
Hence
\[ A \approx \frac{R_L}{R_L + R_1} \frac{1 + sC_1R_2}{\left[1 + C_1(R_2 + R_L R_1) s\right]\left[1 + C_2(R_1 R_2 + R_L) s\right]} \]
\[ = A_0 \frac{(1 + \frac{s}{\omega_1})}{(1 + \frac{s}{\omega_2})(1 + \frac{s}{\omega_3})} \]

where
\[ A_0 \equiv \frac{R_L}{R_L + R_1} \]
\[ \omega_1 \equiv \frac{1}{C_1(R_2 + R_L R_1)} \]
\[ \omega_2 \equiv \frac{1}{C_1 R_2} \]
\[ \omega_3 \equiv \frac{1}{C_2(R_1 R_2 + R_L)} \]
Hence

\[ A \propto \frac{R_L}{R_L + R_1} \frac{1 + 3C_1R_L}{[1 + C_1(R_2 + R_{11}R_L) + 3][1 + C_2(R_{11}R_{11}R_L) + 3]} \]

\[ = A_0 \frac{(1 + \frac{\omega_1}{\omega_0})}{(1 + \frac{\omega_1}{\omega_0})\left(1 + \frac{\omega_2}{\omega_0}\right)} \]

where

\[ A_0 = \frac{R_L}{R_L + R_1} = \frac{100}{100 + 47} = 0.68 \Rightarrow -3.4 \text{ dB} \]

\[ \omega_1 = \frac{1}{C_1(R_2 + R_{11}R_L)} \quad f_1 = \frac{159}{0.1(1 + \frac{471100}{32})} = 4.8 \text{ Hz} \]

\[ \omega_2 = \frac{1}{C_1R_L} \quad f_2 = \frac{159}{0.1 \times 1} = 1.6 \text{ kHz} \]

\[ \omega_3 = \frac{1}{C_2(R_{11}R_{11}R_L)} \quad f_3 = \frac{159}{0.002(47111100)} = 82 \text{ kHz} \]
The conventional quadratic formula for the two poles \( w_1 \) and \( w_3 \) is much higher entropy (gives much less useful information) than does the modified formula.

Conventional:

\[
\frac{1}{w_{1,3}} = \frac{C_1(R_x+R_{11}R_L) \pm \sqrt{C_1^2(R_x+R_{11}R_L)^2 - 4C_1C_2R_2(R_{11}R_L)}}{2}
\]

Modified:

\[
w_1 = \frac{1}{C_1(R_x+R_{11}R_L)} \quad \quad w_3 = \frac{1}{C_2(R_{11}R_{21}R_L)}
\]

Note, in particular, (when the two roots are real and well-separated) that the modified formula is much lower entropy and not only gives both roots with equal numerical accuracy, but also exposes the fact that \( C_1 \) affects only \( w_1 \), and \( C_2 \) affects only \( w_3 \) — which is useful information for design purposes.
A still better solution:
Apply the mental frequency sweep

Look at the original circuit and consider response as frequency increases:

At low frequencies, both capacitances are open, so have flat response. As frequency increases, the reactance of $C_1$, the larger capacitance, comes down causing a pole. When the reactance of $C_1$ drops below $R_2$, the response flattens causing a zero. However, at this frequency the reactance of $C_2$ is still 50 times higher than $R_2$, so $C_2$ has negligible effect.
4. Improved Quadratic Roots
At still higher frequencies, the reactance of $C_2$ drops below $R_2$, causing a second pole.

Hence, the solution can be obtained in two parts, each containing only one reactance (one pole).
4. Improved Quadratic Roots

Low-frequency range (C₂ open)

\[ A = \frac{R_L}{R_L + R_1} \left( \frac{R_L + \frac{1}{sC_1}}{R_L + \frac{1}{sC_1} + R_1R_L} \right) = A_0 \left( 1 + \frac{\frac{3}{4\pi R_L}}{1 + \frac{s}{\omega_1}} \right) = A_m \left( 1 + \frac{\omega_2}{\omega_1} \right) \]

where

\[ A_0 = \frac{R_L}{R_L + R_1} = \frac{100}{100 + 47} = 0.65 \Rightarrow -3.4 \text{ dB} \]

\[ \omega_1 = \frac{1}{C_1(R_L + R_1R_L)} \]

\[ f_1 = \frac{159}{0.1(1 + 4.7\times 100)} \approx 48 \text{ Hz} \]

\[ \omega_2 = \frac{1}{C_1R_2} \]

\[ f_2 = \frac{159}{0.1 \times 1} = 1.6 \text{ kHz} \]

\[ A_m = A_0 \frac{\omega_1}{\omega_2} = \frac{R_L}{R_L + R_1} \cdot \frac{R_2}{R_L + R_1R_L} = 0.65 \cdot \frac{0.045}{1.6} = 0.02 \Rightarrow -34 \text{ dB} \]
High-frequency range (C₁ short)

\[ A = A_m \left( \frac{1}{1 + \frac{\omega}{\omega_3}} \right) \]

where \( \omega_3 = \frac{1}{C_2 (R_{\text{L}} R_{\text{L}} R_{\text{L}})} \)

\[ f_3 = \frac{150}{0.002 (47 \text{kHz} 1100)} = 82 \text{kHz} \]

Hence, overall response is

\[ A = A_0 \left( \frac{1 + \frac{\omega}{\omega_3}}{(1 + \omega/\omega_1)(1 + \omega/\omega_3)} \right) = A_m \left( \frac{1 + \omega/\omega_3}{(1 + \omega/\omega_1)(1 + \omega/\omega_3)} \right) \]
4. Improved Quadratic Roots

\[ A_0 = \frac{R_L}{R_L + R_1} = -3.4 \text{ dB} \]
\[ \omega_1 = \frac{1}{C_1(R_L+R_1+11R_C)} \]
\[ f_1 = 48 \text{ Hz} \]

\[ \omega_2 = \frac{1}{C_1R_L} \]
\[ f_2 = 1.6 \text{ kHz} \]

\[ \omega_3 = \frac{1}{C_2(R_1+11R_L)} \]
\[ f_3 = 52 \text{ kHz} \]

\[ A_m = \frac{R_L}{R_L+R_1} \frac{R_2}{R_L+R_1+11R_C} = -34 \text{ dB} \]

\[ 45^\circ \log \left( \frac{a_1}{a_1+b_1} \right) = 68^\circ \]
\[ a_1 \]
\[ 10 b_1 \]

\[ 45^\circ \log \left( \frac{10 b_1}{a_1^2} \right) = 13^\circ \]

\[ 0^\circ \]
\[ 45^\circ \]
\[ 90^\circ \]
\[ = \frac{A_0 \left(1 + \frac{s}{\omega_b}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_3}\right)} \]

where \[ \frac{1}{\omega_{\text{eq}}} = \frac{c_1 (\omega_2 + R_1 R_4) \pm \sqrt{c_1^2 (\omega_2 + R_1 R_4)^2 - 4 c_1 c_2 R_2 (R_1 R_4)}}{2} \]

This is useless for design, and in any case is inaccurate numerically.
Generalization: Presentation of Results

Sketch magnitude and phase by straight-line asymptotes, and label salient features (flat gains, corner frequencies, Q's, etc.) with both analytic expressions and numerical values.

This is a compact summary so that both the analytic and numerical results can be interpreted at a glance, which is especially useful for reports, design reviews, etc. so that managers can easily and quickly see and understand the results obtained by others.

For design, the element values that must be changed to give different numerical results can easily be seen.
5. APPROXIMATIONS AND ASSUMPTIONS

How to build Low Entropy Expressions with minimum work
Double-pole low-pass LC filter

\[ \frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \]

in which

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad \text{(resonant frequency)} \]

\[ Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{R_0}{R} \quad \text{where } R_0 = \frac{L}{C} \quad \text{(characteristic resistance)} \]

\( Q < 0.5 \): roots \( \omega_1 \) and \( \omega_2 \) are real
\( Q > 0.5 \): roots \( \omega_1 \) and \( \omega_2 \) are complex
\[
\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q} \left( \frac{\omega}{\omega_0} \right) + \left( \frac{\omega}{\omega_0} \right)^2}
\]

\[
\left| \frac{v_2}{v_1} \right| \xrightarrow{\omega \to 0} 1 \Rightarrow 0 \text{ dB}
\]

\[
\omega \to \infty \Rightarrow \left( \frac{\omega_0}{\omega} \right)^2
\]

Asymptotes intersect at \( \omega_0 \)

Asymptotes are independent of \( Q \);

\( Q \) affects shape only in neighborhood of \( \omega_0 \)
\[
\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{\omega}{\omega_0}\right) + \left(\frac{\omega}{\omega_0}\right)^2}
\]

\[
\frac{|v_2|}{|v_1|} \xrightarrow{\omega \to 0} 1 \implies 0 \text{ dB}
\]

\[
\frac{v_2}{v_1} \xrightarrow{\omega \to \infty} \left(\frac{\omega}{\omega_0}\right)^2
\]

Asymptotes intersect at \( \omega_0 \)
Asymptotes are independent of \( Q \); \( Q \) affects shape only in neighborhood of \( \omega_0 \)

\{-40 \text{ dB/dec}
\}
\{-12 \text{ dB/oct}\}
\[
\frac{v_2}{v_1} = \frac{1}{1 + \frac{Q}{2\pi} \left(\frac{\omega}{\omega_0}\right) + \left(\frac{\omega}{\omega_0}\right)^2}
\]

\[
\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1 - \left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2} \left(\frac{\omega}{\omega_0}\right)^4}}
\]

At \( w = \omega_0 \):
\[
\left|\frac{v_2}{v_1}\right| = Q
\]
5. Approxs & Assumptions

\[ \frac{v_2}{v_1} = \frac{1}{1 + Q \left( \frac{\omega}{\omega_0} \right)^2 + \left( \frac{\omega}{\omega_0} \right)^2} \]

\[ | \frac{v_2}{v_1} | = \frac{1}{\sqrt{\left[ 1 - \left( \frac{\omega}{\omega_0} \right)^2 \right]^2 + \frac{1}{Q^2} \left( \frac{\omega}{\omega_0} \right)^2}} \]

At \( \omega = \omega_0 \):

\[ | \frac{v_2}{v_1} | = Q \]

\[ Q_{\text{db}} = 20 \log Q \]
5. Approxs & Assumptions

\[
\frac{v_2}{v_1} = \frac{1}{1 + Q \left( \frac{\omega}{\omega_0} \right)^2}
\]

\[
\left| \frac{v_2}{v_1} \right| = \frac{1}{\sqrt{\left[ 1 - \left( \frac{\omega}{\omega_0} \right)^2 \right]^2 + \frac{1}{Q^2} \left( \frac{\omega}{\omega_0} \right)^2}}
\]

At \( w = w_0 \):
\[\left| \frac{v_2}{v_1} \right| = Q\]

\[Q_{\text{dB}} = 20 \log Q\]

\[-40 \text{ dB/dec}\]
\[-12 \text{ dB/dec}\]
\[
\frac{V_2}{V_1} = \frac{1}{1 + \frac{\Omega}{Q_0} + \frac{\Omega^2}{Q_0^2}}
\]
\[
\left| \frac{V_2}{V_1} \right| = \frac{1}{\sqrt{\left[1 - \left(\frac{\Omega}{Q_0}\right)^2\right]^2 + \frac{\Omega^2}{Q_0^2}}} 
\]

At \( \omega = \omega_0 \):
\[
\left| \frac{V_2}{V_1} \right| = Q 
\]

which is not the maximum; the maximum moves to the left for lower \( Q \).
\[ \frac{v_2}{v_1} = \frac{1}{1 + \frac{Q}{Q_0} \left( \frac{v}{v_0} \right)^2} \]

\[ \frac{|v_2|}{v_1} = \frac{1}{\sqrt{\left[1 - \left( \frac{v}{v_0} \right)^2 \right]^2 + \frac{1}{Q^2} \left( \frac{v}{v_0} \right)^2}} \]

At \( \omega = \omega_0 \):

\[ \frac{|v_2|}{v_1} = Q \]

\[ -40 \text{dB/dec} \]

\[ -12 \text{dB/oct} \]
Phase shape:

\[ \phi_{\text{phase}} = -\tan^{-1}\left( \frac{\frac{1}{a}( \frac{w}{w_0} )}{1 - ( \frac{w}{w_0} )^2} \right) \]

\[ \frac{w \to \infty}{w \to 0} \quad 0^\circ \quad -90^\circ \quad -180^\circ \]

independent of Q
Phase shape:

$$\frac{\phi}{\pi} = -\tan^{-1}\left[\frac{\frac{1}{Q}(\frac{\omega}{\omega_0})}{1 - (\frac{\omega}{\omega_0})^2}\right]$$

$$\omega \to \infty \quad 0^\circ$$

$$\omega = \omega_0 \quad -90^\circ$$

$$\omega = \infty \quad -180^\circ$$

independent of Q
Increased Q causes sharper phase change between the $0^\circ$ and $-180^\circ$ asymptotes.

Need: a straight-line approximation.
Choose same slope at $w = w_0$:

$$\frac{\omega_i}{\omega_0} = (2^{1/4})^{1/2} = (4.81)^{1/4}$$
Better choice:

\[
\frac{\omega_i}{\omega_0} \approx 5^{-\frac{1}{2q}}
\]
An even better choice is

\[ \frac{\omega_i}{\omega_0} = 10^{2Q} \]

because for \( Q = 0.5 \) (two equal real roots)

\[ \frac{\omega_i}{\omega_0} = 10 \]

and the slope is \(-90^\circ/\text{dec}\), the same as twice the \(-45^\circ/\text{dec}\) slope for a single pole.
Second-order response:

\[ A = A_1 \frac{1}{1 + \frac{1}{Q} \left( \frac{x}{\omega_0} \right) + \left( \frac{x}{\omega_0} \right)^2} \]

\[ A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} = A_1 \frac{1}{1 + \frac{a_1}{a_2} \left( \frac{x}{\omega_0} \right)} \]

- \[ x = \frac{\omega_0}{\omega_0} \]
- \[ a_2 = 1 \]
- \[ a_1 = \frac{1}{Q} \]
- \[ a_2 = \frac{1}{\omega_0} \]
- \[ a_1 = \frac{1}{\omega_0 Q} \]
Second-order response:

\[ A = A_1 \frac{1}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \]

\[ A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} \]

\[ = A_1 \frac{1}{1 + \frac{a_1}{\omega_0^2} (\sqrt{a_2} x) + (\sqrt{a_2} x)^2} \]

\[ x = \frac{s}{\omega_0} \quad x = s \]

\[ a_2 = 1 \quad a_2 = \omega_0^2 \]

\[ a_1 = \frac{1}{Q} \quad a_1 = \frac{1}{\omega_0 Q} \]

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4a_2/a_1^2} \]

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \]

\[ A = A_1 \frac{1}{\left( 1 + a_1 F x \right) \left( 1 + \frac{a_1}{a_2} F x \right)} \]
Second-order response:

\[ A = A_i \frac{1}{1 + \frac{1}{Q} \left( \frac{S}{\omega_0} \right) + \left( \frac{S}{\omega_0} \right)^2} \]

\[ A = A_i \frac{1}{1 + a_1(x) + a_2(x)^2} \]

\[ A = A_i \frac{1}{1 + \frac{a_1}{a_2} (\frac{\omega_2 x}{\omega_0}) + \left( \frac{\omega_2 x}{\omega_0} \right)^2} \]

\[ x = \frac{S}{\omega_0} \]

\[ a_2 = 1 \]

\[ a_1 = \frac{1}{Q} \]

\[ a_1 = \frac{1}{\omega_0 Q} \]

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4a_2/a_1^2} \]

\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4Q^2} \]

\[ A = A_i \frac{1}{\left( 1 + a_1 F x \right) \left( 1 + \frac{a_1}{a_2} x \right)} \]
5. Approxs & Assumptions

For $\theta = 0.5$, $F = 0.5$

\[ A = A_0 \left( \frac{1 + \frac{a_1 F}{a_2}}{(1 + \frac{a_1 F}{a_2})^2} \right) \]

Roots:

\[ A = A_0 \left( \frac{1}{\left( \frac{a_1 F}{a_2} \right)^2} + \frac{\left( \frac{a_1 F}{a_2} \right)}{a_2} \right) \]

Second order response:

\[ A = A_0 \left( 1 + \frac{1}{\left( \frac{a_1 F}{a_2} \right)^2} \right) \]
Second-order response:
\[ A = A_1 \frac{1}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \]

For \( Q < 0.5 \):
\[ A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_0} \right)^2} \]

For \( Q \approx 1 \):
\[ A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_0} \right) \left(1 + \frac{s}{\omega_0} \right)} \]

For \( Q = 0.5 \):
\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4a_2/a_1^2} \]

For \( Q = 1 \):
\[ F = \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4a_2^2} \]
Low-pass 2-pole characteristic:

\[ \frac{V_2}{V_1} = \frac{1}{1 + \frac{1}{Qw_b} + \left(\frac{1}{w_b/Q}\right)^2} \approx \frac{1}{1 + \frac{1}{Qw_b} + \left(\frac{1}{w_b/Q}\right)^2} \]

\[ \approx \frac{\omega_b}{Q} \]
3. Approxs & Assumptions
Normal and inverted poles and zeros:

\[ A = A_m \left( 1 + \frac{w_1}{\frac{\omega_i}{2}} \right)^{-1} \]

\[ A = A_m \left( 1 + \frac{w_1}{\frac{\omega_i}{2}} \right)^{1} \]

\[ A = A_m \left( 1 + \frac{w_1}{\frac{\omega_i}{2}} \right)^{2} \]

\[ A = A_m \left( 1 + \frac{w_1}{\frac{\omega_i}{2}} \right)^{3} \]
Input and Output Impedances of low-pass filter

\[ \frac{Z_i}{Z_o} = \frac{\frac{1}{sC} + R + sL}{sC} \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ Q = \frac{R_0}{R} = \frac{1}{\omega_0 CR} = \frac{\omega_0 L}{R} \]

\[ R_0 = \sqrt{\frac{L}{C}} \]

Express in terms of \( \omega_0 \), \( Q \), \( R_0 \):

\[ Z_i = \frac{1}{\omega_0 C} \left( 1 + \frac{\omega_0 CR}{s} + \left( \frac{s}{\omega_0} \right)^2 \right) \]

\[ Z_o = \omega_0 L \left( \frac{s}{\omega_0} \right) \left( 1 + \frac{R}{s} \right) \]

\[ = R_0 \left( \frac{s}{\omega_0} \right) \left( 1 + \frac{\omega_0 R}{s} \right) \]

\[ = R_0 \frac{\left( \frac{s}{\omega_0} \right) \left( 1 + \frac{\omega_0 R}{s} \right) \left( \frac{s}{\omega_0} \right)^2}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) \left( \frac{s}{\omega_0} \right)^2} \]
Input and Output Impedances of low-pass filter

\[ V_1 \quad R = \frac{R_0}{Q} \quad sL = R_0 \frac{\omega_0}{Q} \quad V_2 \]

\[ \omega_0 = \frac{1}{\sqrt{LC}} \quad Q = \frac{R_0}{R} \]

\[ R_0 = \sqrt{\frac{L}{C}} \]

\[ Z_c = \frac{R_0}{Q} + R_0 \frac{s}{\omega_0} + R_0 \frac{\omega_0}{3} \]

\[ Z_o = \frac{(\frac{R_0}{Q} + R_0 \frac{s}{\omega_0}) R_0 \frac{\omega_0}{3}}{\frac{R_0}{Q} + R_0 \frac{s}{\omega_0} + R_0 \frac{\omega_0}{3}} \]

\[ = R_0 \left( 1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right) \]

\[ = R_0 \left( \frac{\omega_o}{\omega_0} \right) \left( 1 + \frac{\omega_0/Q}{1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \right) \]

Note how the algebra is shortened when the analysis starts with the normalized element values.
5. Approxs & Assumptions

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.
Input and Output Impedances of low-pass filter

\[ Z_c = \frac{R_0}{Q} + R_0 \frac{\xi}{\omega_0} + R_0 \frac{\omega_0}{3} \]

\[ Z_0 = \frac{\left( \frac{R_0}{Q} + R_0 \frac{\xi}{\omega_0} \right) R_0 \frac{\omega_0}{3}}{\frac{R_0}{Q} + R_0 \frac{\xi}{\omega_0} + R_0 \frac{\omega_0}{3}} \]

\[ = R_0 \frac{1 + \frac{1}{Q} \left( \frac{\xi}{\omega_0} \right) + \left( \frac{\xi}{\omega_0} \right)^2}{1 + \frac{1}{Q} \left( \frac{\xi}{\omega_0} \right) + \left( \frac{\xi}{\omega_0} \right)^2} \]

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.
\[ Z_i = R_0 \times \left[ 1 + \frac{1}{Q(\frac{s}{\omega_0})} + \left(\frac{s}{\omega_0}\right)^2 \right] \times \left[ \frac{1}{\frac{s}{\omega_0}} \right] \]

ref. value
\[ \frac{1}{H} \]
single slope

\[ R_0 \]
5. Approxs & Assumptions

\[ |H| = \left| \frac{V_2}{V_1} \right| \]
\[ Z_i = R_0 \times \left[ 1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right] \times \left[ \frac{1}{\frac{s}{\omega_0}} \right] \]
\[ Z_i = R_0 \times \left[ 1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right] \times \left[ \frac{1}{\frac{s}{\omega_0}} \right] \]
\[ Z_i = R_0 \times \left[ 1 + \frac{1}{Q} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right] \times \left[ \frac{1}{\frac{s}{\omega_0}} \right] \]
Asymptote sketches for high Q (>0.5)

\[ |z_i| = R_0 \]
\[ \frac{R_0}{Q} = R \]

\[ \angle z_i = +180^\circ \quad \angle +90^\circ \quad \angle 0^\circ \quad \angle -90^\circ \]

\[ \omega_0, \quad \omega_0 \leq 120^\circ \]
Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.
\[ Z_0 = R_0 \times \left[ \frac{\frac{3}{\bar{w}_0}}{1 + \frac{1}{\bar{Q}} \left( \frac{\bar{f}}{\bar{w}_0} \right)^2 + \left( \frac{\bar{f}}{\bar{w}_0} \right)^2} \right] \times \left( 1 + \frac{\bar{w}_0/\bar{Q}}{\bar{f}} \right) \]

**Ref. Value**

\[ \frac{1}{Z_i} \]

**Inverted Zero**

\[ R_0 \]
5. Approxs & Assumptions
\[ Z_0 = R_0 \times \left[ \frac{\omega_0^2}{1 + \frac{1}{\alpha^2} \left( \frac{\omega}{\omega_0} \right) + \left( \frac{\omega}{\omega_0} \right)^2} \right] \times \left( 1 + \frac{\omega_0 / \alpha}{s} \right) \]
\[ \bar{Z}_o = R_o \times \left[ \frac{\frac{1}{\bar{Z}_i}}{1 + \frac{1}{\bar{Q}}\left(\frac{s}{\bar{w}_0}\right) + \left(\frac{s}{\bar{w}_0}\right)^2} \right] \times \left(1 + \frac{\bar{w}_0/\bar{Q}}{s} \right) \]
\[
Z_0 = R_0 \times \left[ \frac{2\omega_0}{1 + \frac{1}{\alpha}(\frac{\xi}{\omega_0}) + (\frac{\xi}{\omega_0})^2} \right] \times \left(1 + \frac{\omega_0/\Omega}{s}\right)
\]
5. Approxs & Assumptions
Exercise 5.1
Sketch asymptotes for $Z_i$ and $Z_o$ for low $Q$.

Exercise
For the two-pole low-pass LC filter, sketch the magnitude and phase asymptotes of $Z_i$ and $Z_o$ for low $Q$ ($<0.5$).
(But take $Q > 0.1$)
Exercise 5.1 - Solution

Asymptotes for $Z_i$ for low $Q$ ($\ll 0.5$):

\[
\frac{r_i}{Q} = R
\]

$Q, \omega_0$

$+180^\circ$

$+90^\circ$

$0^\circ$

$-90^\circ$
Exercise 5.1 - Solution

Asymptotes for \( Z_i \) for low \( Q \) (\( < 0.5 \)):
Exercise 5.1 - Solution

Asymptotes for $E_0$ for low Q ($<0.5$):
Exercise 5.1 - Solution

Asymptotes for $Z_0$ for low $Q$ ($\ll 0.5$):

\[ \frac{R_0}{Q} = R \]

\[ |Z_0| \]

\[ Q_{w_0} \quad w_0 \quad \bar{w_0} \]

\[ +90^\circ \quad 0^\circ \quad -90^\circ \]
Since there are now two resistances, re-name
\[ R \rightarrow R_e, \quad Q \rightarrow Q_e \]

By analogy, define \( Q_L = \frac{R_L}{R_o} \)
Why is $Q_L$ defined "upside down" relative to $Q_e$?
5. Approxs & Assumptions

\[ v_i \] \quad \Rightarrow \quad \text{Loaded low-pass LC filter}

\[ \begin{align*}
\text{Note: } R &\Rightarrow R_e, \quad Q \Rightarrow Q_e \\
\omega_0 &= \frac{1}{\sqrt{LC}} \quad R_0 = \sqrt{\frac{L}{C}} \\
Q_e &= \frac{R_e}{R} \quad \frac{Q_L}{Q_e} = \frac{R_L}{R_0}
\end{align*} \]

\[ H = \frac{R_0 Q_L}{1 + Q_L \frac{1}{\omega_0}} + \frac{R_0 \frac{1}{Q_e} + R_0 \frac{1}{\omega_0}}{1 + Q_L \frac{1}{\omega_0}} + \frac{Q_L}{1 + \frac{1}{Q_e} + \left( \frac{Q_L}{Q_e} + 1 \right) \frac{1}{\omega_0} + Q_L \left( \frac{1}{\omega_0} \right)^2}
\]

\[ = \frac{1}{1 + \frac{1}{Q_L Q_0}} \quad \frac{1}{1 + \frac{\left( \frac{1}{Q_e} + \frac{1}{Q_L} \right)}{1 + 1/Q_L Q_0} \left( \frac{1}{\omega_0} \right) + \frac{1}{1 + 1/Q_L Q_0} \left( \frac{1}{\omega_0} \right)^2} \]
Conventional result:

\[
H = \frac{R_L}{1 + sC R_L} + R_e + sL = \frac{R_L}{R_L + R_e + s(C R_L R_e + L) + s^2 L C R_L}
\]

Reveals no insight
Conventional result:

This high entropy result can be converted into the desired low entropy version by application of mental energy, but it takes quite an effort, and you have to know where you’re going!
Conventional result:

\[
H = \frac{R_L}{R_L + R_e} \frac{1}{1 + s \left( \frac{R_e}{R_L} \right) \frac{R_L}{R_L + R_e} + s \omega_c \frac{R_L}{R_L + R_e}}
\]

\[
= \frac{1}{1 + \frac{1}{Q_e} \frac{R_L}{R_L + R_e} \left( \frac{\omega}{\omega_0} \right)^2 + 1 + \frac{1}{Q_e} \frac{R_L}{R_L + R_e} \left( \frac{\omega}{\omega_0} \right)^2}
\]
\[ H = \frac{1}{1+1/Q_e Q_L} \frac{1}{1+1/Q_e Q_L} \left( \frac{s}{\omega_0 \sqrt{1+1/Q_e Q_L}} \right) + \left( \frac{s}{\omega_0 \sqrt{1+1/Q_e Q_L}} \right)^2 \]

Result, compared with unloaded case:

1. Low-freq. asymptote is \( \frac{1}{1+1/Q_e Q_L} = \frac{R_L}{R_L + R_e} \) (resistive divider)

2. The corner frequency is changed to \( \frac{1}{\sqrt{1+1/Q_e Q_L}} \omega_0 \)

3. The damping coefficient is changed to \( \frac{1}{Q_e + \frac{1}{Q_e}} \frac{1}{\sqrt{1+1/Q_e Q_L}} \)

This is a good example of how a low entropy format can allow one equation to disclose more than one useful piece of information.
\[ H = \frac{1}{1 + \frac{1}{Q_e Q_L}} \left( \frac{1}{Q_e} + \frac{1}{Q_L} \right) \left( \frac{s}{\omega_0 \sqrt{1+1/Q_e Q_L}} \right) + \left( \frac{s}{\omega_0 \sqrt{1+1/Q_e Q_L}} \right)^2 \]

Second order

1. Low-freq. asymptote is \( \frac{1}{1 + 1/Q_e Q_L} = \frac{R_L}{R_L + R_e} \) (resistive divider)

Second order

2. The corner frequency is changed to \( \frac{1}{\sqrt{1+1/Q_e Q_L}} \omega_0 \)

First order

3. The damping coefficient is changed to
\[
\sqrt{\frac{1}{Q_e} + \frac{1}{Q_L}} \sqrt{1+1/Q_e Q_L}
\]

For the high-Q case, \( Q_e Q_L \gg 0.5 \), \( Q_e Q_L \gg 1 \) and the first two effects are negligible, and the damping coefficient becomes
\[
\frac{1}{Q_e} + \frac{1}{Q_L}
\]

Result, compared with unloaded case:

5. Approxs & Assumptions
Hence, for the high-$Q$ case,

$$H \approx \frac{1}{1 + \frac{1}{Q_t} \left( \frac{\xi}{\omega_0} \right) + \left( \frac{\xi}{\omega_0} \right)^2}$$

where $Q_t$ is a "total" $Q$-factor given by the "parallel combination"

$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L}$$
Loaded low-pass LC filter

\[ sL = R_e \frac{Q_e}{\omega_0} \]

\[ v_2 = H v_1 \]

\[ R_e = R_0/Q_e \]

\[ Z_i = \frac{R_0 Q_e}{1 + Q_e \left( \frac{s}{\omega_0} \right)} + \frac{R_0}{Q_e} + R_0 \frac{s}{\omega_0} = R_0 \frac{Q_e + \frac{1}{Q_e} + (\frac{Q_e+1}{Q_e})(\frac{s}{\omega_0}) + Q_e \left( \frac{s}{\omega_0} \right)^2}{1 + Q_e \left( \frac{s}{\omega_0} \right)} \]

\[ = R_0 \left( 1 + 1/Q_e \right) \frac{1 + \frac{1}{Q_e} + \frac{s}{\omega_0} \left( \frac{s}{\omega_0} \right)}{(\frac{s}{\omega_0}) \left( 1 + \frac{\omega_0 Q_e}{s} \right)} \]

Note: \( R \to R_0, \ Q \to Q_e \)

\[ \omega_0 = \sqrt{L/C} \quad R_0 = \sqrt{\frac{L}{C}} \]

\[ Q_e = \frac{R_0}{R_e} \quad Q_L = \frac{R_L}{R_0} \]

Same three effects as for \( H \), but with addition of an inverted pole at \( \omega_0/Q_e \).
5. Approxs & Assumptions

Loaded low-pass LC filter

\[ \text{Note: } R \rightarrow R_e, \quad Q \rightarrow Q_e \]

\[ R_e = \frac{R_0}{Q_e} \]

\[ \frac{1}{j\omega C} = R_e \frac{Q_e}{2} \]

\[ sL = R_0 \frac{s}{\omega_0} \]

\[ u_2 = H v_1 \]

\[ \omega_0 = \sqrt{\frac{L}{C}} \]

\[ R_0 = \sqrt{\frac{L}{C}} \]

\[ Q_e = \frac{R_e}{R_e} \]

\[ Q_L = \frac{R_L}{R_0} \]

Hence, for the high-\( Q \) case,

\[ Z_i \approx R_0 \left(1 + \frac{1}{Q_e} \left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2 \right) \]

\[ \frac{1}{(\frac{s}{\omega_0})(1 + \frac{\omega_0 Q_e}{s})} \]
For high-$Q$ case:

\[ \frac{Q_e R_0}{Q_l} = R_L \]

\[ \frac{1}{Q_t} \frac{1}{\sqrt{1 + 1/Q_e Q_l^2}} \]

\[ \frac{R_0}{Q_e} = R_e \]

\[ \omega_0 \]

\[ +90^\circ \]

\[ 0^\circ \]

\[ -90^\circ \]

\[ Z_i \]

\[ 5 \sqrt{2} Q_e \omega_0 \]
Compare the $Z_i$ asymptotes with and without $R_L$:

**Without $R_L$:**

$(Q_L = \infty)$

**With $R_L$:**

$(Q_L \neq \infty)$
The appearance of the new corner frequency $\omega_0/Q_L$ can be confirmed by a mental frequency sweep:

Without $R_L$, $Z_i \to \infty$ as $\omega \to 0$ because of the capacitive reactance.

With $R_L$, $Z_i$ flattens, so a concave downwards corner is introduced, which is an inverted pole.
5. Approxs & Assumptions

 Loaded low-pass LC filter

\[ v_1 \quad R_e = \frac{R_0}{Q_e} \quad \frac{1}{\omega C} = R_s \frac{s}{\omega} \quad u_2 = H \cdot u_1 \]

\[ sL = R_s \frac{s}{\omega} \quad u_2 = H \cdot u_1 \]

\[ \omega_0 = \frac{1}{\sqrt{L}} \quad R_0 = \sqrt{\frac{L}{C}} \]

\[ Q_e = \frac{R_e}{R_0} \quad Q_L = \frac{R_L}{R_0} \]

\[ Z_0 = \frac{Q_e \cdot R_0}{1 + Q_e \cdot (\frac{s}{\omega_0})} \cdot \left( \frac{R_0}{Q_e} + R_e \frac{s}{\omega_0} \right) = R_0 Q_L \cdot \frac{\frac{1}{Q_e} + \frac{s}{\omega_0}}{Q_L + \frac{1}{Q_e} + \left( \frac{Q_e}{Q_L} + 1 \right) (\frac{s}{\omega_0}) + (\frac{s}{\omega_0})^2} \]

\[ = R_0 \cdot \frac{1}{1 + 1/QQ_L} \cdot \frac{\left( \frac{s}{\omega_0} \right) \left( 1 + \frac{\omega_0/Q_e}{s} \right)}{1 + \left( \frac{Q_L + 1}{Q_L} \right) \left( \frac{s}{\omega_0} \right) + \frac{1}{1 + 1/QQ_L} \left( \frac{s}{\omega_0} \right)^2} \]

Same three effects as for H, so for high-Q case

\[ Z_0 \approx R_0 \cdot \frac{\left( \frac{s}{\omega_0} \right) \left( 1 + \frac{\omega_0/Q_e}{s} \right)}{1 + \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2} \]
For high-Q case:

\[ \frac{R_0}{Q_e} \frac{Q_e}{Q_e} \sqrt{1 + 1/Q_e^2} = R_0 \frac{Q_e}{Q_e} \frac{1}{Q_e^2} \]

\[ R_0 \frac{Q_e}{Q_e} \sqrt{1 + 1/Q_e^2} = R_0 \]

\[ \theta = 90^\circ \]

\[ \theta = 0^\circ \]

\[ \theta = -90^\circ \]

\[ t = \frac{1}{2} \alpha \varepsilon_{\omega_0} \]
When an LC filter is loaded, a 4th effect needs to be accounted for:

- Second order
  1. Low-freq. asymptote is \( \frac{1}{1+1/QQL} = \frac{R_L}{R_L + R_2} \) (resistive divider)

- Second order
  2. The corner frequency is changed to \( \frac{1}{\sqrt{1+1/QQL} \cdot \omega_0} \)

- First order
  3. The damping coefficient is changed to \( \frac{1/Qe + 1/QL}{\sqrt{1+1/QQL}} \)

- First order
  4. New corner frequencies may appear in some transfer functions
A third damping resistance $R_c$ may be present, representing the capacitor esr:

\[
\begin{align*}
\text{By analogy with } Q_e, \text{ define } Q_c & \equiv \frac{R_0}{R_c} \\
\text{The analysis for } H, Z_i, \text{ and } Z_o \text{ could be re-done in the same way.} \\
\text{Instead, let's } \textit{build} \text{ the result by applying what we already know about the two simpler cases.} \\
The price we are willing to pay, in order to leap-frog directly to the result, is that the second-order effects will be omitted.\end{align*}
\]
A third damping resistance $R_c$ may be present, representing the capacitor esr:

One first-order effect of adding a second damping resistance was to lower the total $Q_t$ to the parallel combination

\[
\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L}
\]
A third damping resistance $R_c$ may be present, representing the capacitor esr:

\[ \frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c} \]

A good guess would be that adding a third damping resistance would lower the total $Q_t$ to the triple parallel combination.
Another possible first-order effect of adding a third damping resistance is the appearance of additional corner frequencies.

A mental frequency sweep can be used to verify an analytical result, but it can also be used "in reverse" to expose new corner frequencies.

The strategy is to determine whether or not the addition of the third damping resistance changes the asymptote slope as frequency approaches either zero or infinity.
For the voltage transfer function $H$:

$\omega \to 0$: no change of slope, so no new inverted pole or zero;

$\omega \to \infty$: a concave upwards corner appears, so there is a new normal zero.

Further, the value of the corner is where $1/\omega C = R_c$, which is $1/RC = Q_c \omega_0$. 
Assembled results:

Consider additional damping: Capacitor esr $R_c$

$$
\begin{align*}
\frac{sL}{R_0} &= \frac{R_0}{Q_c} \\
\frac{1}{sC} &= \frac{s}{R_0} \frac{\omega_0}{s}
\end{align*}
$$

$$
\begin{align*}
\frac{R_e}{R_0} &= \frac{R_0}{Q_c} \\
\frac{R_c}{R_0} &= \frac{R_0}{Q_c}
\end{align*}
$$

For the high-\(Q\) case, the previous results can be extended by inspection:

$$
H = \frac{1 + \frac{1}{Q_c} \left( \frac{s}{\omega_0} \right)}{1 + \frac{1}{Q_e} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2}
$$

$$
\frac{1}{Q_T} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}
$$

triple parallel combination:
Assembled results:

Consider additional damping: Capacitor esr $R_c$

\[ sL = \frac{R_o}{2} \]
\[ v_i = u_i \]
\[ \omega_0 = \sqrt{\frac{L}{C}} \]
\[ R_0 = \frac{1}{\sqrt{C}} \]
\[ R_e = \frac{R_0}{Q_e} \]
\[ R_c = \frac{R_0}{Q_c} \]

\[ sL = \frac{R_o}{2} \]
\[ v_i = u_i \]
\[ \omega_0 = \sqrt{\frac{L}{C}} \]
\[ R_0 = \frac{1}{\sqrt{C}} \]
\[ R_e = \frac{R_0}{Q_e} \]
\[ Q_e = \frac{R_0}{\omega_0} \]
\[ Q_c = \frac{R_0}{\omega_0} \]
\[ Q_c = \frac{R_0}{\omega_0} \]

For the high-$Q$ case, the previous results can be extended by inspection:

\[ H = \frac{1 + \frac{1}{Q_c}\left(\frac{s}{\omega_0}\right)}{1 + \frac{1}{Q_e}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \]
\[ \frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_c} \]

A similar process leads to the assembled result for $Z_i$:

\[ Z_i = R_o \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0}{Q_t}\right)} \]

(No new corners)
5. Approxs & Assumptions

\[
\frac{R_0}{Q_L \sqrt{1 + 1/Q_v^2}} \quad \text{and} \quad Q_L R_0 = R_v
\]
Principle for extension of results to a more complicated case:

1. Determine the new total Q(e).

2. Add any additional pole or zero factors (Is there any change in the \(w \to 0\) or \(w \to \infty\) asymptotes?)

Exercise:
Obtain the corresponding results for \(Z_o\).
Exercise solution:

\[ Z_0 = R_0 \frac{\left( \frac{q}{\omega_0} \right) \left( 1 + \frac{\omega_0}{Q_e} \right) \left( 1 + \frac{1}{Q_t} \frac{q}{\omega_0} \right)}{1 + \frac{1}{Q_t} \left( \frac{q}{\omega_0} \right) + \left( \frac{q}{\omega_0} \right)^2} \]

Check high-frequency limit:

\[ Z_0 \xrightarrow{\omega \to \infty} \frac{R_0}{Q_e} = R_c \]
The key step is now to determine $T_{12}$ and $T_{22}$ from the small signal model for the condition $\dot{v} = 0$:

$$
T_{12} = \left[ \frac{VO \times (RL - D^2 \times N^2 \times Z) \times (1 + sC \times RC)}{(D \times N \times \Delta)} \right]
$$

$$
T_{22} = \left[ \frac{VO \times (RL - D^2 \times N^2 \times Z) \times (1 + sC \times RL)}{(D \times N \times RL \times \Delta)} \right]
$$

where

$$
Z = \left[ \frac{(sL_1 + R_{L1}) (s \times C_1 R_{C1} + 1)}{(s^2 L_1 \times C_1 \times RL1 + sC_1 \times RL1 \times RL1) + sL_1 + RL1} \right]
$$

$$
\Delta = \left[ a_1 \times (D^2 \times N^2 \times Z) \times (1 + sC \times RL) \right]
$$

$$
a_1 = \left[ (s^2 L \times C \times RL) + sC \times RL \times (RL \times RC + \frac{L}{(C \times RL)}) \times RL \right]
$$

At the resonant frequency of the input filter, the impedance $Z$ will attain a very high value, limited only by the series resistances $RL1$ and $RC1$. The peaking in the value of $Z$ will affect both the numerators and denominators of the transfer functions $T_{12}$ and $T_{22}$, as shown in equations 15 and 16. The net effect will be a reduction in the loop gain $G_l$ and a corresponding phase margin reduction.
6. PRODUCTS AND SUMS OF FACTORED POLE-ZERO EXPRESSIONS

Doing the Algebra on the Graph
Functions expressed in factored pole-zero form often need to be combined, either by multiplication or addition.

Multiplication is straightforward:

\[ A_1 = A_{1\text{ref}} \left( \frac{1+s/\omega_{z11}}{1+s/\omega_{p11}} \right) \left( \frac{1+s/\omega_{z12}}{1+s/\omega_{p12}} \right) \ldots \]

\[ A_2 = A_{2\text{ref}} \left( \frac{1+s/\omega_{z21}}{1+s/\omega_{p21}} \right) \left( \frac{1+s/\omega_{z22}}{1+s/\omega_{p22}} \right) \ldots \]

\[ A = A_{1\text{ref}} A_{2\text{ref}} \left( \frac{1+s/\omega_{z11}}{1+s/\omega_{p11}} \right) \left( \frac{1+s/\omega_{z12}}{1+s/\omega_{p12}} \right) \ldots \left( \frac{1+s/\omega_{z11}}{1+s/\omega_{p11}} \right) \left( \frac{1+s/\omega_{z21}}{1+s/\omega_{p21}} \right) \left( \frac{1+s/\omega_{z22}}{1+s/\omega_{p22}} \right) \ldots \]

The product contains the poles and zeros of both functions.
Double-pole low-pass RC filters

\[ \frac{v_3}{v_1} = \frac{1}{(1 + \frac{\omega}{\omega_1})(1 + \frac{\omega}{\omega_2})} \quad \text{where} \quad \omega_1 = \frac{1}{C_1 R_1}, \quad \omega_2 = \frac{1}{C_2 R_2} \]

\[ \left| \frac{v_3}{v_1} \right|_{\text{dB}} = -20 \log \left[ 1 + \left( \frac{\omega}{\omega_1} \right)^2 \right] - 20 \log \left[ 1 + \left( \frac{\omega}{\omega_2} \right)^2 \right] \]

\[ \begin{align*}
\frac{v_3}{v_1} &= -\tan^{-1} \left( \frac{\omega}{\omega_1} \right) - \tan^{-1} \left( \frac{\omega}{\omega_2} \right)
\end{align*} \]

superposition
6. Products & Sums
Addition is more complicated:

\[ A = A_1 + A_2 \]

\[ A = A_{1ref} \left( 1 + \frac{s}{\omega_{z11}} \right) \left( 1 + \frac{s}{\omega_{z12}} \right) \ldots + A_{2ref} \left( 1 + \frac{s}{\omega_{z21}} \right) \left( 1 + \frac{s}{\omega_{z22}} \right) \ldots \]

\[ A = \frac{A_{1ref} \left( 1 + \frac{s}{\omega_{z11}} \right) \left( 1 + \frac{s}{\omega_{z12}} \right) \ldots \left( 1 + \frac{s}{\omega_{p21}} \right) \left( 1 + \frac{s}{\omega_{p22}} \right) \ldots + A_{2ref} \left( 1 + \frac{s}{\omega_{z21}} \right) \left( 1 + \frac{s}{\omega_{z22}} \right) \ldots \left( 1 + \frac{s}{\omega_{p11}} \right) \left( 1 + \frac{s}{\omega_{p22}} \right) \ldots}{\left( 1 + \frac{s}{\omega_{p11}} \right) \left( 1 + \frac{s}{\omega_{p22}} \right) \ldots \left( 1 + \frac{s}{\omega_{p21}} \right) \left( 1 + \frac{s}{\omega_{p22}} \right) \ldots} \]
Addition is more complicated:

\[
A = A_1 + A_2
\]

The sum contains the poles of both functions, but the numerator consists of the sums of cross-products of poles and zeros, and is a new polynomial that has to be renormalized and refactored.
This can be very tedious, and requires approximations if the numerator is higher than a quadratic in $s$.

"Doing the algebra on the graph" makes suitable approximations obvious.
\[ A_1 = 1 \]

\[ A_2 = \frac{\omega_0}{s} \]
A guess is that the sum follows the larger:

\[ A_1 = 1 \]

\[ A_2 = \frac{\omega_0}{s} \]
A guess is that the sum follows the larger:

\[ A_1 = 1 \]

\[ A_2 = \frac{\omega_0}{s} \]
A guess is that the sum follows the larger:

\[ A = A_1 + A_2 = 1 + \frac{\omega_0}{s} \]

This is confirmed algebraically:

\[ A = A_1 + A_2 = 1 + \frac{\omega_0}{s} \]
If the sum follows the larger, the result is:

\[ \omega_0 = A_{20} \omega_1 \]

\[ A_2 = A_{20} \frac{1}{1 + \frac{s}{\omega_1}} \]

\[ A_1 = 1 \]

\[ A_{20} \quad \omega_1 \]

\[ 0 \text{db} \]
If the sum follows the larger, the result is:

\[ A = A_{20} \left( 1 + \frac{s}{A_{20} \omega_1} \right) \]

However, the algebra shows that this is an approximation:

\[ A = 1 + A_{20} \left( 1 + \frac{s}{\omega_1} \right) = 1 + A_{20} + \frac{s}{\omega_1} \]

\[ = (1 + A_{20}) \frac{1 + \frac{s}{(1+A_{20})\omega_1}}{1 + \frac{s}{\omega_1}} \]
This is the exact answer.
This is the exact answer.

It's your decision as to whether the approximate answer is good enough.
Exercise 6.1

Guess an exact sum

Guess the exact sum $A = A_1 + A_2$ on the graph, then find it algebraically.

$A_1 = 1$

$A_2$

$\omega_0$

$\omega_2$
Exercise 6.1 - Solution

Guess the exact sum \( A = A_1 + A_2 \) on the graph, then find it algebraically.

\[
1 + \frac{\omega_0}{\omega_2} = \frac{\omega_0}{\omega_2}
\]

\( A_1 = 1 \)
Exercise 6.1 - Solution

\[ A_1 = 1 \]

\[ 0 \text{db} \]

\[ A = 1 + \frac{\omega_0}{s} \left( 1 + \frac{s}{\omega_2} \right) = 1 + \frac{\omega_0}{\omega_2} + \frac{\omega_0}{s} \]

\[ = \left( 1 + \frac{\omega_0}{\omega_2} \right) \left( 1 + \frac{\omega_0 \| \omega_2}{s} \right) \]
Guidelines:
In ranges where both functions have the same slope, the combination has the same slope and is the sum of the separate values.

The poles of the sum are the poles of the two functions.

The "gain-bandwidth tradeoff" relates the corner frequencies to the flat values.
Exercise 6.2
Find an exact sum graphically
Use the Guidelines to construct the exact sum of the two functions, without doing any algebra:
Exercise 6.2 - Solution

\[ A_{20} = \frac{\omega_0}{\omega_1} \]

\[ A_1 = 1 \]
Exercise 6.2 - Solution

\[ A_{20} = \frac{\omega_0}{\omega_1} \]

\[ A_1 = 1 \]

\[ 1 + \frac{\omega_0}{\omega_1} \]

\[ \omega_0 + \omega_1 \]

\[ \omega_2 \]

\[ \frac{\omega_0}{\omega_2} \]
Exercise 6.2 - Solution

\[ A_{20} = \frac{\omega_0}{\omega_1} \]

\[ A_1 = 1 \]

\[ 1 + \frac{\omega_0}{\omega_1} \]

\[ \omega_0 + \omega_1 \]

\[ A_2 \]

\[ \frac{\omega_0}{\omega_2} \]

\[ \frac{\omega_0}{\omega_2} + 1 \]
Exercise 6.2 - Solution

\[ A_{20} = \frac{\omega_0}{\omega_1} \]

\[ A_1 = 1 \]

\[ \omega_0 + \omega_1 = \omega_0 \frac{\omega_0}{\omega_2} \]

\[ \omega_0 = \frac{1 + \frac{\omega_0}{\omega_2}}{1 + \frac{\omega_1}{\omega_2}} \]

\[ \omega_1 = \frac{1 + \frac{\omega_0}{\omega_1}}{1 + \frac{\omega_0}{\omega_2}} \]

\[ 1 + \frac{\omega_0}{\omega_2} = \frac{1 + \omega_0}{\omega_0} \]

\[ \omega_0 / \omega_2 \]
Difference of two functions:

\[ A = A_1 - A_2 \]

\[ \omega_0 = \frac{A_{10}}{A_{20}} \omega_1 \]

\[ A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}} \]
Difference of two functions:

\[ A = A_1 - A_2 \]

\[ \omega_0 = \frac{A_{10}}{A_{20}} \omega_1 \]

\[ A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}} \]
Difference of two functions:

\[- A_2 + A_1 = A = A_1 - A_2\]

\(A_0 = A_{10} - A_{20}\)

\(\omega = \frac{A_{10}}{A_{20}}\)

\(\omega_0 = \frac{1}{1 + \frac{s}{\omega_1}}\)

\(A_1 = A_{10} \cdot \frac{1}{1 + \frac{s}{\omega_1}}\)
Difference of two functions:

\[ A = A_1 - A_2 \]

\[ \omega_0 = \frac{A_{10}}{A_{20}} \omega_1 \]

\[ \omega_2 = \frac{A_{10} - A_{20}}{A_{20}} \omega_1 \]

\[ A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}} \]

\( A_0 = A_{10} - A_{20} \)
Difference of two functions:

\[ A = A_1 - A_2 \]

\[ A_0 = A_{10} - A_{20} \]

\[ \omega_1 \]

\[ \omega_2 = \frac{A_{10} - A_{20}}{A_{20}} \omega_1 \]

\[ \omega_0 = \frac{A_{10}}{A_{20}} \omega_1 \]

\[ A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}} \]
Difference of two functions:

\[ A = A_1 - A_2 \]

The result is

\[ A = A_0 \frac{1 - \frac{s}{\omega_2}}{1 + \frac{s}{\omega_1}} \]

The corner \( \omega_2 \) is a right half plane (rhp) zero: it has a concave upward magnitude response, but a phase lag, not a phase lead.
Difference of two functions:

\[ A = A_1 - A_2 \]

\[ A = A_0 \frac{1 - \frac{s}{\omega_2}}{1 + \frac{s}{\omega_1}} \]

\[ A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}} \]

\[ \omega_2 = \frac{A_{10} - A_{20}}{A_{20}} \omega_1 \]

\[ \omega_0 = \frac{A_{10}}{A_{20}} \omega_1 \]

\[ A_0 = A_{10} - A_{20} \]

\[ \omega_1 \]

\[ \omega_2 \]

\[ \omega_0 \]

\[ 0^\circ \]

\[ -45^\circ / \text{dec} \]

\[ -90^\circ \]
Difference of two functions:

\[ A = A_1 - A_2 \]

A rhp zero occurs when a signal can go from input to output by two paths, one inverting and one not, with one path dominating at low frequencies, and the other dominating at high frequencies.

Every common-emitter or common-source amplifier stage potentially exhibits a rhp zero.
Consider sums of functions that result in quadratics in $s$

\[ Z_1 = R_0 \frac{s}{\omega_0} \]
\[ Z_2 = R_0 \frac{\omega_0}{s} \]
Consider sums of functions that result in quadratics in $s$

$$Z_1 = R_0 \frac{s}{\omega_0} \quad Z_2 = R_0 \frac{\omega_0}{s}$$

$$Z = Z_1 + Z_2 = R_0 \left( \frac{s}{\omega_0} + \frac{\omega_0}{s} \right) = R_0 \left[ 1 + \left( \frac{s}{\omega_0} \right)^2 \right]$$

The numerator is a quadratic pair of zeros with infinite $Q$
Consider sums of functions that result in quadratics in $s$

In any realistic case, there will be at least one additional corner:

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o / Q}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}$$
Consider sums of functions that result in quadratics in $s$

In any realistic case, there will be at least one additional corner:

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o / Q}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}$$

$$Z = R_o \left[1 + \frac{1}{Q} \left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right] = R_o \left[\frac{\omega_o}{s} + \frac{1}{Q} + \frac{s}{\omega_o}\right]$$

The second version exposes the symmetry.
Consider sums of functions that result in quadratics in $s$

In any realistic case, there will be at least one additional corner:

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o}{Q}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}$$

$$Z = R_o \left[1 + \frac{1}{Q} \left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right]$$

Conclusion: The $Q$ of a quadratic corner is affected by a nearby corner.
Short cut to find the quadratic Q-factor:

Evaluate $Z_1$ and $Z_2$ separately at $s = j\omega_o$:

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o}{Q}\right)$$

$$Z_1(j\omega_o) = R_o j \left(1 + \frac{1}{jQ}\right) = R_o \left(\frac{1}{Q} + j\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}$$

$$Z_2(j\omega_o) = R_o \frac{1}{j} = R_o \left(1 - j\right)$$

When the two are added, the imaginary parts cancel, and the real part is the sum of the separate real parts:

$$Z(j\omega_o) = \frac{R_o}{Q}$$
In the above example, $Z_1$ and $Z_2$ are the series and parallel branches of the single-damped LC low-pass filter, and $Z$ is the input impedance $Z_i$:

$$Z_i = \frac{R_o}{Q} R_o \frac{s}{\omega_o} R_o \frac{\omega_o}{s}$$

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o / Q}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}$$

$$\omega_o \equiv \frac{1}{\sqrt{LC}} \quad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q \equiv \frac{R_o}{R}$$

$$Z = \frac{R_o}{\omega_o}$$
Doing the algebra on the graph can be extended to the double- and triple-damped filters.
Exercise 6.3
Find $Z_i$ for the double-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[
\begin{align*}
Z_1 &= \frac{R_o}{Q_e} + \frac{s}{\omega_o} + \frac{R_o}{Q_c} \\
Z_2 &= \frac{R_o}{\omega_o/s} \\
Z_i &= \frac{R_o}{Q_e} + \frac{R_o}{Q_c} \\
\end{align*}
\]

\[
\begin{align*}
\omega_o &= \frac{1}{\sqrt{LC}} \\
R_o &= \sqrt{\frac{L}{C}} \\
Q_e &= \frac{R_o}{R_e} \\
Q_c &= \frac{R_o}{R_c}
\end{align*}
\]
Exercise 6.3 - Solution
Find $Z_i$ for the double-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[
Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o}{Q_e}\right)
\]

\[
Z_2 = R_o \frac{\omega_o}{s} \left(1 + \frac{s}{Q_c \omega_o}\right)
\]

\[
\omega_o = \frac{1}{\sqrt{LC}} \quad R_o = \sqrt{\frac{L}{C}}
\]

\[
Q_e = \frac{R_o}{R_e} \quad Q_c = \frac{R_o}{R_c}
\]
Exercise 6.3 - Solution

Find $Z_i$ for the double-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[
\omega_o = \frac{1}{\sqrt{LC}} \quad R_o = \sqrt{\frac{L}{C}}
\]

\[
Q_e = \frac{R_o}{R} \quad Q_c = \frac{R_o}{R_c}
\]

\[
Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o}{Q_e}\right)
\]

\[
Z_2 = R_o \frac{\omega_o}{s} \left(1 + \frac{s}{Q_c \omega_o}\right)
\]
**Exercise 6.3 - Solution**

To find the quadratic Q-factor, evaluate $Z_1$ and $Z_2$ separately at $s = j\omega_o$:

$$Z_1(j\omega_o) = R_o j \left(1 + \frac{1}{jQ_e}\right) = R_o \left(\frac{1}{Q_e} + j\right)$$

$$Z_2(j\omega_o) = R_o (-j) \left(1 + j\frac{1}{Q_c}\right) = R_o \left(\frac{1}{Q_c} - j\right)$$

Hence $Z_i(j\omega_o) = Z_1(j\omega_o) + Z_2(j\omega_o) = R_o \left(\frac{1}{Q_e} + \frac{1}{Q_c}\right)$
Exercise 6.3 - Solution
Find $Z_i$ for the double-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[
\frac{1}{\omega_o} = \sqrt{\frac{L}{C}} \quad R_o = \sqrt{\frac{L}{C}}
\]

\[
Q_e \equiv \frac{R_o}{R_e} \quad Q_c \equiv \frac{R_o}{R_c}
\]

\[
Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o / Q_e}{s}\right)
\]

\[
Z_2 = R_o \frac{\omega_o}{s} \left(1 + \frac{s}{Q_c \omega_o}\right)
\]

\[
\frac{R_o}{Q_t} = \frac{R_o}{Q_e} + \frac{R_o}{Q_c}
\]
Exercise 6.4
Find $Z_i$ for the triple-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[ Z_i \]

\[ Z_1 \]

\[ Z_2 \]

\[ Q_s = \frac{R_o}{Q_e} \quad Q_c = \frac{R_o}{Q_c} \quad Q_L = \frac{R_L}{R_o} \]

\[ \omega_o = \frac{1}{\sqrt{LC}} \quad R_o = \sqrt{\frac{L}{C}} \]

\[ v.0.2 \text{ 10/07 } \text{ http://www.RDMiddlebrook.com} \]

6. Products & Sums
Exercise 6.4 - Solution

Find $Z_i$ for the triple-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o}{Q_e}\right)$$

$$Z_2 = R_o \frac{Q_L}{Q_1} \left(\frac{1}{Q_c} + \frac{\omega_o}{s}\right) \approx R_o \frac{\omega_o}{s} + \frac{1 + \frac{s}{\omega_o / Q_L}}{1 + \frac{s}{\omega_o / Q_L}}$$

$$Z_i(0) = Z_1(0) + Z_2(0) = \frac{R_o}{Q_e} + R_o Q_L \approx R_o Q_L$$

With neglect of the second-order effects, $Z_i$ follows the higher asymptote:
Exercise 6.4 - Solution
Find \( Z_i \) for the triple-damped LC filter. Draw the asymptotes for \( Z_1 \) and \( Z_2 \). Construct the asymptotes for the input impedance \( Z_i = Z_1 + Z_2 \), and find the \( Q_t \) of the quadratic in \( s \). Neglect second-order effects.

\[
\begin{align*}
\omega_o & = \frac{1}{\sqrt{LC}} & R_o & = \sqrt{\frac{L}{C}} \\
Q_e & = \frac{R_o}{R_e} & Q_c & = \frac{R_o}{R_c} & Q_L & = \frac{R_L}{R_o}
\end{align*}
\]

\[
\begin{align*}
Z_1 &= R_o \frac{s}{\omega_o} \left( 1 + \frac{\omega_o}{Q_e} \right) \\
Z_2 &= R_o \frac{\omega_o}{s} \frac{1 + \frac{s}{Q_c \omega_o}}{1 + \frac{s}{Q_L \omega_o}}
\end{align*}
\]
Exercise 6.4 - Solution
To find the quadratic Q-factor, evaluate $Z_1$ and $Z_2$ separately at $s = j\omega_0$:

$$Z_1(j\omega_0) = R_o j \left(1 + \frac{1}{jQ_e}\right) = R_o \left(\frac{1}{Q_e} + j\right)$$

$$Z_2(j\omega_0) = R_o \left(-j\right) \frac{1 + j\frac{1}{Q_c}}{1 - j\frac{1}{Q_L}} = R_o \left(-j + \frac{1}{Q_c}\right) \left(1 + j\frac{1}{Q_L}\right) \left(1 + \frac{1}{Q_L^2}\right)$$

$$\approx R_o \left[\frac{1}{Q_c} + \frac{1}{Q_L} - j \left(1 - \frac{1}{Q_c Q_L}\right)\right] = R_o \left(\frac{1}{Q_c} + \frac{1}{Q_L} - j\right)$$

Hence $Z_i(j\omega_0) = Z_1(j\omega_0) + Z_2(j\omega_0)$

$$= R_o \left(\frac{1}{Q_e} + \frac{1}{Q_c} + \frac{1}{Q_L}\right)$$
Exercise 6.4 - Solution

Find $Z_i$ for the triple-damped LC filter. Draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[
\begin{align*}
\omega_o & \equiv \frac{1}{\sqrt{LC}} \\
R_o & \equiv \sqrt{\frac{L}{C}} \\
Q_e & \equiv \frac{R_o}{R_e} \\
Q_c & \equiv \frac{R_o}{R_c} \\
Q_L & \equiv \frac{R_L}{R_o}
\end{align*}
\]

\[
\begin{align*}
Z_1 &= R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o/Q_e}{s}\right) \\
Z_2 &= R_o \frac{\omega_o}{Q_c \omega_o} \left(1 + \frac{s}{\omega_o/Q_L}\right)
\end{align*}
\]
Exercise 6.5

Construct \( A = A_1 + A_2 \) in both magnitude and phase asymptotes, starting from \( A_1 \) and \( A_2 \) in suitable factored pole-zero forms.
**Exercise 6.5 - Solution**

Construct $A = A_1 + A_2$ in both magnitude and phase asymptotes, starting from $A_1$ and $A_2$ in suitable factored pole-zero forms.

\[
A_1 = \frac{s}{\omega_0} \quad A_2 = \frac{\omega_0}{s} \left(1 + \frac{\omega_0}{Q}\right)
\]
Exercise 6.5 - Solution

With neglect of second-order effects, the sum will follow the higher function:

\[ A_1 = \frac{s}{\omega_0} \]

\[ A_2 = \frac{\omega_o}{s} \left(1 + \frac{\omega_o}{Q} \right) \]

The form is:

\[ A = \frac{\omega_o}{s} \left(1 + \frac{\omega_o}{Q} \right) \left[ 1 + \frac{1}{Q_t} \left( \frac{s}{\omega_0} \right) + \left( \frac{s}{\omega_0} \right)^2 \right] \]
Exercise 6.5 - Solution

Find the quadratic $Q$-factor $Q_t$:

$$A = \frac{s}{\omega_o} + \frac{\omega_o}{s} \left( 1 + \frac{\omega_o / Q}{s} \right)$$

$$A(j\omega_o) = j + \frac{1}{j} \left( 1 + \frac{1}{jQ} \right) = j - j \left( 1 - j \frac{1}{Q} \right) = -\frac{1}{Q}$$

So $Q_t = -Q$ and the final form is

$$A = \frac{\omega_o}{s} \left( 1 + \frac{\omega_o / Q}{s} \right) \left[ 1 - \frac{1}{Q} \left( \frac{s}{\omega_o} \right) + \left( \frac{s}{\omega_o} \right)^2 \right]$$
Exercise 6.5 - Solution

The negative $Q_t$ means that the quadratic is two rhp zeros, so the magnitude asymptotes have a concave upwards corner at $\omega_0$, and the phase is a $180^\circ$ lag, not a lead.
Exercise 6.5 - Solution

\[
\frac{1}{Q_t} = -\frac{1}{Q}
\]

\[
A_1 = \frac{s}{\omega_0}
\]

\[
A_2 = \frac{\omega_0}{s} \left( 1 + \frac{\omega_0}{Q} \right)
\]
Exercise 6.5 - Solution

\[ A_1 = \frac{s}{\omega_0} \]

\[ A_2 = \frac{\omega_0}{s} \left(1 + \frac{\omega_0}{Q} \right) \]

\[ \frac{1}{Q_t} = -\frac{1}{Q} \]

[Diagram showing frequency response with annotations]

- 0db
- -90°
- +45°/dec
- -180°
- -270°

\[ 10^{2Q} \omega_0 \]
Exercise 6.5 - Solution

\[ A_1 = \frac{s}{\omega_0} \]

\[ \frac{1}{Q_t} = -\frac{1}{Q} \]

\[ A_2 = \frac{\omega_0}{s} \left( 1 + \frac{\omega_0}{Q} \right) \]

- 90°
- 180°
+ 45°/dec
- 270°

\[ 10^{2Q} \omega_0 \]
Exercise 6.5 - Solution
By doing the algebra on the graph to set up

\[
A = \frac{\omega_0}{s} \left(1 + \frac{\omega_0}{Q}\right) \left[1 - \frac{1}{Q} \left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2\right]
\]

you have effectively found the symbolic roots of a cubic equation!

Algebraically,

\[
A = \frac{s}{\omega_0} + \frac{\omega_0}{s} \left(1 + \frac{\omega_0}{Q}\right)
\]

\[
= \frac{1}{Q} \left(\frac{\omega_0}{s}\right)^2 + \frac{\omega_0}{s} + \frac{s}{\omega_0}
\]

which is a cubic in \((s/\omega_0)\).
Extensions of the graphical method

1. In the sum of two functions, any one can be extracted to reduce the sum to the form $1 + T$:

$$Z = Z_1 + Z_2 = Z_1 \left(1 + \frac{Z_2}{Z_1}\right)$$

etc.
Extensions of the graphical method

1. In the sum of two functions, any one can be extracted to reduce the sum to the form $1 + T$:

$$Z = Z_1 + Z_2 = Z_1 \left(1 + \frac{Z_2}{Z_1}\right) \text{ etc.}$$

2. The sum of any number of impedances in series can be found graphically:

$$Z = Z_1 + Z_2 + Z_3 + \ldots$$

The result follows whichever contribution is the largest.
Extensions of the graphical method

1. In the sum of two functions, any one can be extracted to reduce the sum to the form $1 + T$:

$$Z = Z_1 + Z_2 = Z_1 \left(1 + \frac{Z_2}{Z_1}\right)$$ etc.

2. The sum of any number of impedances in series can be found graphically:

$$Z = Z_1 + Z_2 + Z_3 + ...$$

The result follows whichever contribution is the largest.

3. Similarly, the sum of any number of impedances in parallel can be found graphically:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + ...$$

The result follows whichever contribution is the smallest.
For the triple-damped LC filter, draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the output impedance $Z_o = Z_1 \parallel Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

\[
\begin{align*}
Z_i & = Z_1 + Z_2 \\
\frac{1}{Z_0} & = \frac{1}{Z_1} + \frac{1}{Z_2}
\end{align*}
\]

\[
\begin{align*}
\omega_0 & = \frac{1}{\sqrt{LC}} \\
R_o & = \frac{\sqrt{L}}{C} \\
Q_e & = \frac{R_o}{R_e} \\
Q_c & = \frac{R_o}{R_c} \\
Q_L & = \frac{R_L}{R_o}
\end{align*}
\]
For the triple-damped LC filter, draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the output impedance $Z_o = Z_1 \parallel Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

$$Z_1 = R_o \frac{s}{\omega_o} \left(1 + \frac{\omega_o}{Q_e} \frac{s}{\omega_o} \right)$$

$$Z_2 = R_o \frac{\omega_o}{s} \frac{1 + s \frac{\omega_o}{Q_c \omega_o}}{1 + \frac{s}{\omega_o} \frac{Q_c \omega_o}{Q_L}}$$

$$\omega_o = \frac{1}{\sqrt{LC}} \quad R_o = \sqrt{\frac{L}{C}}$$

$$Q_e = \frac{R_o}{R_e} \quad Q_c = \frac{R_o}{R_c} \quad Q_L = \frac{R_l}{R_o}$$
For the triple-damped LC filter, draw the asymptotes for $Z_1$ and $Z_2$. Construct the asymptotes for the output impedance $Z_o = Z_1 \parallel Z_2$, and find the $Q_t$ of the quadratic in $s$. Neglect second-order effects.

$$Z_1 = R_0 \frac{s}{\omega_o} \left(1 + \frac{\omega_o/Q_e}{s}\right)$$

$$Z_2 = R_0 \frac{\omega_o}{s} \left(1 + \frac{s}{\omega_o/Q_L}\right)$$

$$Q_o = \frac{1}{\sqrt{LC}} \quad R_0 = \sqrt{\frac{L}{C}}$$

$$Q_e = \frac{R_0}{R_e} \quad Q_c = \frac{R_0}{R_c} \quad Q_L = \frac{R_L}{R_o}$$
"Doing the algebra on the graph" applies to any transfer functions, whether they be voltage gains, current gains, impedances, or admittances.
"Doing the algebra on the graph" applies to any transfer functions, whether they be voltage gains, current gains, impedances, or admittances.

In particular, the last example of impedances in parallel also applies to reciprocal sums of voltage gains or current gains, which will be valuable in the later applications of the Dissection Theorem.
7. THE I/O IT:
The Input/Output Impedance Theorem

How to find them directly from the Gain, thereby saving almost two-thirds of the work
Why do we need to deal with input and output impedances?

1. They may be part of the specifications.

2. They describe the interaction between two system blocks, and are therefore components of the Divide and Conquer approach, specifically incorporated in the Chain Theorem.

Definitions of "input" and "output:"

Input and output impedances are transfer functions (TFs), just as is the gain.

A TF is a ratio of one signal in a circuit to another, so the most general definition of "input" and "output" is that the "input" is the signal in the denominator, and the "output" is the signal in the numerator:

\[
\frac{\text{"output"}}{\text{"input"}} = \text{transfer function TF}
\]
If numerator and denominator are both voltages or currents, the TF is a voltage gain or a current gain; if the numerator is a voltage and the denominator is a current, the TF is a transimpedance (and vice versa for a transadmittance).

If the numerator is the voltage across the same port into which the denominator current flows, the TF is a self-impedance.

The denominator of a TF is not necessarily an independent excitation; the independent excitation may be elsewhere.

Thus, there are three kinds of "input":

1. A signal at a port designated as "input"
2. An independent excitation
3. The denominator of a TF
Driving Point Impedance

The port at which a circuit is driven is the driving point.

One of the many TFs of interest is the driving point impedance, which is the self-impedance "seen" at the driving point:

\[
Z_{dp} \equiv \frac{v_d}{i_d}
\]

A system usually has designated signal "input" and "output" ports:
Input Impedance

\[ Z_i \equiv \frac{v_i}{i_i} \]

\[ Z_{dp} \equiv \frac{v_i}{i_i} = Z_i \equiv \text{input impedance at the signal input port} \]

Note: the "input" signal for the input impedance TF is \( i_i \), although the input signal for the gain TF may be \( v_i \) or \( i_i \), depending upon the definition of the gain.
Output Impedance

\[ Z_{dp} = \frac{v_o}{i_o} = Z_o \equiv \text{output impedance} \]

at the signal output port
Conventional Approach

Calculate the gain $H$
Calculate the output impedance $Z_o$
Calculate the input impedance $Z_i$

Usually these are done separately, each starting from scratch, and they may be equally lengthy analyses (especially if there is feedback present).

However, much of the analysis is the same in each case, so there is motivation to find a short cut that avoids the repetitions.
Consider the simple voltage divider:

\[ v_o = H e_i \]

The three analyses lead to:

\[ H = \frac{Z_2}{Z_1 + Z_2} \]
\[ Z_i = Z_1 + Z_2 \]
\[ Z_o = Z_1 || Z_2 \]

The "hard part" in each case is calculation of \( Z_1 + Z_2 \).

However, \( Z_i \) and \( Z_o \) can be written in terms of \( H \):

\[ Z_i = \frac{Z_2}{H} \]
\[ Z_o = Z_1 H \]

Thus, the sum \( Z_1 + Z_2 \) need be calculated only once to find \( H \), and then \( Z_i \) and \( Z_o \) can be found as products or quotients of \( H \).
This trick doesn't work for more complex circuits, but there is still motivation to find a way to calculate $Z_i$ and $Z_o$ from $H$ instead of starting from scratch.
Inner and Outer Input and Output Impedances

Forward voltage gain $A_v = \frac{v_L}{e_S}$

There are two kinds of input and output impedances, depending on whether the system is defined to include the source and load impedances or not.

**Outer**
- **Input Impedance** $\equiv Z_i$
- **Output Impedance** $\equiv Z_o$

**Inner**
- **Input Impedance** $\equiv Z_i^*$
- **Output Impedance** $\equiv Z_o^*$
Output Impedance Theorem

Forward voltage gain $A_v = \frac{v_L}{e_S}$

Forward transadmittance gain $Y_t = \frac{i_L}{e_S} = \frac{v_L}{Z_L e_S} = \frac{A_v}{Z_L}$

Short-circuit forward transadmittance gain $Y_t^{sc} = \frac{A_v}{Z_L} \bigg|_{Z_L \to 0}$
Output Impedance Theorem

Forward voltage gain \( A_v = \frac{v_L}{e_S} \)

\[
Z_o = \frac{\text{oc output voltage}}{\text{sc output current}}
\]

\[
Z_o = \frac{A_v}{Y_t^{sc}} = \frac{\text{fwd voltage gain}}{\text{sc fwd transadmittance}}
\]

This is the Output Impedance Theorem
Input Impedance Theorem

Convert the Thevenin independent source $e_S,Z_S$ to a Norton equivalent:

$$j_S = \frac{e_S}{Z_S}$$

Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{j_S|_{Z_S \to \infty}}$$
Input Impedance Theorem

\[ Z_t = \frac{v_L}{i_S} = \frac{v_L}{jS \left| Z_S \rightarrow \infty \right.} = \frac{v_L}{\frac{e_S}{Z_S} \left| Z_S \rightarrow \infty \right.} \]

Forward transimpedance gain

\[ v_L = A_v e_S \]
Input Impedance Theorem

Forward voltage gain \( A_v = \frac{v_L}{e_S} \)

Forward transimpedance gain

\[
Z_t = \frac{v_L}{i_S} = \frac{v_L}{jS|_{Z_S \to \infty}} = \frac{e_S}{Z_S|_{Z_S \to \infty}} = \frac{v_L}{v_L / A_v |_{Z_S \to \infty}} = Z_S A_v |_{Z_S \to \infty}
\]
Input Impedance Theorem

Forward voltage gain $A_v = \frac{v_L}{e_S}$

$Z_i = \frac{\text{input voltage}}{\text{input current}}$ for the same $v_L$

$Z_i = \frac{Z_t}{A_v} = \frac{\text{fwd transimpedance}}{\text{fwd voltage gain}}$

$Z_i = \left. \frac{Z_S A_v}{A_v} \right|_{Z_S \rightarrow \infty}$ This is the Input Impedance Theorem
Inner and Outer Input and Output Impedances

The value of the formulas is that once the gain is known, only a simple limit with respect to either $Z_L$ or $Z_S$ need be calculated to find the outer output or input impedances $Z_o$ or $Z_i$. 
Inner Input and Output Impedances

\[ i_S = Z_i v_i \]

\[ i_L = \frac{v_L}{Z_L} \]

\[ v_L = A_v e_S \]

It is obvious that

\[ Z_o^* = Z_o \bigg|_{Z_L \to \infty} = \frac{A_v}{Z_L \bigg|_{Z_L \to 0} Z_L \to \infty} = \frac{A_v}{Z_L \bigg|_{Z_L \to 0} Z_L \to \infty} \]

\[ Z_i^* = Z_i \bigg|_{Z_S \to 0} = \frac{Z_S A_v}{A_v \bigg|_{Z_S \to \infty} Z_S \to 0} = \frac{Z_S A_v}{A_v \bigg|_{Z_S \to 0} Z_S \to 0} \]
Inner Input and Output Impedances

Thus, all four input and output impedances can be found by taking simple limits upon the gain $A_v$. 

$$v_L = A_v e_S$$
This has been treated already. The result for the gain $A$ is:

\[ A = A_0 \frac{1 + \frac{s}{\omega_1} \left(1 + \frac{s}{\omega_2}\right)}{1 + \frac{s}{\omega_z}} \]

\[ A_0 = \frac{R_L}{R_1 + R_L} \]

\[ \omega_1 = \frac{1}{C_1 \left(R_2 + R_1 \parallel R_L\right)} \]

\[ \omega_2 = \frac{1}{C_2 \left(R_1 \parallel R_2 \parallel R_L\right)} \]

The formula for the outer input impedance is

\[ Z_i = \frac{Z_S A \big|_{Z_S \to \infty}}{A} \]

Since $R_1$ is a surrogate $Z_S$,

\[ Z_i = \frac{R_1 A \big|_{R_1 \to \infty}}{A} \]
\[ Z_i = \frac{R_1 A}{A} \bigg|_{R_1 \to \infty} = \left. \frac{R_1 A_0}{A_0} \right|_{R_1 \to \infty} \left( 1 + \frac{s}{\omega_z} \right) \bigg|_{R_1 \to \infty} \left( 1 + \frac{s}{\omega_1} \right) \bigg|_{R_1 \to \infty} \left( 1 + \frac{s}{\omega_2} \right) \bigg|_{R_1 \to \infty} \]

The limit can be taken factor by factor:

\[ A = A_0 \left( \frac{1 + \frac{s}{\omega_1}}{1 + \frac{s}{\omega_2}} \right) \]

\[ \omega_1 = \frac{1}{C_1 \left( R_2 + R_1 \parallel R_L \right)} \]

\[ \omega_2 = \frac{1}{C_2 \left( R_1 \parallel R_2 \parallel R_L \right)} \]

\[ \omega_z = \frac{1}{C_1 R_2} \]
The limit can be taken factor by factor:

\[ Z_i = \frac{R_1 A}{A} = \frac{R_1 A_0}{A_0} \left( 1 + \frac{s}{\omega_z} \right) \left( 1 + \frac{s}{\omega_1} \right) \left( 1 + \frac{s}{\omega_2} \right) \]

A huge simplification emerges: any factor in \( A \) that does not contain \( R_1 \) is unaffected by the limit and therefore cancels.
The limit can be taken factor by factor:

\[
Z_i = \lim_{R_1 \to \infty} \frac{R_1 A}{A} = \lim_{R_1 \to \infty} \frac{R_1 A_0}{A_0} \left(1 + \frac{s}{\omega_z}\right) \left(1 + \frac{s}{\omega_1}\right) \left(1 + \frac{s}{\omega_2}\right)
\]

A huge simplification emerges: any factor in A that does not contain \(R_1\) is unaffected by the limit and therefore cancels.
The limit can be taken factor by factor:

\[
Z_i = \frac{R_1 A|_{R_1 \to \infty}}{A} = \frac{R_1 A_0|_{R_1 \to \infty}}{A_0} \frac{1 + \frac{s}{\omega_1}}{1 + \frac{s}{\omega_1}|_{R_1 \to \infty}} \frac{1 + \frac{s}{\omega_2}}{1 + \frac{s}{\omega_2}|_{R_1 \to \infty}}
\]
The limit can be taken factor by factor:

\[ Z_i = \frac{R_1 A}{A} \bigg|_{R_1 \to \infty} = \frac{R_1 A_0}{A_0} \bigg|_{R_1 \to \infty} \left(1 + \frac{s}{\omega_1}\right) \left(1 + \frac{s}{\omega_2}\right) \]

You can take advantage of this knowledge in advance, by highlighting \( R_1 \) in the parameter definitions and omitting these factors when substituting into the formula.
Rewrite the definitions highlighting $R_1$:

$$Z_i = R_{i0} \frac{\left(1 + \frac{s}{\omega_1} \right) \left(1 + \frac{s}{\omega_2} \right)}{\left(1 + \frac{s}{\omega_1 \left| R_1 \rightarrow \infty \right.} \right) \left(1 + \frac{s}{\omega_2 \left| R_1 \rightarrow \infty \right.} \right)}$$

where

$$R_{i0} \equiv \frac{1}{\frac{1}{R_1} + \frac{1}{R_L}} = R_1 + R_L$$

$$\omega_1 \left| R_1 \rightarrow \infty \right. \equiv \frac{1}{C_1 \left( R_2 + R_1 \parallel R_L \right) \left| R_1 \rightarrow \infty \right.} < \omega_1$$

$$\omega_2 \left| R_1 \rightarrow \infty \right. \equiv \frac{1}{C_2 \left( R_1 \parallel R_2 \parallel R_L \right) \left| R_1 \rightarrow \infty \right.} < \omega_2$$

$$A = A_0 \frac{1 + \frac{s}{\omega_1}}{(1 + \frac{s}{\omega_1})(1 + \frac{s}{\omega_2})}$$

$$A_0 = \frac{R_L}{R_1 + R_L}$$

$$\omega_1 = \frac{1}{C_1 \left( R_2 + R_1 \parallel R_L \right)}$$

$$\omega_2 = \frac{1}{C_2 \left( R_1 \parallel R_2 \parallel R_L \right)}$$
Exercise 7.1
Find the outer output impedance $Z_o$

\[ v_1 \quad R_1 \quad 47k \quad C_1 \quad 0.1\mu F \quad R_2 \quad 1k \quad C_2 \quad 0.002\mu F \quad R_L \quad 100k \quad v_2 = A v_1 \]

\[
A = A_0 \frac{1 + \frac{s}{\omega_1}}{(1 + \frac{s}{\omega_2})(1 + \frac{s}{\omega_z})} \\
A_0 = \frac{R_L}{R_1 + R_L} \\
\omega_1 = \frac{1}{C_1 (R_2 + R_1 \parallel R_L)} \\
\omega_2 = \frac{1}{C_2 (R_1 \parallel R_2 \parallel R_L)} \\
\omega_z = \frac{1}{C_1 R_2} \]
Rewrite the definitions highlighting $R_L$:

$$\omega_0 = \left. \frac{1}{C_1 R_2} \right|_{R_L \to 0} = \omega_z$$

$$\omega_1 = \left. \frac{1}{C_1 (R_2 + R_1||R_L)} \right|_{R_L \to 0} = \infty$$

$$\omega_2 = \left. \frac{1}{C_2 (R_1||R_2||R_L)} \right|_{R_L \to 0}$$

$$Z_0 = \left. \frac{A}{R_L} \right|_{R_L \to 0} = R_{00} \left( \frac{1 + \frac{s}{\omega_1|R_L \to 0}}{1 + \frac{s}{\omega_2|R_L \to 0}} \right) \left( 1 + \frac{s}{\omega_1} \right) \left( 1 + \frac{s}{\omega_2} \right)$$

$$A = A_0 \left( \left. \frac{1}{R_L} \right|_{R_L \to 0} + \frac{s}{\omega_1|R_L \to 0} \right) \left( 1 + \frac{s}{\omega_2|R_L \to 0} \right)$$
Exercise 7.2
Find the inner output impedance $Z_o^*$

$$v_1 \quad R_1 \quad 47k \quad C_1 \quad 0.1\mu F \quad R_2 \quad 0.002\mu F \quad 1k \quad R_L \quad 100k \quad v_2 = A v_1$$

$$Z_o = R_{o0} \frac{ \left( 1 + \frac{s}{\omega_z} \right) \left( 1 + \frac{s}{\omega_1} \right) \left( 1 + \frac{s}{\omega_2} \right) }{ \left( 1 + \frac{s}{\omega_1} \right) \left( 1 + \frac{s}{\omega_2} \right) }$$

$$R_{o0} \equiv R_1 \parallel R_L \quad \omega_1 \equiv \frac{1}{C_1(R_2 + R_1 \parallel R_L)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \quad \omega_2 \equiv \frac{1}{C_2 \left( \frac{R_1}{R_2} \parallel R_L \right)}$$
Exercise 7.2 - Solution

\[ Z_0^* = Z_0 \bigg|_{R_L \to \infty} = R_{00}^* \left(1 + \frac{s}{\omega_z}\right) \left(1 + \frac{s}{\omega_1_{R_L \to \infty}}\right) \left(1 + \frac{s}{\omega_2_{R_L \to \infty}}\right) \]

where

\[ R_{00}^* \equiv R_{00} \bigg|_{R_L \to \infty} = R_1 \left| R_L \right|_{R_L \to \infty} = R_1 \]

\[ \omega_1 \bigg|_{R_L \to \infty} = \frac{1}{C_1 \left( R_2 + R_1 \left| R_L \right|_{R_L \to \infty} \right)} < \omega_1 \]

\[ \omega_2 \bigg|_{R_L \to \infty} = \frac{1}{C_2 \left( R_1 \left| R_2 \right|_{R_L \to \infty} \right)} < \omega_2 \]

\[ Z_0 = R_{00} \left(1 + \frac{s}{\omega_z}\right) \left(1 + \frac{s}{\omega_1_{R_L \to \infty}}\right) \left(1 + \frac{s}{\omega_2_{R_L \to \infty}}\right) \]

\[ \omega_1 \equiv \frac{1}{C_1 \left( R_2 + R_1 \left| R_L \right| \right)} \]

\[ \omega_2 \equiv \frac{1}{C_2 \left( R_1 \left| R_2 \right|_{R_L \to \infty} \right)} \]
Bottom Line

The Input/Output Impedance Theorem allows you to find the input and output impedances of a circuit by taking simple limits upon the already known gain. This saves almost two-thirds of the work required to obtain the three results separately.

Taking one limit upon the gain gives the outer impedances; Taking two limits upon the gain gives the inner impedances.

A huge simplification occurs by anticipating that factors in the gain that do not contain the source or load impedance, do not appear in the result.
8. NDI AND THE EET:
Null Double Injection and the Extra Element Theorem

How to find the contribution of a particular element to the transfer function
Null Double Injection (ndi)

Usually, a transfer function (TF) is calculated as a response to a single independent excitation.

However, large analysis benefits accrue when certain constraints are imposed on several excitations present simultaneously.
The input is an independent signal, the output is a dependent signal. The gain is $A_1$. 
Consider a second dependent signal, a voltage $v$ at some internal port:

\[ v = B_1 u_i \]

The gain from $u_i$ to $v$ is $B_1$. 
Apply a second independent signal, a current $i$ at the same internal port:

$$u_i = 0$$

The "gain" from $i$ to $v$ is a driving point impedance $B_2$.
Apply both independent signals simultaneously:

\[ v = B_1 u_i + B_2 i \]

For a linear system model, the two dependent signals are the superposition of the values they would have for each independent signal separately.
Apply both independent signals simultaneously:

\[ v = B_1 u_i + B_2 i \]

For a linear system model, the two dependent signals are the superposition of the values they would have for each independent signal separately.

By adjustment of \( u_i \) and \( i \), \( u_o \) can be made to have any value we like.

In particular:

\[ u_o \] can be made zero by adjustment of the relative values of \( u_i \) and \( i \), namely

\[ u_i \bigg|_{u_o=0} = -\frac{A_2}{A_1} i \]
There is now a null double injection (ndi) condition:

\[
v = B_1 u_i|_{u_o=0} + B_2 i
\]

The voltage at the internal port is

\[
v = \left( B_2 - B_1 \frac{A_2}{A_1} \right) i
\]

so the driving point impedance is

\[
Z_{dp}|_{u_o=0} = \left( B_2 - B_1 \frac{A_2}{A_1} \right)
\]
Recap:

The driving point impedance (dpi) $Z_{dp}$ at the internal port can have two different values, one when the input is zero, and another when the input is not zero, but is adjusted to null the output:

$$Z_{dp}|_{u_i=0} = B_2$$

from $i$

$$Z_{dp}|_{u_o=0} = B_2 - B_1 \frac{A_2}{A_1}$$

from $u_i$
Recap:

The driving point impedance (dpi) $Z_{dp}$ at the internal port can have two different values, one when the input is zero, and another when the input is not zero, but is adjusted to null the output:

$$Z_{dp}\bigg|_{u_i=0} = B_2 \quad \equiv Z_d$$

from $i$

$$Z_{dp}\bigg|_{u_o=0} = B_2 - B_1 \frac{A_2}{A_1} \quad \equiv Z_n$$

from $u_i$
The dpi $Z_d$ is calculated under single injection (si) conditions:

\[ u_i = 0 \] (zero)

\[ u_o = A_1 u_i \]

The dpi $Z_n$ is calculated under null double injection (ndi) conditions:

\[ u_i \mid u_o = 0 \] (nulled)

\[ u_o = 0 \]
The Extra Element Theorem (EET)

Replace the current source by an impedance $Z$:

\[ \begin{align*}
\text{signal input} & \quad u_i \\
\text{Z} & \\
\text{signal output} & \quad u_o = A_1 u_i + A_2 i
\end{align*} \]

The same linear superposition equations still apply to the rest of the circuit.

However, a relation between $v$ and $i$ is now enforced by $Z$, namely

\[ v = -Zi \]

Substitute for $v$ to find $i$ in terms of $u_i$:

\[ v = B_1 u_i + B_2 i = -Zi \]

so

\[ i = -\frac{B_1}{B_2 + Z} u_i \]
The Extra Element Theorem (EET)

\[ v = B_1 u_i + B_2 i \]

Now substitute for \( i \) in the equation for \( u_o \) to find \( u_o \) in terms of \( u_i \):

\[ u_o = A_1 u_i + A_2 i = A_1 u_i - A_2 \frac{B_1}{B_2 + Z} u_i \]

\[ u_o = A_1 \left( \frac{B_2 - B_1 \frac{A_2}{A_1} + Z}{B_2 + Z} \right) u_i = A_1 \left( 1 + \frac{B_2 - B_1 \frac{A_2}{A_1}}{Z} \right) u_i \]
The Extra Element Theorem (EET)

The two combinations of the linear circuit parameters are precisely what have just been defined as $Z_d$ and $Z_n$, so

$$u_o = A_1 \left( \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \right) u_i$$
The Extra Element Theorem (EET)

The two combinations of the linear circuit parameters are precisely what have just been defined as $Z_d$ and $Z_n$, so

$$u_o = A_1 \left( \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \right) u_i$$

However,

$$\frac{u_o}{u_i} = \text{gain in presence of } Z$$

$$A_1 = \text{gain when } Z = \infty$$
The Extra Element Theorem (EET)

The two combinations of the linear circuit parameters are precisely what have just been defined as \(Z_d\) and \(Z_n\), so

\[
u_o = A_1 \left( \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \right) u_i
\]

However,

\[
\frac{u_o}{u_i} = \text{gain in presence of } Z \equiv H
\]

\[
A_1 = \text{gain when } Z = \infty \equiv H_{\infty}
\]
The Extra Element Theorem (EET)

Hence, the Extra Element Theorem (EET) is:

\[
H = H_{\infty} \left( 1 + \frac{Z_n}{Z} \right) \left( 1 + \frac{Z_d}{Z} \right)
\]

- \(H = \text{gain in presence of } Z\)
- \(H_{\infty} = \text{gain when } Z = \infty\)

- \(Z_d = Z_{dp} \bigg|_{u_i=0} \quad \text{with } u_i = \text{zero}\)
- \(Z_n = Z_{dp} \bigg|_{u_o=0} \quad \text{with } u_i \neq \text{zero, } u_o \text{ nulled}\)
Hence
\[ u_{o1} = A_1 \frac{1 + \frac{Z}{Z_0}}{1 + \frac{Z}{Z_0}} u_{i1} \]

or
\[ \text{gain}|_{Z} = \text{gain}|_{Z=0} \frac{1 + \frac{Z}{Z_0}}{1 + \frac{Z}{Z_0}} \]

This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.
Hence

\[ u_{o1} = A_1 \frac{1 + \frac{2n}{2}}{1 + \frac{2n}{2}} u_{i1} \]

or

\[ \text{gain}_{12} = \text{gain}_{2=0} \frac{1 + \frac{2n}{2}}{1 + \frac{2n}{2}} \]

This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.

The Theorem also proves that any transfer function (e.g. gain) of a linear system is a bilinear function of any single element (e.g. \( \frac{1}{s} \)).
NETWORK ANALYSIS

forming $A_{11}$ the terms by which $x$ is multiplied must be the
minor $A_{11}$ obtained by omitting both the first and $j$th rows and columns. If we let
$A^0$ and $A_{11}^0$ represent, respectively, $A$ and $A_{11}$ when $x = 0$, therefore, we have

$$Z = \frac{A^0 + zA_{jj}}{A_{11}^0 + zA_{11}}.$$  \hspace{1cm} (1-11)

Since $A_{jj}$ and $A_{11}$ are evidently independent of $x$ they can equally well
be written as $A^0_{jj}$ and $A_{11}^0$. This will occasionally be done in later analysis
in order to facilitate further transformations.

The relation between $Z_T$ and $z$ can be found in similar fashion. It is
given by

$$Z_T = \frac{A^0 + zA_{jj}}{A_{11}^0 + zA_{11}}.$$  \hspace{1cm} (1-12)

If $z$ represents a unilateral coupling term, instead of a bilateral element,
the expansion is essentially the same. Thus, if we suppose that $z$ is a part of
$Z_{ij}$ in the original determinant, we readily find

$$Z = \frac{A^0 + zA_{ij}}{A_{11}^0 + zA_{11}}.$$  \hspace{1cm} (1-13)

and

$$Z_T = \frac{A^0 + zA_{ij}}{A_{11}^0 + zA_{11}}.$$  \hspace{1cm} (1-14)
The "brute-force" method: loop analysis

\[
(R_s + R_E) i_1 - R_E i_B = v_1
\]

\[
-R_E i_1 + \left[ R_E + (1 + \beta) R_E \right] i_B = 0
\]

\[
i_B = \frac{\begin{vmatrix}
R_s + R_E & v_1 \\
-R_E & 0
\end{vmatrix}}{\begin{vmatrix}
R_s + R_E & -R_E \\
-R_E & R_E + (1 + \beta) R_E
\end{vmatrix}} = \frac{R_E v_1}{(R_s + R_E) [R_E + (1 + \beta) R_E] - R_E^2}
\]

Finally, \( v_2 = R_L \beta i_B \)

which leads to:

\[
A_m = \frac{v_2}{v_1} = \frac{\beta R_E R_L}{(1 + \beta) R_E R_s + (1 + \beta) R_E R_E + R_s R_E}
\]
Implementation:
All that is needed is to calculate the driving point impedance across the terminals to which the extra element is to be added, under two conditions:

\[ Z_d = Z_{dp} \bigg|_{u_{01}=0} \quad \text{(original input zero)} \]

\[ Z_n = Z_{dp} \bigg|_{u_{01}=0} \quad \text{(original output nulled)} \]
Example: The previously designed CE amplifier.

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \rightarrow 1/j\omega C_2$ is desired.

Original gain:

$$A_0 = \frac{R_B}{R_s + R_B} \frac{\alpha R_e}{R + R_e + (R_s||R_b)(1+\beta)}$$

$$= 0.46 \frac{10}{2.2 + 0.036 + 0.039}$$

$$= 2.0 \Rightarrow 6\, \text{dB}$$
Step 1. Calculate $Z_d$ by shorting $u_{ci} = u_i$, and applying a second injected signal across $R$:

$$Z_d = R_d = R \| \left[ R_e + \frac{(R_s R_b)}{(1 + \beta)} \right]$$

$$= 2.2 \| \left[ \frac{0.036 + (10 \| 6)}{120} \right]$$

$$= 75 \Omega$$
Step 2. Calculate $Z_n$ by applying a second injected signal across $R$, and adjusting it with respect to $v_1$ to null $v_{o1} = v_2 = 0$. Then, since $v_1 = 0$, $i_E = 0$, hence:

$Z_n = R_n = R = 2.2k$

Original gain:

$$A_o = \frac{R_B}{R_s + R_0 + R_e + (R_{11}R_0)/(1\mu F)}$$

$$= 0.46 \frac{10}{2.2 + 0.036 + 0.039}$$

$$= 2 \Rightarrow 6\text{dB}$$
Reference gain:
\[ A_0 = \frac{v_2}{v_1} \]
\[ Z_d = R \parallel (r_E + \frac{R_s}{1+\beta}) \]
8. NDI & the EET
\( -v_2 = 0 \)

\( Z_n = R \)

These voltages are equal.
Example: The previously designed CE amplifier

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element \( Z \rightarrow 1/sC_2 \) is desired.

Original gain:

\[
A_0 = \frac{R_B}{R_s + R_0} \frac{\alpha R_L}{R + r_e + \left(R_s || R_0\right) || (1 + \beta)}
\]

\[
= 0.46 \frac{10}{2.2 + 0.036 + 0.039}
\]

\[
= 2.0 \Rightarrow 6 \, \text{dB}
\]
Original gain:

\[ A_0 = \frac{R_B}{R_S + R_0} \frac{\alpha R_L}{R + R_E + (R_S || R_B)/(1+\beta)} \]

\[ = 0.46 \frac{10}{2.2 + 0.036 + 0.039} \]

\[ = 2.0 \Rightarrow 6 \text{ dB} \]

Hence, corrected gain in presence of \( C_2 = 1 \mu F \) bypass capacitance is:

\[ A = A_0 \left( 1 + \frac{R_n}{1 + \frac{R_d}{\frac{sC_2R_n}{1 + sC_2R_d}}} \right) = A_0 \left( 1 + \frac{\frac{sC_2R_n}{1 + sC_2R_d}}{\frac{sC_2R_n}{1 + sC_2R_d}} \right) = A_0 \left( 1 + \frac{\frac{sC_2R_n}{1 + sC_2R_d}}{\frac{sC_2R_n}{1 + sC_2R_d}} \right) \]

where

\[ \omega_1 = \frac{1}{C_2R_n} \quad f_1 = \frac{159}{1 \times 2.2} = 72 \text{ Hz} \]

\[ \omega_2 = \frac{1}{C_0 R_d} = \frac{159}{1 \times 0.075} = 211 \text{ kHz} \]

\[ A_m = A_0 \frac{\omega_2}{\omega_1} \]

\[ = \frac{R_B}{R_S + R_0} \frac{\alpha R_L}{R + R_E + (R_S || R_B)/(1+\beta)} \]

\[ = \frac{R_B}{R_S + R_0} \frac{\alpha R_L}{R_E + (R_S || R_B)/(1+\beta)} \]

\[ = 62 \Rightarrow 36 \text{ dB} \]
8. NDI & the EET
Note: Nulling a voltage is not the same as shorting it!

Note: the null double injection calculation is easier than the single injection calculation!
Dual forms of the EET

The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element.

There is a corresponding form to find the correction factor resulting from an extra series element:

\[
\text{gain}_{\infty} = \text{gain}_{\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z}{Z_d}} = \frac{Z_n}{Z_d} \text{gain}_{\infty} \frac{1 + \frac{Z_n}{Z_d} + 1}{1 + \frac{Z}{Z_d} + 1}
\]

\[
= \frac{Z_n}{Z_d} \text{gain}_{\infty} \frac{1 + \frac{Z_n}{Z_d}}{1 + \frac{Z}{Z_d}}
\]
The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element. There is a corresponding form to find the correction factor resulting from an extra series element:

\[
\text{reference gain}
\quad \downarrow
\quad \text{gain} \bigg|_{Z} = \text{gain} \bigg|_{Z=0} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_n}{Z_d}}
\]

\[
\text{reference gain}
\quad \downarrow
\quad = \text{gain} \bigg|_{Z=0} \frac{\frac{Z_n}{Z}}{\frac{Z_n}{Z_d}} + 1
\]

\[
\frac{Z_n}{Z_d} \quad \frac{Z_n}{Z} \quad \frac{Z_n}{Z_d} + 1
\]

This must be the gain when \( Z = 0 \)
Example: An alternative to the method of the previous example is to find the correction factor to the midband gain $A_m$ resulting from addition of the series “extra element” $Z \rightarrow R \parallel 1/sC$. 
8. NDI & the EET

Original gain:

\[ A_m = \frac{R_e}{R_s + R_B} \left( \frac{\beta}{\beta + 1} \right) + \left( \frac{R_s(R_B)}{1 + \beta} \right) \]

\[ = 62 \Rightarrow 36 \text{ dB} \]
8. NDI & the EET

Original gain:
\[ A_m = \frac{R_e}{R_s + R_b} \frac{\alpha R_e}{r_e + (R_s || R_b)/(1+\beta)} \]
\[ = 62 \Rightarrow 36 \text{dB} \]

Step 1. Calculate \( Z_d \) by shorting \( u_{ii} = u_i \) and applying a second injected signal in series with \( r_e \):
\[ Z_d = R_d' = r_e + (R_s || R_b)/(1+\beta) \]
Step 2. Calculate $Z_n$ by applying a second injected signal in series with $r_E$, and adjusting it with respect to $v_1$ to null $u_0 = v_2 = 0$. Then, since $v_2 = 0$, $i_E = 0$, hence

$$Z_n = R_n' = \infty$$
Original gain:

\[ A_m = \frac{R_e}{R_s + R_e} \frac{\alpha R_L}{1 + (R_s/R_e)(1 + B)} \]

\[ = 62 \Rightarrow 36 \text{dB} \]

Hence, corrected gain in presence of \( R/Ls/C_2 \) is:

\[ A = A_m \frac{1 + \frac{R}{R_s}}{1 + \frac{1}{sC_2}} = A_m \frac{1 + \frac{R}{R_d}}{1 + \frac{1}{sC_2R \cdot (RIR_d')}} \]

However, \( RIR_d' = R_d \), so

\[ A = A_m \frac{1 + 1/sC_2R}{1 + 1/sC_2R d} \Rightarrow \text{same result as before} \]
The Parallel and Series forms of the EET

Generalization: Extra Element Theorem - #1

There are two forms of Extra Element Theorem:

1. \[ \text{gain} \big|_{Z=0} = \text{gain} \big|_{Z=\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \]

   Provides a correction factor for an extra element added in shunt across a node pair.

2. \[ \text{gain} \big|_{Z=0} = \text{gain} \big|_{Z=0} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \]

   Provides a correction factor for an extra element added in series with a branch.

The "extra element" Z can be any two-terminal combination of impedances.
Note that in all cases the null double injection calculation is easier than the single injection calculation.

This results from use of the null condition (which makes several other quantities zero); and because the relation between \( u_{e1} \) and \( u_{e2} \) to produce the null is never needed — only the null itself is used.
Exercise 8.1
Insert $C_1$ by the EET

Exercise
In the CE amplifier stage, find the correction factor to the midband gain $A_m$ resulting from inclusion of the coupling capacitance $C_1 = 0.05 \mu F$.

\[ A_m = 36 \text{dB} \]
Exercise 8.1 - Solution

Exercise Solution

\[ A_m = 36 \text{dB} \]
Exercise 8.1 - Solution

\[ V_i = 0 \]

\[ R_s = 10k \]

\[ R_c = 8.6k \]

\[ R_L = 10k \]

\[ \beta = 120 \]

\[ A_m = 36dB \]
Exercise 8.1 - Solution

Step 1.

\[ Z_d = R_d = R_s + R_B \| (1 + \beta) R_E \]
\[ = 10 + 8.6 \| (120 \times 0.036) \]
\[ = 10 + 8.6 \| 4.3 \]
\[ = 13 \text{k} \]

\[ A_m = 36 \text{dB} \]
Exercise 8.1 - Solution

\[ A_m = 36 \text{dB} \]
Exercise 8.1 - Solution

Step 2.

\[ Z_n = R_n = \infty \]

\[ A_m = 36 \text{dB} \]
Exercise 8.1 - Solution

Exercise Solution

Hence corrected gain in the presence of $Z \rightarrow \frac{1}{sC_1}$ is

$$A = A_m \frac{1 + \frac{Z}{2A}}{1 + \frac{Z}{2A}} = A_m \frac{1}{1 + \frac{1}{sC_1R_d}}$$

where

$$\omega_3 = \frac{1}{C_1R_d} \quad f_3 = \frac{15\pi}{0.05 \times 13} = 245 \text{ Hz}$$
Generalization: Extra Element Theorem - #2

If the reference circuit is purely resistive, 
\( Z_d = R_d \) and \( Z_n = R_n \) are pure resistances.

If, also, the extra element is a pure reactance, 
the Extra Element Theorem correction factor gives the corner frequencies directly.
Generalization: Extra Element Theorem - #3

The Extra Element Theorem can profitably be used to divide the analysis of a complicated circuit into successive simpler steps:

Designate one element as "extra," and the circuit without the element as the "reference circuit." Calculate the gain of the (simpler) reference circuit, then restore the omitted element by the Extra Element Theorem correction factor.

This is a particularly useful approach when the designated "extra" element is a reactance and the reference circuit is purely resistive.
Exercise 8.2
Lag-lead network: Find $A$ by designating $C_1$ as an extra element.

Find the transfer function $A = v_2/v_1$ by designating $C_1$ as an "extra" element.
Exercise 8.2 - Solution

\[ \frac{V_2}{V_1} = \frac{R_L}{R_1 + R_L} \]
Exercise 8.2 - Solution

\[ \frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L} \]

\[ R_d = R_2 + R_1 || R_L \]
Exercise 8.2 - Solution

\[ \frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L} \]

\[ R_d = R_2 + R_1 || R_L \]

\[ R_n = R_2 \]
Exercise 8.2 - Solution

\[ \frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L} \]

\[ R_d = R_2 + R_1 || R_L \]

\[ R_n = R_2 \]

\[ \frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L} \frac{1 + sC_1 R_2}{1 + sC_1 (R_2 + R_1 || R_L)} \]
Special case: The EET for a self-impedance

The Extra Element Theorem may be used to find an extra element correction factor for any transfer function of a linear circuit. It is necessary merely to identify the "input" and "output" signals; $Z_d$ and $Z_n$ are then calculated as the driving point impedance seen by the extra element with the "input" zero and with the "output" nulled, respectively.

Examples of transfer functions:

- "output" $\rightarrow$ current drawn from power supply  
  "input" $\rightarrow$ input voltage  
  (a transadmittance)

- "output" $\rightarrow$ output voltage ripple component  
  "input" $\rightarrow$ power supply ripple voltage  
  (a voltage gain; audio susceptibility of a power supply)

- "output" $\rightarrow$ corresponding driving voltage  
  "input" $\rightarrow$ any driving current  
  (a self-impedance, e.g. input or output impedance)
Example: Input impedance $Z_i$ of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":

"Reference transfer function":

$$\left. \frac{V_{i1}}{I_{i1}} \right|_{t=\infty} = Z_i \left|_{t=\infty} = R_s + R_B \| (1+\beta)(e+R) \right.$$
Example: Input impedance $Z_i$ of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":

```
\[
\frac{v_i}{i_1} \bigg|_{Z_i} = R_s + R_B \frac{1}{(1+\beta)(r_e+R)}
\]
```

```
\[
Z_d = Z_{dp} \bigg|_{input} = Z_{dp} \bigg|_{i_1=0} = R_d = R \frac{R_E + \frac{R_B}{1+\beta}}{1+\beta}
\]
```

```
\[
Z = \frac{1}{sC_2}
\]
```
Example: Input impedance $Z_i$ of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":

\[ \left. \frac{v_i}{i_1} \right|_{Z_i} = \frac{Z_i}{Z_i} = R_s + R_B || (1+\beta)(r_e+R) \]

\[ Z_n = Z_{dp} \mid_{v_i=0} = R_n = R || \left( r_e + \frac{R_s R_B}{1+\beta} \right) \]
Example: Input impedance $Z_i$ of a CE amplifier stage with emitter bypass capacitance as “extra” element.

“Reference circuit”:

“Reference transfer function”:

$$\left. \frac{v_i}{i_i} \right|_{z=0} = \left. \frac{v_i}{i_i} \right|_{z=0} = R_s + R_b \left( 1 + \beta \right) \left( r_e + R \right)$$

Hence, $Z_i = \left[ R_s + R_b \left( 1 + \beta \right) \left( r_e + R \right) \right] \frac{1 + sC_2R_e}{1 + sC_2R_d}$
Example: Input impedance $Z_i$ of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":

$$
\frac{v_i}{i_i} \bigg|_{s=0} = Z_i \bigg|_{s=0} = R_s + R_b || (1+\beta)(r_e + R)
$$

"Reference transfer function":

$$
Z = \frac{1}{sC_2}
$$

$$
Z_d = Z_{dp} \bigg|_{\text{input, \_i_1}=0} = Z_{dp} \bigg|_{i_1=0} = R_d = R || (r_e + \frac{R_b}{1+\beta})
$$
"Reference circuit":

\[ \frac{v_1}{i_1} \mid_{s=0} = Z_i \mid_{s=\infty} = R_s + R_B \left( 1 + \beta \right) (r_b + R) \]

\[ Z = \frac{1}{sC_2} \]

**NOTE:** In the special case of a self-impedance, nulling the "output" voltage is the same as shorting the "input" current, because the "output" and "input" are at the same node pair.
Exercise 8.3
Lag-lead network: Find $Z_i$ by designating $C_1$ as an extra element.

Example: Lag-lead network

Find the input impedance $Z_i = v_i / i_1$ by designating $C_1$ as an "extra" element.
Exercise 8.3 - Solution

"Reference circuit:

\[ i_1 \]

\[ v_1 \]

\[ R_1 \]

\[ R_2 \]

\[ R_L \]

"Reference input impedance:

\[ Z_i \mid_{z=\infty} = \frac{v_1}{i_1} \mid_{z=\infty} = R_1 + R_L \]
Exercise 8.3 - Solution

"Reference circuit:

\[ Z_d = R_d = R_2 + R_L \]

"Reference" input impedance:

\[ Z_i \bigg|_{Z_0} = \frac{V_i}{I_i} \bigg|_{Z_0} = R_1 + R_L \]
Exercise 8.3 - Solution

"Reference circuit:"

\[ Z_n = R_n = R_2 + R_1 || R_L \]

Hence:

\[ Z_i = \frac{\frac{V_1}{I_1}}{Z_n} = R_1 + R_L \]

\[ Z_i = (R_1 + R_L) \frac{1 + sC_1R_n}{1 + sC_1R_d} \]
Exercise 8.4

Lag-lead network: Find $Z_o$ by designating $C_1$ as an extra element.

Find the output impedance $Z_o = v_o/i_o$ by designating $C_1$ as an "extra" element.
Exercise 8.4 - Solution

\[ Z_0 \bigg|_{Z = \infty} \equiv \frac{V_o}{I_o} \bigg|_{Z = \infty} = R_1 || R_L \]
Exercise 8.4 - Solution

\[ R_d = R_2 + R_1 \parallel R_L \]

\[ Z_0 \bigg|_{Z = \infty} = \frac{V_0}{i_0} \bigg|_{Z = \infty} = R_1 \parallel R_L \]
Exercise 8.4 - Solution

\[ R_{dl} = R_2 + R_1 \parallel R_L \]

\[ R_n = R_2 \]

With \( C_1 \) replaced:

\[ Z_0 = R_1 \parallel R_L \frac{1 + s C_1 R_2}{1 + s C_1 (R_2 + R_1 \parallel R_L)} \]
Generalization: Extra Element Theorem - #4

The Extra Element Theorem can be used to find an extra element correction factor for any transfer function; \( Z_d \) and \( Z_e \) are then the driving point impedances seen by the extra element with the "input" zero and with the "output" nulled, respectively.

When the transfer function is a self-impedance, such as the input impedance \( Z_i \) or the output impedance \( Z_o \), nulling the "output" is the same as shorting the "input," hence

\[
Z_d = Z_{dp|\text{"input" zero}} = Z_{dp|\text{"input" open}}
\]

\[
Z_e = Z_{dp|\text{"output" nulled}} = Z_{dp|\text{"input" shorted}}
\]
1CE: The basic Common-Emitter amplifier stage

High-frequency properties of CE amplifier

Measurement indicates that there is a high-frequency pole \( \omega_4 \):

\[ A_m = 36 \text{dB} \]

\[ \omega_4 \]
High-frequency properties of CE amplifier

Measurement indicates that there is a high-frequency pole \( \omega_4 \):

\[
A_m = 36 \text{dB}
\]

The expectation is that this is caused by the collector-base transition-layer capacitance \( C_t \).

A typical value is \( C_t = 5 \text{pF} \). The resulting corner frequency with \( R_L = 10k \) is

\[
\frac{159}{5 \times 10^{-6} \times 10} = 3.2 \text{MHz}.
\]

Since the actual corner frequency is much lower, there must be a multiplying effect on \( C_t \) resulting from its connection to the transistor base instead of to ground.
High-frequency properties of CE amplifier

Measurement indicates that there is a high-frequency pole \( \omega_4 \):

\[ A_m \approx 36 \text{dB} \]

Since the midband gain \( A_m = 36 \text{dB} \) has already been determined, use the Extra Element Theorem to find the correction factor resulting from inclusion of \( Z \rightarrow 1/sC_e \).
Midband model after Thevenin reduction of $R_s, R_f, R_k$:
Midband model after Thevenin reduction of $R_s, R_L, R_2$:

$$Z_d = \frac{v_f}{i} \bigg|_{v_f=0}$$

$R_g = R_s || R_L || R_2 = 4.6k$

Input: $v_i = 0$
Midband model after Thevenin reduction of $R_s, R_L, R_2$:

The current generator $i$ can be divided into two equal current generators in series.
Midband model after Thevenin reduction of $R_s, R_b, R_z$:

The current generator $i$ can be divided into two equal current generators in series.

Since the voltage at the junction of the two current generators $i$ is immaterial, the junction can be grounded.

\[ R_g = R_s + R_b + R_z = 4.6k \]

\[ \beta i_b \]

\[ -v_2 \]

\[ R_L / 10k \]

\[ v_1 = 0 \]
Midband model after Thevenin reduction of $R_s, R_b, R_2$:

A separate ground can be identified for each current generator $i$. 

$R_g = R_s + R_1 + R_2 \approx 4.6k$

$\beta i_B$
Midband model after Thevenin reduction of $R_s, R_b, R_2$:

Rearranged diagram:

$Z_d = R_d = \frac{V}{i} = \frac{V_{BE}}{i} + \frac{V_2}{i}$

$V_{BE} = [R_g \| (1+\beta)R_E]i$

$V_2 = R_L (\alpha i_E + i) = R_L \left( \frac{\alpha}{R_E} V_{BE} + i \right)$

$R_d = \frac{V_{BE}}{i} + R_L \left( \frac{\alpha}{R_E} V_{BE} + 1 \right)$

\[
R_d = (1 + \frac{\alpha R_g}{R_E}) \left[ R_g \| (1+\beta)R_E \right] + R_L = R_L \left[ \frac{1}{R_L} + \frac{\alpha}{R_E} + \frac{1}{R_g \| (1+\beta)R_E} \right]
\]

\[
= R_L \left[ \frac{1}{R_L} + \frac{\beta}{(1+\beta)R_E} + \frac{1}{R_g + \frac{1}{(1+\beta)R_E}} \right]
\]

\[
= \frac{R_g \| (1+\beta)R_E}{R_g \| \frac{1}{E} \| R_L} R_L = \frac{4.6 \| 4.3}{4.6 \| 0.036 \| 10} R_L = 62 R_L = 620 k
\]
Generalization: Floating Current Generator

A floating current generator can be replaced by two separate, equal, grounded current generators. This is a useful technique in "doing the algebra on the circuit diagram."
Midband model after Thevenin reduction of $R_s, R_f, R_x$:

\[
Z_n = R_n = \frac{v}{i} = \frac{(1+\beta)R_E i_B}{-\beta i_B} = -\frac{R_E}{\alpha} = -36 \Omega
\]
Note that the $Z_n$ calculation is much shorter and easier than the $Z_d$ calculation!
Midband model after Thevenin reduction of $R_s, R_b, R_2$:
Hence corrected gain after inclusion of $C_t$ is
\[
A = A_m \frac{1 + \frac{2\omega}{\omega_c}}{1 + \frac{2\omega}{Z}} = A_m \frac{1 + SC_tR_n}{1 + SC_tR_d}
\]
\[
= A_m \frac{1 - \frac{3}{\omega_4}}{1 + \frac{3}{\omega_4}}
\]
where
\[
\omega_4 = \frac{1}{C_tR_d}
\]
\[
f_4 = \frac{159}{5 \times 10^{-8} \times 620} = 51 \text{kHz}
\]
\[
\omega_5 = \frac{1}{C_tR_n}
\]
\[
f_5 = \frac{159}{5 \times 10^{-6} \times 0.036} = 880 \text{kHz}
\]
Midband model after Thevenin reduction of $R_s, R_j, R_z$:

Hence corrected gain after inclusion of $C_t$ is

$$A = A_m \frac{1 + \frac{1}{2\pi f_3}}{1 + \frac{1}{2\pi f_2}} = A_m \frac{1 + \frac{1}{2\pi f_1}}{1 + \frac{1}{2\pi f_4}}$$

where

$$f_4 = \frac{159}{5 \times 10^{-3} 	imes 620} = 51 \text{kHz}$$

$$f_5 = \frac{159}{5 \times 10^{-3} 	imes 0.086} = 880 \text{kHz}$$

Note that the zero $w_5 = \frac{1}{C_t R_n} = \frac{C_t}{C_{ree}}$ is negative (right half-plane), and is at a very high frequency unless there is substantial external emitter resistance and/or there is substantial external collector-base capacitance (as often exists).
Note that the pole $w_4 = \frac{1}{C_4 R_4} = \frac{1}{C_4 R_2} \frac{R_{gII} R_L}{R_{gII}(1+\beta)RL}$ is at a much lower frequency than $w_5 = \frac{1}{C_5 R_L}$, and can be ascribed to an effective multiplication of $C_4$ by a factor

$$m = \frac{R_{gII}(1+\beta)RL}{R_{gII} R_L} = \frac{R_{gII}(1+\beta)RL}{R_{gII} R_L} \left(1 + \frac{R_{gII} R_L}{R_{gII}}\right) \frac{R_{gII}(1+\beta)RL}{R_{gII} R_L} \xrightarrow{R_g \ll R_L} 1 + \beta$$
Midband model after Thevenin reduction of $R_s, R_1, R_2$:

$$R_y = R_s || R_1 || R_2 = 4.6k$$

$$\beta = 62$$

This example: $m = 62$
Alternative method for calculation of $Z_d$

There are two forms of the Extro Element theorem:

$$A = A|_{z=\infty} \frac{1 + \frac{z_n}{2d}}{1 + \frac{2d}{z}} = A|_{z=0} \frac{1 + \frac{z_n}{2d}}{1 + \frac{2d}{z}}$$

where

$$A|_{z=0} = \frac{z_n}{2d} A|_{z=\infty}$$

Hence in general

$$\frac{A|_{z=0}}{A|_{z=\infty}} = \frac{z_n}{2d}$$

It may be easier to find $A|_{z=0}$, $A|_{z=\infty}$, and $z_n$ than to find $Z_d$ directly.

Example: Addition of collector-base capacitance $C_t$ to the CE amplifier stage. $A|_{z=\infty}$ and $z_n$ were easily found:

$$A|_{z=0} = A_m = \frac{R_B}{R_s + R_B + R_g + (1+\beta)R_E} \quad z_n = R_n = -\frac{r_e}{\alpha}$$
The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element. There is a corresponding form to find the correction factor resulting from an extra series element:

\[
\begin{align*}
\text{reference gain} & \quad \downarrow \\
\text{gain} \bigg|_{Z} & = \text{gain} \bigg|_{\frac{Z}{Z_o^\infty}} \frac{1 + \frac{Z}{Z_o}}{1 + \frac{Z}{Z_d}} \\
\text{reference gain} & \quad \downarrow \\
& = \text{gain} \bigg|_{Z=0} \frac{1 + \frac{Z}{Z_o}}{1 + \frac{Z}{Z_d}} \\
& = \left( \frac{Z_o}{Z_d} \right) \text{gain} \bigg|_{\frac{Z}{Z_o^\infty}} \frac{1 + \frac{Z}{Z_o}}{1 + \frac{Z}{Z_d}} \\
& = \frac{\frac{Z_o}{Z_d}}{\frac{Z_o}{Z_d} + 1}
\end{align*}
\]

This must be the gain when \( Z = 0 \)
Alternative method for calculation of \( Z_{d} \)

There are two forms of the Extro Element theorem:

\[
A = A_{1} \left|_{z = \infty} \frac{1 + \frac{z_{n}}{Z_{d}}}{1 + \frac{2A_{1}z_{n}}{Z_{d}}} \right. = A_{1} \left|_{z = 0} \frac{1 + \frac{z_{n}}{Z_{d}}}{1 + \frac{2A_{1}z_{n}}{Z_{d}}} \right.
\]

where

\[
A_{1} \left|_{z = 0} = \frac{z_{n}}{Z_{d}} A_{1} \left|_{z = \infty} \right.
\]

Hence in general

\[
\frac{A_{1} \left|_{z = 0}}{A_{1} \left|_{z = \infty}} = \frac{z_{n}}{Z_{d}}
\]

It may be easier to find \( A_{1} \left|_{z = 0} \right. \), \( A_{1} \left|_{z = \infty} \right. \), and \( z_{n} \) than to find \( Z_{d} \) directly.

Example: Addition of collector-base capacitance \( C_{t} \) to the CE amplifier stage. \( A_{1} \left|_{z = \infty} \right. \) and \( z_{n} \) were easily found:

\[
A_{1} \left|_{z = \infty} = A_{m} = \frac{R_{B}}{R_{S} + R_{B} \cdot \frac{B R_{L}}{R_{g} + (C B) R_{E}}} \quad z_{n} = R_{n} = -\frac{R_{E}}{\alpha}
\]
Midband model after Thevenin reduction of $R_s, R_1, R_2$: 

$$R_g = R_s || R_1 || R_2 = 4.6k$$

$$R_e = 36k$$
Model for calculation of $A|_{z=0}$

\[ A|_{z=0} = -\frac{R_B}{R_s+R_B} \frac{r_{E\|RL}}{R_g + r_{E\|RL}} = -\frac{R_B}{R_s+R_B} \frac{R_{g\|R_{E\|RL}}}{R_g} \]

Hence:
\[ Z_D = R_D = R_n \frac{A|_{z=0}}{A|_{z=0}} = \frac{r_E}{R_g + (1+\beta)r_E} \frac{R_g}{R_{g\|R_{E\|RL}}} = \frac{R_{g\|R_{E\|RL}}}{R_{g\|R_{E\|RL}}} \]

This is much easier than was the direct calculation of $Z_D$!
**Generalization:** Extra Element theorem — #5

The two reference gains and the two driving point impedances are related by:

\[
\frac{A_1|_{Z=0}}{A_1|_{Z=\infty}} = \frac{Z_n}{Z_d}
\]

One reference gain is always known or is easily found. 
\(Z_n\) is always easier to find than \(Z_d\). 

Therefore: 
It is often easier to find the other reference gain and to use the above ratio relation for \(Z_d\), than to find \(Z_d\) directly.
Common-emitter (1CE) amplifier stage

Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$. 

**Previous results:**

$$ A_v = A_{vm} \frac{1-s/\omega_z}{1+s/\omega_p} = 36dB \frac{1-s/2\pi}{880MHz} \frac{1-s/2\pi}{51kHz} $$

$$ A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_m + \frac{R_S}{1+\beta}} = 62 \Rightarrow 36dB $$

$$ R_n = r_E/\alpha = 36\Omega \quad R_d = mR_L = 620k $$

$$ \omega_z \equiv \frac{1}{C_t R_n} \quad \omega_p \equiv \frac{1}{C_t R_d} \quad m \equiv \frac{R_S \| R_B \| (1+\beta)r_m}{R_S \| R_B \| r_m \| R_L} = 62 $$

Outer input impedance $Z_i$:

Use the parallel EET with reference value $Z \equiv 1/sC_t$ infinite:

$$ Z_i \equiv \frac{v_S}{j_S} = R_{im} \frac{1+sC_t R_{ni}}{1+sC_t R_{di}} $$

$$ R_{im} \equiv R_S + R_B \| (1+\beta)r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega $$

$$ R_{ni} = R_{dp} \text{ with output } v_S \text{ nulled} $$

$$ = R_{dp} \text{ with input } j_S \text{ shorted} $$

$$ = R_{dp} \text{ for the voltage gain } A $$
Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$

Previous results:

$$A_v = A_{vm} \frac{1 - s / \omega_z}{1 + s / \omega_p} = 36 \text{dB} \quad 1 - \frac{s/2\pi}{880 \text{MHz}}$$
$$A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_m + \frac{R_S}{1 + \beta}} = 62 \Rightarrow 36 \text{dB}$$

$$R_n = \frac{r_E}{\alpha} = 36 \Omega$$
$$R_d = m R_L = 620 k$$

Outer input impedance $Z_i$:

$$Z_i \equiv \frac{v_S}{j S} = \frac{R_{im}}{1 + s C t R_{ni}}$$

$$R_{im} \equiv R_S + R_B \parallel (1 + \beta) r_m = 13k \Rightarrow 82 \text{dB} \text{ ref 1\Omega}$$

$$R_{ni} = m R_L \equiv \frac{R_S \parallel R_B \parallel (1 + \beta) r_m}{R_S \parallel R_B \parallel r_m \parallel R_L} R_L = 620k$$
Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$

**Previous results:**

$$A_v = A_{vm} \frac{1 - s / \omega_z}{1 + s / \omega_p} = 36dB \quad \frac{1 - \frac{s}{2\pi \text{MHz}}}{1 + \frac{s}{2\pi \text{kHz}}}$$

$$A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_m + \frac{R_S \| R_B}{1 + \beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E / \alpha = 36\Omega \quad R_d = m R_L = 620k$$

$$\omega_z = \frac{1}{C_t R_n} \quad \omega_p = \frac{1}{C_t R_d} \quad m = \frac{R_S \| R_B \| (1 + \beta) r_m}{R_S \| R_B \| r_m \| R_L} = 62$$

Outer input impedance $Z_i$:

$$Z_i = \frac{v_S}{j_S} = R_{im} \frac{1 + s C_t R_{ni}}{1 + s C_t R_{di}}$$

$$R_{im} \equiv R_S + R_B \| (1 + \beta) r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega$$

$$R_{ni} = m R_L \equiv \frac{R_S \| R_B \| (1 + \beta) r_m}{R_S \| R_B \| r_m \| R_L} R_L = 620k$$

$$R_{di} = R_{dp} \text{ with input } j_S \text{ open}$$
Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$

**Previous results:**

\[
A_v = A_{vm} \frac{1 - s / \omega_z}{1 + s / \omega_p} = 36dB \quad 1 - \frac{s}{2\pi 880MHz} \quad 1 + \frac{s}{2\pi 51kHz}
\]

\[
A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{1 + \beta} = 62 \Rightarrow 36dB
\]

\[
R_n = \frac{r_E}{\alpha} = 36\Omega \quad R_d = m R_L = 620k
\]

\[
\omega_z = \frac{1}{C_t R_n} \quad \omega_p = \frac{1}{C_t R_d} \quad m = \frac{R_S \parallel R_B \parallel (1 + \beta) r_m}{R_S \parallel R_B \parallel r_m \parallel R_L} = 62
\]

Outer input impedance $Z_i$:

\[
Z_i \equiv \frac{v_S}{j_S} = \frac{R_{im}}{1 + s C_t R_{ni}} \frac{1 + s C_t R_{di}}{1 + s C_t R_{ni}}
\]

\[
R_{im} \equiv R_S + R_B \parallel (1 + \beta) r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega
\]

\[
R_{ni} = m R_L = \frac{R_S \parallel R_B \parallel (1 + \beta) r_m}{R_S \parallel R_B \parallel r_m \parallel R_L}
\]

\[
R_{di} = R_{dp} \text{ with input } j_S \text{ open}
\]

\[
R_{ni} \bigg|_{R_S \to \infty} = \frac{R_B \parallel (1 + \beta) r_m}{R_B \parallel r_m \parallel R_L} R_L = 820k
\]

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Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$

**Previous results:**

$$A_v = A_{vm} \frac{1 - s / \omega_z}{1 + s / \omega_p} = 36dB \quad \frac{1 - s / 2\pi}{880MHz} \quad \frac{1 + s / 2\pi}{51kHz}$$

$$A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_m + \frac{R_S \| R_B}{1 + \beta}} = 62 \implies 36dB$$

$$R_n = r_E / \alpha = 36\Omega$$

$$R_d = mR_L = 620k$$

$$\omega_z = \frac{1}{C_t R_n} \quad \omega_p = \frac{1}{C_t R_d}$$

$$m = \frac{R_S \| R_B \| (1 + \beta) r_m}{R_S \| R_B \| r_m \| R_L} = 62$$

Outer input impedance $Z_i$:

$$Z_i \equiv \frac{v_S}{j_S} = R_{im} \frac{1 + s C_t R_{ni}}{1 + s C_t R_{di}} = 82dB \quad \frac{1 + \frac{s/2\pi}{51kHz}}{1 + \frac{s/2\pi}{39kHz}}$$

Check:

By GB trade-off: $R_{i\infty} = R_{im} \frac{R_n}{R_d} = 13k \frac{620k}{820k} = 10k \implies 80dB$

By inspection: $R_{i\infty} = R_S \| R_{im} \| R_{i\infty} = 10k \implies 80dB$
Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$

Previous results:

$$A_v = A_{vm} \frac{1 - s / \omega_z}{1 + s / \omega_p} = 36dB \frac{1 - s/2\pi}{1 + s/2\pi} \frac{880\text{MHz}}{51kHz}$$

$$A_{vm} = \frac{R_B}{R_S + R_B} + \frac{\alpha R_L}{1 + \beta} = 62 \Rightarrow 36dB$$

$$R_n = r_E / \alpha = 36\Omega \quad R_d = mR_L = 620k$$

$$\omega_z = \frac{1}{C_t R_n} \quad \omega_p = \frac{1}{C_t R_d} \quad m = \frac{R_S \parallel R_B \parallel (1 + \beta)R_m}{R_S \parallel R_B \parallel \frac{R_m}{R_L}} = 62$$

Inner input impedance $Z_i^*$:

$$Z_i^* = Z_i \bigg|_{R_S \rightarrow 0} = R_{im} \frac{1 + sC_t R_{ni}}{1 + sC_t R_{di}} \bigg|_{R_S \rightarrow 0} = R_{im} \frac{1 + sC_t R_{ni}^*}{1 + sC_t R_{di}^*}$$

$$R_{im} = R_{im} \bigg|_{R_S \rightarrow 0} = R_S + R_B \parallel (1 + \beta)R_m \bigg|_{R_S \rightarrow 0} = R_B \parallel (1 + \beta)R_m = 2.9k \Rightarrow 69dB$$

$$R_{ni} = R_{ni} \bigg|_{R_S \rightarrow 0} = \frac{R_S \parallel R_B \parallel (1 + \beta)R_m \parallel R_L}{R_S \parallel R_B \parallel R_m \parallel R_L} \bigg|_{R_S \rightarrow 0} = R_L = 10k \Rightarrow 80dB$$
Use the EET to find the outer and inner input impedances $Z_i$ and $Z_i^*$

$$Z_i^* = 69dB \frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$$

Check:

By GB trade-off: $R_{i\infty}^* = R_{im}^* \frac{R_{ni}^*}{R_{di}^*}$

$$= 2.9k \frac{39kHz}{3.2MHz} = 35\Omega \Rightarrow 31dB$$

By inspection: $R_{i\infty}^* = R_B r_m Z_o - 35\Omega \Rightarrow 31dB$
Exercise 8.5

Use the EET to find $Z_o$ and $Z_o^*$ for the 1CE amplifier stage

![Amplifier Circuit Diagram]

Previous results:

$$A_v = A_{vm} \frac{1 - s/\omega_z}{1 + s/\omega_p} = 36\text{dB} \quad 1 - \frac{s/2\pi}{880\text{MHz}}$$

$$A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_m + \frac{R_S R_B}{1 + \beta}} = 62 \Rightarrow 36\text{dB}$$

$$R_n = r_E/\alpha = 36\Omega$$

$$R_d = mR_L = 620k$$

$$\omega_z = \frac{1}{C_t R_n} \quad \omega_p = \frac{1}{C_t R_d} \quad m = \frac{R_S R_B (1 + \beta) r_m}{R_S R_B R_M R_L} = 62$$

Outer output impedance $Z_o$:

Use the parallel EET with reference value $Z \equiv 1/sC_t$ infinite:

$$Z_o \equiv \frac{v_o}{j_S} = R_{om} \frac{1 + sC_t R_{no}}{1 + sC_t R_{do}}$$

$$R_{om} \equiv R_L = 10k \Rightarrow 80\text{dB} \text{ ref } 1\Omega$$

$$R_{no} = R_{dp} \text{ with output } v_o \text{ nulled } = R_{dp} \text{ with input } j_S \text{ shorted}$$

$$= R_S \| R_B \| (1 + \beta) r_m = 2.2k$$

$$R_{do} = R_{dp} \text{ with input } j_S \text{ open } = R_d \text{ for the voltage gain } A$$

$$\Rightarrow 620k$$

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8. NDI & the EET
Exercise 8.5: - Solution
Use the EET to find $Z_o$ and $Z_i^*$ for the 1CE amplifier stage

$$Z_o = 80 dB \frac{1 + \frac{s}{2\pi} \frac{1}{14 MHz}}{1 + \frac{s}{2\pi} \frac{1}{51 kHz}}$$

Check:

By GB trade-off: $R_{oo\infty} = R_{om} \frac{R_{no}}{R_{do}}$

$$= 2.9k \frac{51 kHz}{14 MHz} = 36 \Omega \Rightarrow 31 dB$$

By inspection: $R_{oo\infty} = R_1 R_2 \frac{1}{r_m} R_L = 36 \Omega \Rightarrow 31 dB$
Exercise 8.5: - Solution

Use the EET to find \( Z_o \) and \( Z^*_o \) for the 1CE amplifier stage

\[
\begin{align*}
R_{om} & = R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega \\
R_{no} & = R_S \| R_B \| (1 + \beta)r_m = 2.2k \\
R_{do} & = \frac{R_S \| R_B \| (1 + \beta)r_m}{R_S \| R_B \| r_m \| R_L} R_L \rightarrow 620k
\end{align*}
\]

Inner output impedance \( Z^*_o \):

\[
Z^*_o = Z_o \bigg|_{R_L \rightarrow \infty} = R_{om} \frac{1 + sC_t R_{no}}{1 + sC_t R_{do}} \bigg|_{R_L \rightarrow \infty} = R_{om} \frac{1 + sC_t R_{no}}{1 + sC_t R^*_o}
\]

\[
R^*_o = R_{om} \bigg|_{R_L \rightarrow \infty} = R_L \bigg|_{R_L \rightarrow \infty} = \infty
\]

\[
R^*_o = R_{do} \bigg|_{R_L \rightarrow \infty} = \frac{R_S \| R_B \| (1 + \beta)r_m}{R_S \| R_B \| r_m \| R_L} R_L \bigg|_{R_L \rightarrow \infty} = \infty
\]
Exercise 8.5: Solution

Use the EET to find $Z_o^*$ and $Z_o$ for the 1CE amplifier stage

\[
\begin{align*}
R_{om} &= R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega \\
R_{no} &= R_S \parallel R_B \parallel (1 + \beta)r_m = 2.2k \\
R_{do} &= \frac{R_S \parallel R_B \parallel (1 + \beta)r_m}{R_S \parallel R_B \parallel r_m \parallel R_L} R_L = 620k
\end{align*}
\]

Inner output impedance $Z_o^*$:

Because $R_{om}^*$ and $R_{do}^*$ both are infinite, change the $Z_o$ reference value from $R_{om}$ to $R_{o\infty}$. Then:

\[
Z_o^* = Z_o \bigg|_{R_L \to \infty} = R_{om} \frac{1 + sCtR_{no}}{1 + sCtR_{do}} \bigg|_{R_L \to \infty} = R_{o\infty} \frac{1 + 1/sCtR_{no}}{1 + 1/sCtR_{do}} \bigg|_{R_L \to \infty} = R_{o\infty} \left(1 + 1/sCtR_{no}\right)
\]

\[
R_{o\infty} = \frac{R_{no}}{R_{do}} \bigg|_{R_L \to \infty} = R_S \parallel R_B \parallel \frac{r_m}{R_L} \bigg|_{R_L \to \infty} = R_S \parallel R_B \parallel r_m = 36\Omega \Rightarrow 31dB
\]
Exercise 8.5: - Solution

Use the EET to find $Z_o$ and $Z_o^*$ for the 1CE amplifier stage

$$Z_o^* = 31dB \left( 1 + \frac{14MHz}{s/2\pi} \right)$$
9. THE DT AND THE CT:
The Dissection Theorem and the Chain Theorem

How to find the gain of a multistage amplifier as the product of separately calculated low entropy factors
Null Double Injection (ndi)

Usually, a transfer function (TF) is calculated as a response to a single independent excitation.

However, large analysis benefits accrue when certain constraints are imposed on several excitations present simultaneously.
For any linear system model:

\[ u_o = A_1 u_i \]

The input is an independent signal, the output is a proportional dependent signal.
Consider a second input, an injected "test signal" $u_z$:

\[ u_o = A_1 u_i + A_2 u_z \]

Since the model is linear, the output is now a linear sum of the values it would have with each input alone.
There are now two more dependent signals, $u_x$ and $u_y$, where $u_x + u_y = u_z$:

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The dependent signal $u_y$ is also a linear sum of the values it would have with each input alone.
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\[
\begin{align*}
  u_y &= B_1 u_i + B_2 u_z \\
  u_x &= -B_1 u_i + (1 - B_2)u_z \\
  u_z &= u_x + u_y \\
  u_o &= A_1 u_i + A_2 u_z
\end{align*}
\]

The dependent signal $u_y$ is also a linear sum of the values it would have with each input alone.

By virtue of $u_z = u_x + u_y$, the independent signal $u_x$ can also be expressed in terms of $B_1$ and $B_2$. 
Several transfer functions (TFs) can be defined:

**Special case 1: \( u_z = 0 \)**

\[
\begin{align*}
\text{signal input} & \quad \hspace{1cm} \text{signal output} \\
\begin{array}{c}
u_i \\
\end{array} & \quad \hspace{1cm} \begin{array}{c}
u_o = A_1 u_i + A_2 u_z \\
\rightarrow \quad \rightarrow \\
\end{array} \\
\end{align*}
\]

\[
\begin{align*}
u_y &= B_1 u_i + B_2 u_z \quad & \quad \begin{array}{c}
u_y = B_1 u_i \\
\end{array} \\
u_x &= -B_1 u_i + (1 - B_2) u_z \quad & \quad \begin{array}{c}
u_x = -B_1 u_i \\
\end{array} \\
u_x &= -B_1 u_i \quad & \quad \begin{array}{c}
u_z = 0 \\
\end{array} \\
\end{align*}
\]

\[
H \equiv \left. \frac{u_o}{u_i} \right|_{u_z=0} = A_1
\]
Several transfer functions (TFs) can be defined:

Special case 2: \( u_i = 0 \)

\[
\begin{align*}
    u_y &= B_1 u_i + B_2 u_z \\
    u_y &= B_2 u_z \\
    u_x &= -B_1 u_i + (1 - B_2) u_z \\
    u_x &= (1 - B_2) u_z \\
    u_o &= A_1 u_i + A_2 u_z \\
    u_o &= A_2 u_z
\end{align*}
\]

These are single injection (si) TFs
Several transfer functions (TFs) can be defined:

**Special case 3: \( u_y = 0 \)**

The two independent signals \( u_i \) and \( u_z \) can be mutually adjusted to null \( u_y \)

\[
\begin{align*}
\text{signal input} & : u_i \\
\text{signal output} & : u_o \\
\text{test signal} & : u_z
\end{align*}
\]

\[
\begin{align*}
0 & = B_1 u_i + B_2 u_z \\
u_y & = B_1 u_i + B_2 u_z \\
u_x & = -B_1 u_i + (1-B_2) u_z \\
u_o & = A_1 u_i + A_2 u_z \\
H^{u_y} & = \left. \frac{u_o}{u_i} \right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}
\end{align*}
\]

**component of \( u_o \) from \( u_i \)**

**component of \( u_o \) from \( u_z \)**

**adjusted to null \( u_y \)**
Several transfer functions (TFs) can be defined:

Special case 4: $u_x = 0$

The two independent signals $u_i$ and $u_z$ can be mutually adjusted to null $u_x$

$$u_y = B_1 u_i + B_2 u_z$$
$$u_x = -B_1 u_i + (1 - B_2) u_z$$
$$0 = -B_1 u_i + (1 - B_2) u_z$$
$$u_o = A_1 u_i + A_2 u_z$$

$$u_o = A_1 u_i + A_2 \frac{B_1}{1 - B_2} u_i$$

$$H^u_y \equiv \left. \frac{u_o}{u_i} \right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}$$

$$H^u_x \equiv \left. \frac{u_o}{u_i} \right|_{u_x=0} = A_1 + A_2 \frac{B_1}{1 - B_2}$$

component of $u_o$ from $u_i$

component of $u_o$ from $u_z$

adjusted to null $u_x$
Several transfer functions (TFs) can be defined:

**Special case 5: \( u_o = 0 \)**

The two independent signals \( u_i \) and \( u_z \) can be mutually adjusted to null \( u_o \)

\[
\begin{align*}
H^{u_y} &= \left. \frac{u_o}{u_i} \right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2} \\
H^{u_x} &= \left. \frac{u_o}{u_i} \right|_{u_x=0} = A_1 + A_2 \frac{B_1}{1 - B_2} \\
T_n &= \left. \frac{u_y}{u_x} \right|_{u_o=0} = \frac{A_1 B_2 - A_2 B_1}{A_1 - (A_1 B_2 - A_2 B_1)}
\end{align*}
\]

These are null double injection (ndi) TFs
Assembled results, so far:

First level TF:

\[ H \equiv \frac{u_o}{u_i} \bigg|_{u_z=0} = A_1 \] (si)

Second level TFs:

\[ H_{u_y}^{u_o} \equiv \frac{u_o}{u_i} \bigg|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2} \] (ndi)

\[ T_n \equiv \frac{u_y}{u_x} \bigg|_{u_o=0} = \frac{A_1 B_2 - A_2 B_1}{A_1 - (A_1 B_2 - A_2 B_1)} \] (ndi)

\[ H_{u_x}^{u_o} \equiv \frac{u_o}{u_i} \bigg|_{u_x=0} = A_1 + A_2 \frac{B_1}{1 - B_2} \] (ndi)

\[ T \equiv \frac{u_y}{u_x} \bigg|_{u_i=0} = \frac{B_2}{1 - B_2} \] (si)

Note that \( A_2 \) and \( B_1 \) occur only as a product \( A_2 B_1 \).
The benefit to be gained from these definitions is that there are useful relations between these several TFs that do not involve the A's and B's.
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The 4 second level TFs are defined in terms of the 4 original parameters $A_1, A_2, B_1, B_2$. Since $A_2$ and $B_1$ occur only as a product $A_2B_1$, there are actually only 3 parameters and there must be a relation between the 4 second level TFs, which is

Redundancy Relation: $$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$
The benefit to be gained from these definitions is that there are useful relations between these several TFs that do not involve the $A$'s and $B$'s.

A consequence of the Redundancy Relation is that the first level TF $H$ can be expressed in terms of any three of the four second level TFs $H^{u_y}, H^{u_x}, T_n, T$.

Two useful versions are:

$$H = H^{u_y} \frac{1 + \frac{1}{T}}{1 + \frac{T_n}{T}}$$

$$H = H^{u_y} \frac{T}{1 + T} + H^{u_x} \frac{1}{1 + T}$$
These two versions, and the redundancy relation, can easily be verified by substitution of the definitions. After this, the A's and B's are no longer required, and will not appear again.
Dissection Theorem (DT)

\[ H \equiv \frac{u_o}{u_i} \bigg|_{u_z=0} \]

\[ H = H^{u_y} \frac{1 + \frac{1}{T_n}}{1 + \frac{1}{T}} = H^{u_y} \frac{T}{1+T} + H^{u_x} \frac{1}{1+T} \]

Notation:
Superscript signal is signal being nulled

Redundancy Relation:
\[ \frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T} \]

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9. The DT & the CT
These results constitute the **Dissection Theorem** (DT), so named because it shows that a first level TF can be "dissected" into three second level TFs established in terms of an injected test signal.

The DT is completely general, and applies to any TF of a linear system model.

For example, $H$ could be a voltage gain, current gain, or an input or output impedance.
There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.
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Why are ndi calculations *always* simpler and easier than si calculations?

Because any element that supports a null signal does not contribute to the result, and because if one signal is nulled, often other signals are automatically nulled as well, and therefore several elements may be absent from the result.
The Dissection Theorem can be represented by the block diagram

Important: The individual blocks do not necessarily represent identifiable parts of the actual circuit!
Check:

\[ \varepsilon = u_i - \frac{1}{H_{uy}} u_o \]

\[ u_o = H_{uy} T \varepsilon + H_{ux} u_i \]

\[ u_o = H_{uy} T \left( u_i - \frac{1}{H_{uy}} u_o \right) + H_{ux} u_i \]

\[ (1 + T) u_o = \left( H_{uy} T + H_{ux} \right) u_i \]

\[ H = H_{uy} \frac{T}{1 + T} + H_{ux} \frac{1}{1 + T} \]

\[ T_n = H_{uy} T \frac{1}{H_{ux}} \]

\[ \frac{H_{uy}}{H_{ux}} = \frac{T_n}{T} \]
So far, nothing has been said about where in the system model the test signal is injected.

Different test signal injection points define different sets of second level TFs. Nevertheless, when a mutually consistent set is substituted into the DT, the same $H$ results:

$$H = H^{u_{y1}} \frac{1 + \frac{1}{T_{n1}}}{1 + \frac{1}{T_1}} = H^{u_{y2}} \frac{1 + \frac{1}{T_{n2}}}{1 + \frac{1}{T_2}} = ...$$
This means that the blocks in the block diagram have different values for different test signal injection points:

Important: The individual blocks do not necessarily represent identifiable parts of the actual circuit!
This means that the blocks in the block diagram have different values for different test signal injection points:

\[ u_i \rightarrow H^u_{x2} \rightarrow T_{n2} \rightarrow H^{u_y2}T_2 \rightarrow \frac{1}{H^{u_y2}} \rightarrow + \rightarrow u_o = Hu_i \]

**Important:** The individual blocks do not necessarily represent identifiable parts of the actual circuit!
Not only does the DT implement the Design & Conquer objective, but the DT is itself a Low Entropy Expression, and much greater benefits accrue if the second level TFs have useful physical interpretations.

Thus, the second level TFs themselves contain the useful design-oriented information and you may never need to actually substitute them into the theorem.

For example, if $T,T_n >> 1$, $H \approx H^u_y$
How to determine the physical interpretations of the second level TFs?

What kind of signal (voltage or current) is injected, and where it is injected, defines an "injection configuration."

Therefore, the key decision in applying the DT is choosing a test signal injection point so that at least one of the second level TFs has the physical interpretation you want it to have.
Specific injection configurations for the DT lead to the:
Extra Element Theorem (EET)
Chain Theorem (CT)
General Feedback Theorem (GFT)

As usual, dual forms of the theorem emerge depending upon whether
the injected signal $u_z$ is a voltage or a current.
The Extra Element Theorem

Inject a test voltage $e_z$ in series with an element $Z$ such that $v_y$ appears across $Z$:

The DT is:

$$H = H^{v_y} \left( 1 + \frac{1}{T_{nv}} \right) \frac{1}{1 + \frac{1}{T_v}}$$

where:

$$H^{v_y} \equiv \left. \frac{u_o}{u_i} \right|_{v_y=0}$$

$$T_v \equiv \left. \frac{v_y}{v_x} \right|_{u_i=0}$$

$$T_{nv} \equiv \left. \frac{v_y}{v_x} \right|_{u_o=0}$$
The Extra Element Theorem

To find \( H^{v_y} \), assume that \( e_z \) and \( u_i \) have been mutually adjusted to null \( v_y \):

![Circuit Diagram]

If \( v_y = 0 \), there is no current through \( Z \), and so the current \( i \) into the test port is also zero, which is the condition that would exist if there were no injected test signal and \( Z \) were open. Therefore,

\[
H_{Z=\infty} = H^{v_y} \equiv \frac{u_o}{u_i} \bigg|_{v_y=0}
\]

where \( H_{Z=\infty} \) is the first level TF \( H \) when \( Z = \infty \).
The Extra Element Theorem

To find $T_v$, set $u_i = 0$:

\[ Z_d \equiv \frac{v_x}{i} \bigg|_{u_i=0} \]

Since $Z$ and $Z_d$ are in series with the same current $i$,

\[ T_v = \left. \frac{v_y}{v_x} \right|_{u_i=0} = \frac{Z}{Z_d} \]
The Extra Element Theorem

To find $T_{nv}$, assume that $e_z$ and $u_i$ have been mutually adjusted to null $u_o$:

![Diagram](image)

The ndi driving point impedance $Z_n$ looking into the test port is

$$Z_n = \frac{v_x}{i} \bigg|_{u_o=0}$$

Since $Z$ and $Z_n$ are in series with the same current $i$,

$$T_{nv} = \left. \frac{v_y}{v_x} \right|_{u_o=0} = \frac{Z}{Z_n}$$
With the second level TFs replaced by the new definitions, the DT morphs into the Extra Element Theorem (EET):

\[ H = H\big|_{Z=\infty} \left(1 + \frac{Z_n}{Z_d} \right) \]
Dissection Theorem (DT)

\[ H \equiv \frac{u_o}{u_i} \bigg|_{u_z=0} \]

\[ H = H^{u_y} \frac{1 + \frac{1}{T_n}}{1 + \frac{1}{T}} = H^{u_y} \frac{T}{1 + T} + H^{u_x} \frac{1}{1 + T} \]

**Notation:**
Superscript signal is signal being nulled

**Redundancy Relation:**
\[ \frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T} \]

\[ H^{u_y} \equiv \frac{u_o}{u_i} \bigg|_{u_y=0} \quad T_n \equiv \frac{u_y}{u_x} \bigg|_{u_o=0} \]

\[ H^{u_x} \equiv \frac{u_o}{u_i} \bigg|_{u_x=0} \quad T \equiv \frac{u_y}{u_x} \bigg|_{u_i=0} \]

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There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.
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Why are ndi calculations always simpler and easier than si calculations?

Because any element that supports a null signal does not contribute to the result, and because if one signal is nulled, often other signals are automatically nulled as well, and therefore several elements may be absent from the result.
Not only does the DT implement the Design & Conquer objective, but the DT is itself a Low Entropy Expression, and *much greater* benefits accrue if the second level TFs have useful physical interpretations.

Thus, the second level TFs themselves contain the useful design-oriented information and you may never need to actually substitute them into the theorem.

For example, if \( T, T_n \gg 1 \), \( H \approx H_{uy} \)
How to determine the physical interpretations of the second level TFs?

What kind of signal (voltage or current) is injected, and where it is injected, defines an "injection configuration."

Therefore, the key decision in applying the DT is choosing a test signal injection point so that at least one of the second level TFs has the physical interpretation you want it to have.
Another special case of the DT leads to the Chain Theorem (CT).

The test signal injection configuration is such that the entire signal from the input flows to the output (no bypass paths).
The Chain Theorem (CT)

The gain \( A_{v12} \equiv \frac{v_o}{e_i} \) is given by the DT:

\[
A_{v12} = A_{v12} \frac{i_y}{v_o} \left( 1 + \frac{1}{T_{ni}} \right)
\]

The TF \( T_{ni} \equiv \frac{i_y}{i_x} \bigg|_{v_o=0} \) is an ndi calculation with the output \( v_o \) nulled.

If \( v_o \) is nulled, so is \( i_x \), so \( T_{ni} = \infty \).

This implies that \( T_{ni} \) is infinite unless the signal can bypass the injection point.
Nulled $i_y$ means that the $A_{v1}$ box is unloaded, so the input voltage to the $A_{v2}$ box is the open-circuit (oc) output voltage of the $A_{v1}$ box.

Thus, $A_{v_{12}}^{i_y} = A_{v1}^{oc}A_{v2}$ is the \textit{voltage-buffered gain} of the two stages.
Also

\[ T_i \equiv \left. \frac{i_y}{i_x} \right|_{e_i=0} = \left. \frac{v}{Z_{o1}} \right|_{e_i=0} = \frac{Z_{i2}}{Z_{o1}}, \]

so the DT becomes

\[ A_{v12} = A_{v1}^{oc} A_{v2} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} \]

This can be interpreted as

\[
\begin{bmatrix}
\text{gain of the two stages} \\
\text{of the two stages}
\end{bmatrix} =
\begin{bmatrix}
\text{voltage buffered gain} \\
\text{of the two stages}
\end{bmatrix} \times
\begin{bmatrix}
\text{voltage loading factor} \\
\text{between the two stages}
\end{bmatrix}
\]
The Chain Theorem (CT)
This is exactly the result that would be obtained directly from the model:

\[ v_o = A_{v12} e_i \]

\[ A_{v12} = A_{v1} A_{v2} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} \]
The Chain Theorem (CT)
A useful application of the DT with $T_{ni} = \infty$ is to assemble the properties of a 2-stage amplifier from the properties of each separate stage.
The Chain Theorem (CT)

This "Divide and Conquer" approach avoids analysis of both stages simultaneously.

\[ v_o = A_{v12}e_i \]
The Chain Theorem (CT)

\[ v_o = A_{v12} e_i \]

\[ A_{v12} = A_{v1}^o A_{v2} \frac{1}{1 + \frac{1}{T_i}} = A_{v1}^o A_{v2} D_i \]

where \(A_{v1}^o A_{v2}\) is the "voltage buffered" gain that would occur if there were a buffer between the two stages, and \(D_i\) is a "discrepancy factor" that accounts for the interaction between the two stages which results from the loading of the first stage by the input of the second stage.
The Chain Theorem (CT)

\[ A_{v12} = A_{v1}^o A_{v2} \frac{1}{1 + \frac{1}{T_i}} = A_{v1}^o A_{v2} D_i \]

Since all TFs will be in factored pole-zero form, the only place where additional approximation may be needed resides inside the \( D_i \), where the sum of two TFs is required.

"Doing the algebra on the graph" can be conducted in two ways:

1. \( 1 + T_i \) can be found as the sum of the TFs 1 and \( T_i \), dominated by the larger;
2. \( D_i \) can be found from \[ \frac{1}{D_i} = 1 + \frac{1}{T_i} = \frac{1}{1 + \frac{1}{T_i}} \] as the reciprocal sum of 1 and \( T_i \), dominated by the smaller.
Let each stage be the 1CE stage previously treated.

\[ Z_i = R_{im} \frac{1 + s C_t R_{ni}}{1 + s C_t R_{di}} = 82dB \quad \frac{1 + \frac{s}{2\pi} 51kHz}{1 + \frac{s}{2\pi} 39kHz} \]

\[ Z_o = R_{om} \frac{1 + s C_t R_{no}}{1 + s C_t R_{do}} \]

\[ R_{im} = R_S + R_B \|(1 + \beta) r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega \]

\[ R_{om} = R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega \]

\[ R_{ni} = mR_L \equiv \frac{R_S \| R_B \|(1 + \beta) r_m}{R_S \| R_B \| r_m \| R_L} \quad R_L = 620k \]

\[ R_{no} = R_S \| R_B \|(1 + \beta) r_m = 2.2k \]

\[ R_{di} = \frac{R_B \|(1 + \beta) r_m}{R_B \| r_m \| R_L} \quad R_L = 820k \]

\[ R_{do} = mR_L = 620k \]

\[ A_v = A_{vm} \frac{1 - s / \omega_z}{1 + s / \omega_p} = 36dB \quad \frac{1 - \frac{s}{2\pi} 880MHz}{1 + \frac{s}{2\pi} 51kHz} \]

\[ A_{vm} = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{\frac{R_S \| R_B}{1 + \beta}} = 62 \Rightarrow 36dB \]

\[ R_n = r_m = 36\Omega \quad R_d = mR_L = 620k \]

\[ \omega_z \equiv \frac{1}{C_t R_n} \quad \omega_p \equiv \frac{1}{C_t R_d} \quad m \equiv \frac{R_S \| R_B \|(1 + \beta) r_m}{R_S \| R_B \| r_m \| R_L} = 62 \]

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However, to make the symbolic equations more compact, without loss of generality, let $R_S \rightarrow 0$ and $\alpha \rightarrow 1$ ($\beta \rightarrow \infty$).

To keep $R_{im}^*$ the same, also let $R_B \|(1 + \beta)r_m = 2.9k \rightarrow R_B$
The new 1CE stage is:

\[
Z_i = R_{im} \frac{1 + s C_t R_L}{1 + s C_t R_L} = 69dB \frac{1 + \frac{s}{2\pi}}{3.2MHz} \frac{1 + \frac{s}{2\pi}}{39kHz}
\]

\[R_{im} = R_B = 2.9k \Rightarrow 69dB \text{ ref. } 1\Omega\]

\[ R_{ni} = mR_L = 10k\]

\[R_{oi} = 0\]

\[R_{di} = \frac{R_B \|(1 + \beta)r_m}{R_B \| r_m \| R_L} = 820k\]

\[R_{do} = mR_L = 10k\]

Note that letting \(R_S \rightarrow 0\) reduces the "miller multiplier" \(m\) to 1.
\[ A_v = \frac{R_L}{r_m} \frac{1 - sC_t r_m}{1 + sC_t R_L} = 49dB \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}} \]

\[ Z_i = R_B \frac{1 + sC_t R_L}{1 + sC_t R_L} = 69dB \frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}} \]

\[ Z_o = R_L \frac{1}{1 + sC_t R_L} = 80dB \frac{1}{1 + \frac{s/2\pi}{3.2MHz}} \]

\[ R_{om} = R_L = 80dB \]

\[ R_{im} = R_B = 69dB \]

\[ A_{vm} = R_L / r_m = 49dB \]
The DT gives

\[ A_{v12} = A_{v1}^{oc} A_{v2} D_i \quad (T_{ni} = \infty) \]

The buffered gain \( A_{v1}^{oc} A_{v2} \) is the product of the two separate gains, where \( A_{v1} \) is already open-circuit:

\[ A_{v1}^{oc} A_{v2} = 98 \text{dB} \left( 1 - \frac{s/2\pi}{880\text{MHz}} \right)^2 \left( 1 + \frac{s/2\pi}{3.2\text{MHz}} \right) \]
The diagram shows the relationship between different frequencies and gains. The expression for the gain $A_{v1}A_{v2}$ is given as:

$$A_{v1}A_{v2} = 98dB \left( 1 - \frac{s/2\pi}{880MHz} \right) \left( 1 + \frac{s/2\pi}{3.2MHz} \right) \left( 1 - \frac{s/2\pi}{880MHz} \right) \left( 1 + \frac{s/2\pi}{3.2MHz} \right)$$

The figure indicates a gain of $98dB$ at $3.2MHz$ with a slope of $-40dB/dec$.
\[ Z_{i2} = 69dB \frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}} \]

\[ Z_{o1} = 80dB \frac{1}{1 + \frac{s/2\pi}{3.2MHz}} \]
\[ T_i \equiv \frac{Z_{i2}}{Z_{o1}} = -11dB \left( 1 + \frac{s/2\pi}{3.2MHz} \right)^2 \]

\[ Z_{i2} = 69dB \frac{1 + \frac{s}{2\pi}}{1 + \frac{s/2\pi}{3.2MHz}} \]

\[ Z_{o1} = 80dB \frac{1}{1 + \frac{s/2\pi}{3.2MHz}} \]
The discrepancy factor \( D_i = \frac{1}{1 + \frac{1}{T_i}} \) or \( \frac{1}{D_i} = \frac{1}{1 + \frac{1}{T_i}} \) or \( D_i = \frac{1}{T_i} \) is dominated by the smaller:
The discrepancy factor \( D_i = \frac{1}{1 + \frac{1}{T_i}} \) or \( \frac{1}{D_i} = \frac{1}{1 + \frac{1}{T_i}} \) or \( D_i = \frac{1}{T_i} \)

is dominated by the smaller:
The discrepancy factor \( D_i = \frac{1}{1 + \frac{1}{T_i}} \) or \( \frac{1}{D_i} = \frac{1}{1 + \frac{1}{T_i}} \) or \( D_i = \frac{1}{T_i} \)

is dominated by the smaller:

\[
D_i = -13\,dB \frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{50kHz}} \frac{1 + \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{880MHz}}
\]

All these graphical constructions can be conducted symbolically to give the result for \( D_i \) in low entropy factored pole-zero form.
Final step: assemble $A_{v12}$ as the product of the buffered gain and the discrepancy factor:

$$A_{v12} = 85 \text{dB}$$

$$A_{v12}^\circ = 98 \text{dB}$$

$$A_{v12}^\circ A_{v2} = 98 \text{dB} \frac{1 - \frac{s}{2\pi} \frac{880 \text{MHz}}{1 + \frac{s}{2\pi} \frac{880 \text{MHz}}{50 \text{kHz}}} 1 - \frac{s}{2\pi} \frac{880 \text{MHz}}{1 + \frac{s}{2\pi} \frac{880 \text{MHz}}{3.2 \text{MHz}}}}{1 + \frac{s}{2\pi} \frac{880 \text{MHz}}{1 + \frac{s}{2\pi} \frac{880 \text{MHz}}{3.2 \text{MHz}}}}$$

$$A_{i1} = -40 \text{dB/dec}$$

$$D_i = -13 \text{dB}$$
The fact that $D_i$ is less than 1 over most of the frequency range indicates that the second stage imposes heavy loading upon the first stage:

\[
D_i = \frac{Z_{i2}}{Z_{i2} + Z_{o1}}
\]
The fact that $D_i$ is less than 1 over most of the frequency range indicates that the second stage imposes heavy loading upon the first stage:

\[ D_i = \frac{Z_{i2}}{Z_{i2} + Z_{o1}} \]

This suggests that the first stage behaves more like a current source than a voltage source, and therefore that the analysis might be better undertaken using the dual form of the DT.
The Chain Theorem (CT)

The gain \( A_{v12} \equiv \frac{v_o}{e_i} \) is given by the DT:

\[
A_{v12} = A_{v12}^{v_y} \frac{1 + \frac{1}{T_{nv}}}{1 + \frac{1}{T}}
\]

The TF \( T_{nv} \equiv \frac{v_y}{v_x} \big|_{v_o=0} \) is an ndi calculation with the output \( v_o \) nulled.

If \( v_o \) is nulled, so is \( v_x \), so \( T_{nv} = \infty \).

This implies that \( T_{nv} \) is infinite unless the signal can bypass the injection point.
The Chain Theorem (CT)

Nulled \( v_y \) means that the \( A_{v1} \) box is shorted, so the input current to the \( A_{v2} \) box is the short-circuit (sc) output current of the \( A_{v1} \) box.

Thus, \( A_{v_{12}}^{v_y} = Y_{t1}^{sc} Z_t^2 \) is the \textit{current-buffered gain} of the two stages.
Also

\[ T_v = \left. \frac{v_y}{v_x} \right|_{e_i=0} = \left. \frac{iZ_{o1}}{iZ_{i2}} \right|_{e_i=0} = \frac{Z_{o1}}{Z_{i2}}, \]

so the DT becomes

\[ A_{v12} = \gamma_{t1}^{sc} Z_{t2} \frac{Z_{o1}}{Z_{i2} + Z_{o1}} \]

This can be interpreted as

\[
\begin{bmatrix}
\text{gain of the two stages} \\
\text{current buffered gain of the two stages}
\end{bmatrix}
\times
\begin{bmatrix}
\text{current loading factor between the two stages}
\end{bmatrix}
\]
The Chain Theorem (CT)
This is exactly the result that would be obtained directly from the model:

\[
A_{v12} = \gamma_{t1} \tau Z_{t2} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}
\]

\[v_o = A_{v12} e_i\]
The Chain Theorem (CT)
A useful application of the DT with $T_{nv} = \infty$ is to assemble the properties of a 2-stage amplifier from the properties of each separate stage.

$$v_o = A_{v12}e_i$$
The Chain Theorem (CT)

This "Divide and Conquer" approach avoids analysis of both stages simultaneously.
The Chain Theorem (CT)

\[ A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{1}{1 + \frac{1}{T_v}} = Y_{t1}^{sc} Z_{t2} D_v \]

\[ T_v \equiv \frac{Z_{o1}}{Z_{i2}} \quad D_v \equiv \frac{1}{1 + \frac{1}{T_v}} = \frac{T_v}{1 + T_v} = \frac{Z_{o1}}{Z_{i2} + Z_{o1}} \]

where \( Y_{t1}^{sc} Z_{t2} \) is the "current buffered" gain that would occur if there were a buffer between the two stages, and \( D_v \) is a "discrepancy factor" that accounts for the interaction between the two stages which results from the loading of the first stage by the input of the second stage.
The Chain Theorem (CT)

\[ A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{1}{1 + \frac{1}{T_v}} = Y_{t1}^{sc} Z_{t2} D_v \]

Since all TFs will be in factored pole-zero form, the only place where additional approximation may be needed resides inside the \( D_v \), where the sum of two TFs is required.

"Doing the algebra on the graph" can be conducted in two ways:

1. \( 1 + T_v \) can be found as the sum of the TFs 1 and \( T_v \), dominated by the larger;
2. \( D_v \) can be found from \( \frac{1}{D_v} = 1 + \frac{1}{T_v} = \frac{1}{1 + \frac{1}{T_v}} \) as the reciprocal sum of 1 and \( T_v \), dominated by the smaller.
\[ \gamma_t^{sc} = \frac{A_v}{Z_o} = \frac{1}{r_m} \left( 1 - sC_t r_m \right) \]

\[ = -31dB \left( 1 - \frac{s/2\pi}{880MHz} \right) \]

\[ Z_t = Z_i A_v = \frac{R_B R_L}{r_m} \left( \frac{1 - sC_t r_m}{1 + sC_t R_L \frac{R_B}{R_B || r_m || R_L}} \right) \]

\[ = 118dB \left( 1 - \frac{s/2\pi}{880MHz} \right) \left( 1 + \frac{s/2\pi}{39kHz} \right) \]

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9. The DT & the CT
The DT gives

$$A_{v12} = \gamma_{t1}^{sc} Z_{t2} D_v \quad (T_{nv} = \infty)$$

The current buffered gain $\gamma_{t1}^{sc} Z_{t2}$ is the product of the two separate gains:

$$\gamma_{t1}^{sc} Z_{t2} = 87dB \left(1 - \frac{s/2\pi}{880MHz}\right)^2 \left(1 + \frac{s/2\pi}{39kHz}\right)$$
\[ \gamma_{t1}^s Z_{t2} = 87 dB \frac{\left(1 - \frac{s/2\pi}{880 \text{MHz}}\right)^2}{\left(1 + \frac{s/2\pi}{39 \text{kHz}}\right)} \]
$Z_{i2} = 69dB \frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$

$Z_{o1} = 80dB \frac{1}{1 + \frac{s/2\pi}{3.2MHz}}$

$Z_{o1}$ and $Z_{i2}$ are the same, but note that $T_v = \frac{Z_{o1}}{Z_{i2}}$ is the reciprocal of $T_i = \frac{Z_{i2}}{Z_{o1}}$.
\[ T_v = \frac{Z_{o1}}{Z_{i2}} = 11dB \left(1 + \frac{s/2\pi}{39kHz}\right) \left(1 + \frac{s/2\pi}{3.2MHz}\right)^2 \]

\( Z_{o1} \) and \( Z_{i2} \) are the same, but note that \( T_v = \frac{Z_{o1}}{Z_{i2}} \) is the reciprocal of \( T_i = \frac{Z_{i2}}{Z_{o1}} \).
The discrepancy factor \( D_v = \frac{1}{1 + \frac{1}{T_v}} \) or \( \frac{1}{D_v} = \frac{1}{1 + \frac{1}{T_v}} \) or \( D_v = 1 \| T_v \)
is dominated by the smaller:

\[
D_v = -11dB \left( \frac{1 + \frac{s/2\pi}{39kHz}}{1 + \frac{s/2\pi}{50kHz}} \right) \left( 1 + \frac{s/2\pi}{880MHz} \right)
\]

All these graphical constructions can be conducted symbolically to give the result for \( D_v \) in low entropy factored pole-zero form.
Final step: assemble $A_{v12}$ as the product of the buffered gain and the discrepancy factor:

$$A_{v12} = 85 dB \left( 1 - \frac{s/2\pi}{880 MHz} \right) \left( 1 + \frac{s/2\pi}{50 kHz} \right)$$

$$= 85 dB \left( 1 - \frac{1}{880 MHz} \right) \left( 1 + \frac{1}{50 kHz} \right)$$
Final step: assemble $A_{v12}$ as the product of the buffered gain and the discrepancy factor:

$$A_{v12} = 85dB \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{50kHz}} \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{880MHz}}$$

The fact that $D_v$ is close to 1 over most of the frequency range confirms the expectation that the first stage behaves more like a current source than a voltage source.
Summary:

The DT allows assembly of the properties of a 2-stage amplifier from the properties of each separate stage.

This can be done by injection of either a test current $j_z$ or a test voltage $e_z$ at the interface:

$$A_{v12} = A_{v12}^{iy} D_i$$

where

$$A_{v12}^{iy} = A_{v12}^{oc} A_{v2}$$

is the voltage buffered gain

and

$$D_i = \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$

are the discrepancy factors representing the interface loading.

$$A_{v12} = A_{v12}^{vy} D_v$$

where

$$A_{v12}^{vy} = Y_{t1}^{sc} Z_{t2}$$

is the current buffered gain

and

$$D_v = \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$
In principle, this procedure can be extended to the addition of extra stages:

In practice, this procedure becomes cumbersome because the discrepancy factor for the first interface changes when a second interface is added.
However, there is an alternative form for the gain of 2 stages that circumvents this problem.

The DT results already obtained are:

\[
A_{v12} = A_{v12}^i y D_i = A_{v12}^i y \frac{Z_{i2}}{Z_{i2} + Z_{o1}} \quad A_{v12} = A_{v12}^v y D_v = A_{v12}^v y \frac{Z_{o1}}{Z_{i2} + Z_{o1}}
\]

Rewrite:

\[
\frac{1}{A_{v12}} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} = \frac{1}{A_{v12}} \frac{Z_{o1}}{A_{v12}} A_{v12}^i y = \frac{1}{A_{v12}} \frac{Z_{o1}}{Z_{i2} + Z_{o1}} = \frac{1}{A_{v12}} A_{v12}^v y
\]

Add the two:

\[
\frac{1}{A_{v12}} = \frac{1}{A_{v12}} A_{v12}^i y + \frac{1}{A_{v12}} A_{v12}^v y
\]
\[ \frac{1}{A_{v12}} = \frac{1}{A_{v12}^{iy}} + \frac{1}{A_{v12}^{vy}} \]

This simple and elegant result says that the interface discrepancy factors \( D_i \) and \( D_v \) are not needed, and the overall gain is a "parallel combination" of the two buffered gains:

\[ A_{v12} = A_{v12}^{iy} \parallel A_{v12}^{vy} \]

where \( A_{v12}^{iy} = A_{v1}^{oc} A_{v2} \) = voltage buffered gain of the 2 stages

and \( A_{v12}^{vy} = \gamma_{t1}^{sc} Z_{t2} \) = current buffered gain of the 2 stages

This result is actually the Chain Theorem (CT), and \( A_{v1}^{oc}, A_{v2}, \gamma_{t1}^{sc}, Z_{t2} \) are the (reciprocals of the) chain parameters (c parameters).
Rework the previous example:
Rework the previous example:

\[ \alpha = 1 \quad -A_{v1}v_S \]

\[ \alpha = 1 \quad A_{v12}v_S \]

\[ v_S \]

\[ R_B \quad 2.9k \]

\[ r_m \quad 36 \]

\[ C_t \quad 5p \]

\[ i_E \]

\[ R_L \quad 10k \]

\[ i_E \]

\[ i_E \]

\[ i_E \]

\[ i_E \]

\[ i_E \]

\[ i_x \]

\[ i_y \]

\[ Z_i \]

\[ Z_{o1} \]

\[ Z_{o2} \]

\[ Z_o \]

\[ A_{v1} A_{v2} = 98dB \]

\[ \frac{1 - \frac{s}{2\pi 880MHz}}{1 + \frac{s}{2\pi 3.2MHz}} \cdot \frac{1 - \frac{s}{2\pi 880MHz}}{1 + \frac{s}{2\pi 3.2MHz}} \]

\[ 98dB \]

\[ 3.2MHz \]

\[ 880MHz \]
Rework the previous example:

\[ A_{v1}v_S \]

\[ A_{v12}v_S \]

\[ v_S \]

\[ Z_i \]

\[ C_t \]

\[ R_B \]

\[ r_m \]

\[ v_{ya} \]

\[ v_x \]

\[ R_L \]

\[ 10k \]

\[ 2.9k \]

\[ 36 \]

\[ 5p \]

\[ 5p \]

\[ 2.9k \]

\[ 36 \]

\[ \alpha = 1 \]

\[ \alpha = 1 \]

\[ v_{ya} = -A_{v1}v_S \]

\[ v_x = A_{v12}v_S \]

\[ Z_{o1} \]

\[ Z_{o2} \]

\[ A_{v1}A_{v2} = 98dB \]

\[ 87dB \]

\[ 98dB \]

\[ 3.2MHz \]

\[ 87dB \]

\[ 39kHz \]

\[ 880MHz \]

\[ Y_t^{sc}Z_{t2} = 87dB \]

\[ v_{0.1} 3/07 \]

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9. The DT & the CT
Rework the previous example:

\[ A_{v1} = \frac{1 - s/2\pi}{880\text{MHz}} \cdot \frac{1 - s/2\pi}{3.2\text{MHz}} \]

\[ A_{v2} = \frac{1 - s/2\pi}{880\text{MHz}} \cdot \frac{1 - s/2\pi}{3.2\text{MHz}} \]

\[ A_{v1} A_{v2} = 98\text{dB} \]

\[ Y_{t1} Z_{t2} = 87\text{dB} \frac{(1 - s/2\pi)^2}{(1 + s/2\pi/39\text{kHz})} \]
The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

The procedure is:

Find \( A_{v1}^{oc} \) and \( Y_{t1}^{sc} \) of stage 1, and \( Z_{t2}^{oc} \) and \( A_{v2}^{oc} \) of stage 2. Combine them by the CT to find \( A_{v12}^{oc} \), as above,
The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

The procedure is:

Find $A_{v1}^{oc}$ and $\gamma_{t1}^{sc}$ of stage 1, and $Z_{t2}^{oc}$ and $A_{v2}^{oc}$ of stage 2. Combine them by the CT to find $A_{v12}^{oc}$, as above, and hence find $\gamma_{t12}^{sc}$. 

\[ \text{Diagram: } \]
The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

The procedure is:

Find $A_{v1}^{oc}$ and $\gamma_{t1}^{sc}$ of stage 1, and $Z_{t2}^{oc}$ and $A_{v2}^{oc}$ of stage 2.

Combine them by the CT to find $A_{v12}^{oc}$, as above, and hence find $\gamma_{t12}^{sc}$.

Find $Z_{t3}^{oc}$ and $A_{v3}^{oc}$ of stage 3.

Combine them by the CT to find $A_{v123}^{oc}$.

and so on...