1. MOTIVATION AND BACKGROUND

The Design-Oriented Analysis (D-OA) Paradigm

Premise:

Not much of what you learned in school has turned out to be much use

Premise:

Not much of what you learned in school has turned out to be much use

So why are you here, listening to another professor?

Premise:

Not much of what you learned in school has turned out to be much use

So why are you here, listening to another professor?

Because you are going to see a completely different approach, from the get-go:

Premise:

Not much of what you learned in school has turned out to be much use

So why are you here, listening to another professor?

Because you are going to see a completely different approach, from the get-go:

You don't solve equations simultaneously; instead:

You solve them sequentially

Premise:

Not much of what you learned in school has turned out to be much use

So why are you here, listening to another professor?

Because you are going to see a completely different approach, from the get-go:

You don't solve equations simultaneously; instead:
You solve them sequentially

An Engineer's story

Premise:

Not much of what you learned in school has turned out to be much use

So why are you here, listening to another professor?

Because you are going to see a completely different approach, from the get-go:

You don't solve equations simultaneously; instead:

You solve them sequentially

An Engineer's story

Falling off a cliff

Most of us "fall off a cliff" when we begin our first job

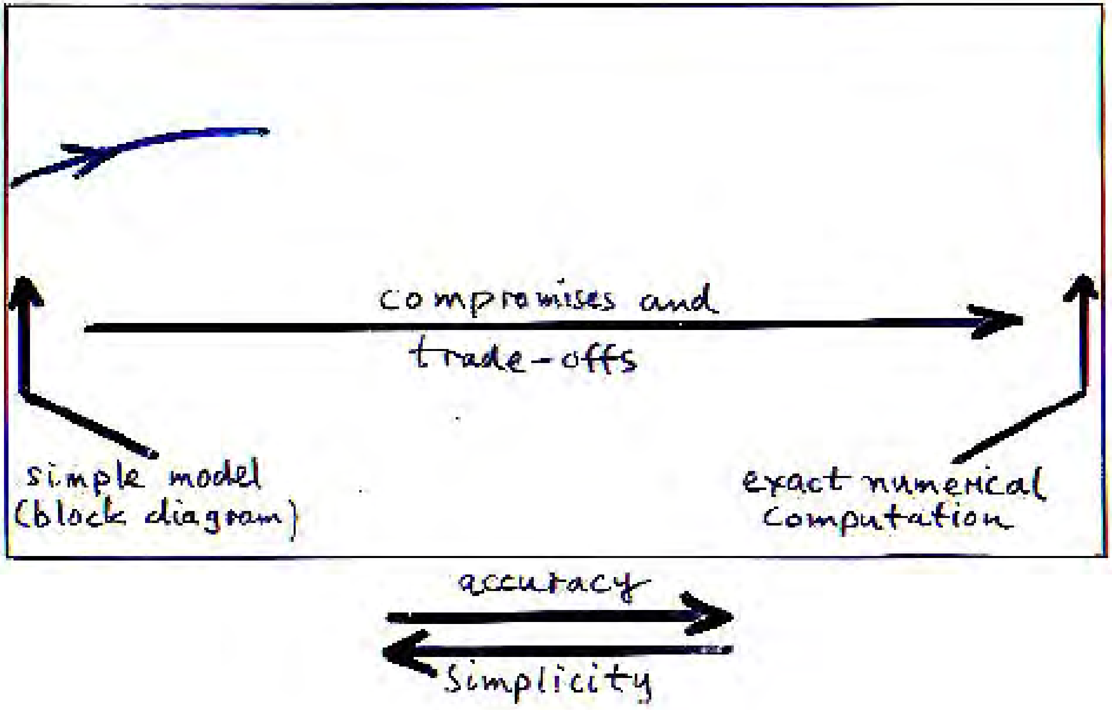
Problem:

New graduate engineers are unable to translate the principles and methods they have learned to the real world.

What can be done?

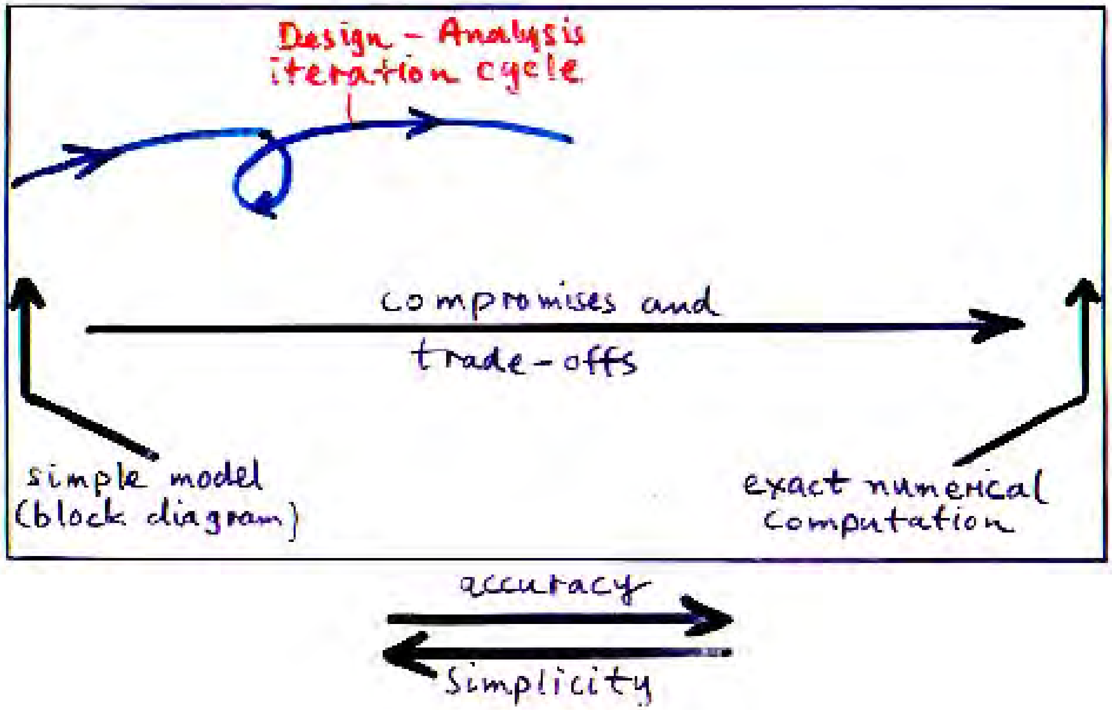
The Design Process

Design iteration loops



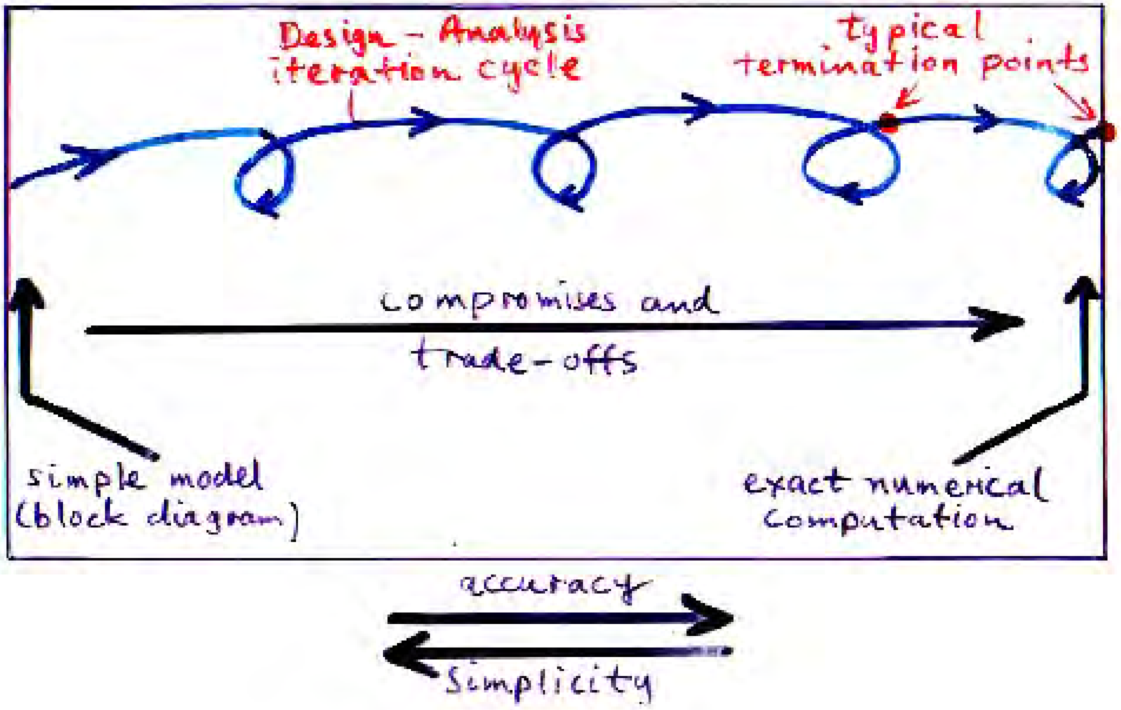
The Design Process

Design iteration loops

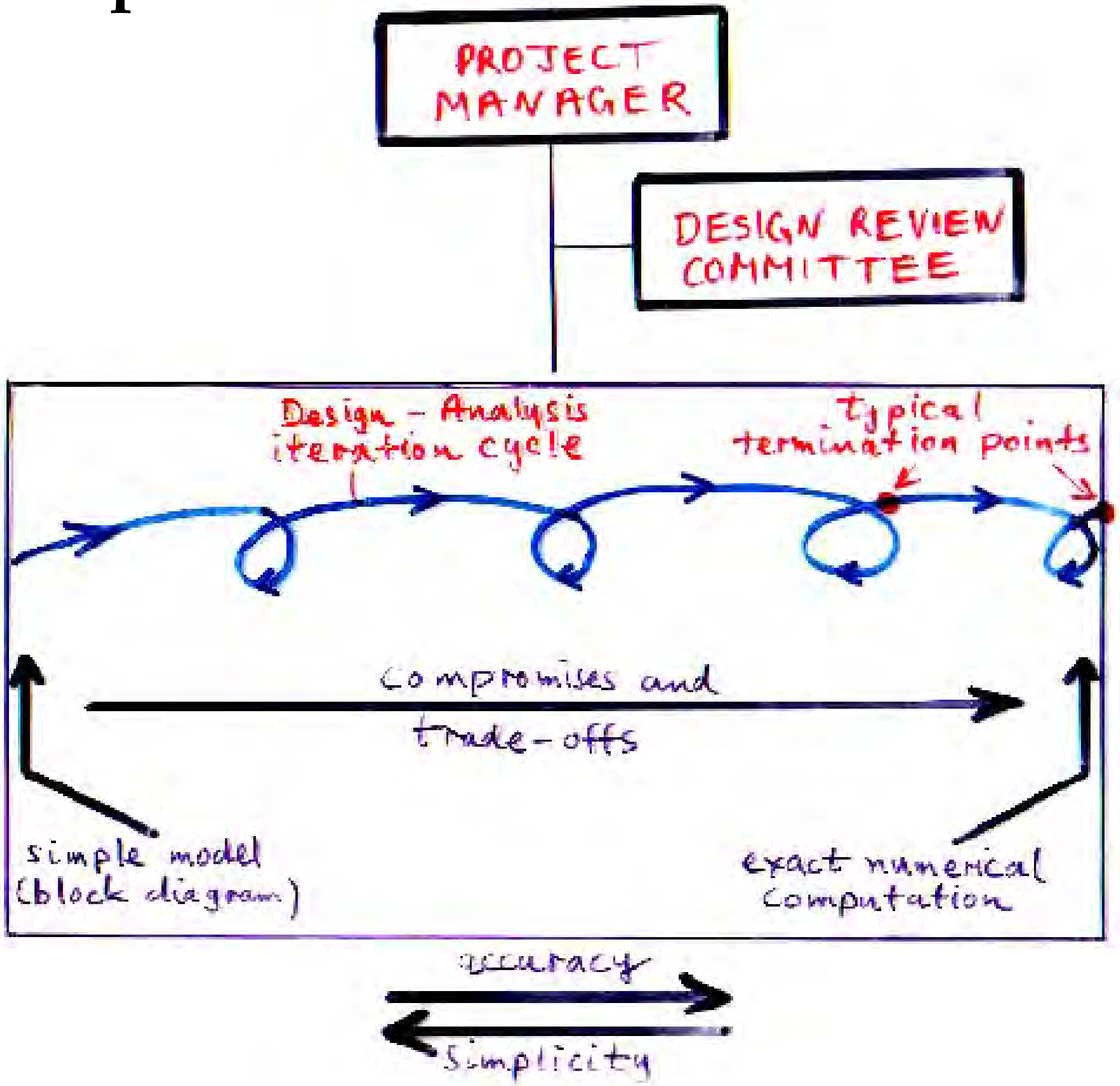


The Design Process

Design iteration loops



The design process consists of a succession of iteration loops:



"How to present the results" is important:

1. If you are a design engineer writing a report or appearing before a design review committee;

2. If you are a Test, Reliability, or System Integration Engineer dealing with someone else's design.

The D-OA approach is valuable for *all* these engineers.

Realization:

Design is the Reverse of Analysis,

Realization:

Design is the Reverse of Analysis, because:

The Starting Point of the Design Problem

(the Specification)

Realization:

Design is the Reverse of Analysis, because:

The Starting Point of the Design Problem

(the Specification) is the

Answer to the Analysis Problem

Conventional problem-solving approach:

1. Put everything into the model and simplify later.

2. Postpone approximation as long as possible, and don't even dare to make an approximation unless you can justify it on the spot.

3. The "answer" is acceptable in whatever form it emerges from the algebra.

4. The more work you do, the more valuable the result.

5. Every problem is a brand-new problem, and requires a brand-new strategy to solve it.

This is a recipe for failure!

Syndromes of Technical Disability:

Algebraic diarrhoea, which leads to

Algebraic paralysis

Fear of approximation

Syndromes of Technical Disability:

Algebraic diarrhoea, which leads to
Algebraic paralysis
Fear of approximation

The negative results of the conventional paradigm are often masked while the student is in school.

Why does the conventional approach fail?

Mathematicians tell us:

# of equations must = # of unknowns

Engineers face:

# of equations << # of unknowns

but have to solve the problem, anyway.

How can we overcome the negative results of the conventional approach?

1. Divide and Conquer:

It's easier to solve many simpler problems than one large one.

2. We must make the equations we do have *work harder* by expressing them in "Low Entropy" form.

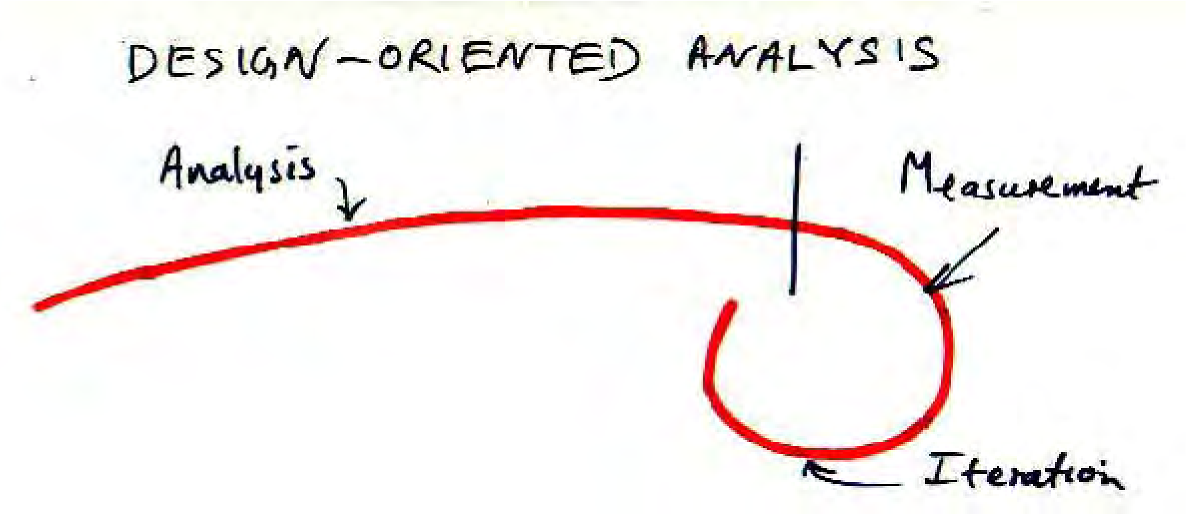
A *High Entropy Expression* is one in which the arrangement of terms and element symbols conveys no information other that obtained by substitution of numbers.

2. We must make the equations we do have *work harder* by expressing them in "Low Entropy" form.

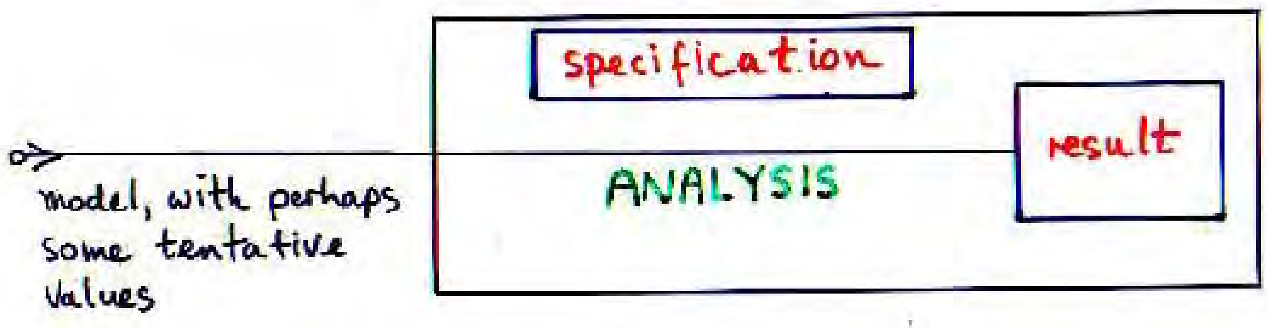
A *High Entropy Expression* is one in which the arrangement of terms and element symbols conveys no information other that obtained by substitution of numbers.

A *Low Entropy Expression* is one in which the terms and element symbols are ordered and grouped so that their physical origin and relative importance are apparent. Only in this way can one *change* the values in an informed manner in order to *change* the analysis answer (that is, to make it meet the Specification).

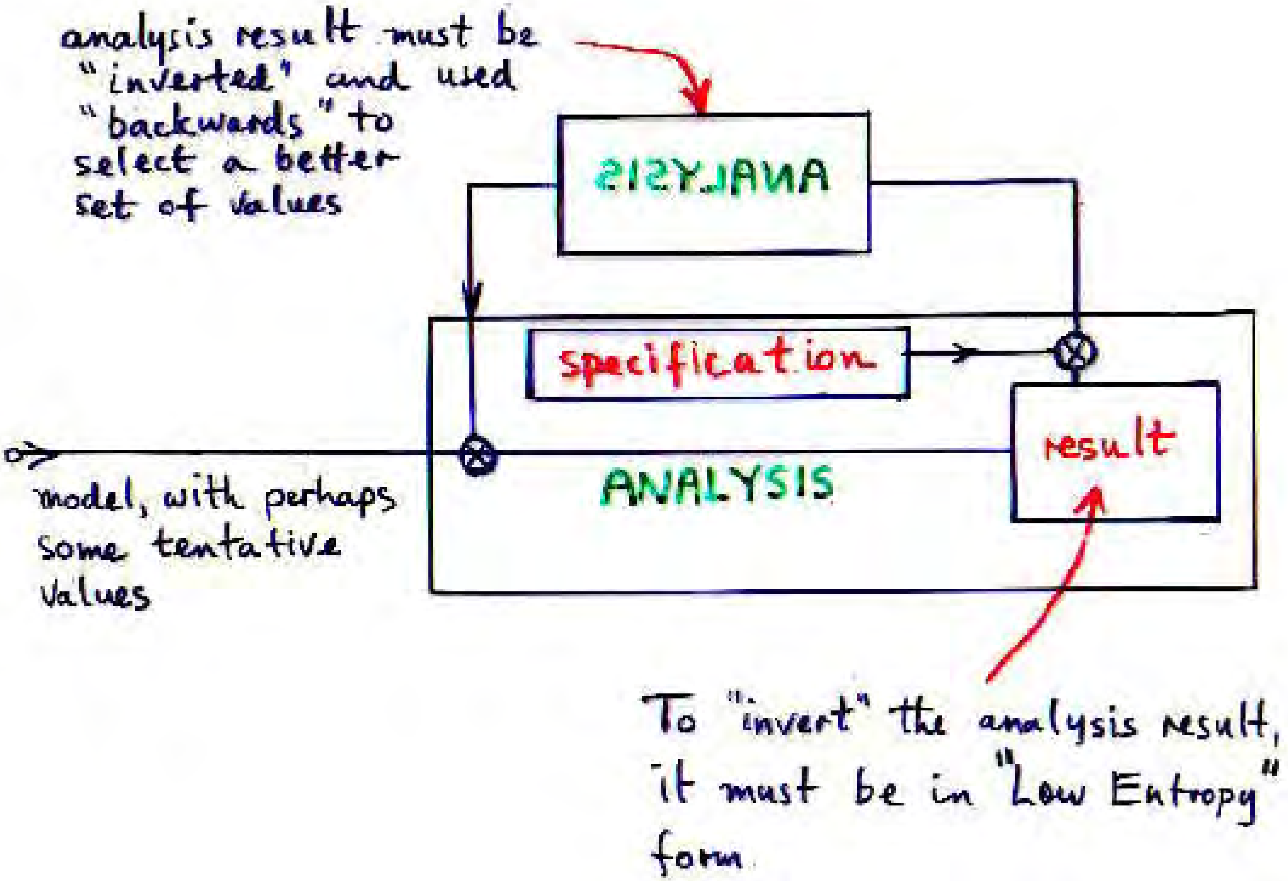
Low Entropy Expressions are essential in order to navigate the Design Iteration Loop:



Low Entropy Expressions are essential in order to navigate the Design Iteration Loop:



Low Entropy Expressions are essential in order to navigate the Design Iteration Loop:



analysis result must be "inverted" and used "backwards" to select a better set of values

model, with perhaps some tentative values

To "invert" the analysis result, it must be in "Low Entropy" form

Design-Oriented Analysis (D-OA) keeps the entropy low at every step along the way to a low entropy result.

Design-Oriented Analysis (D-OA) keeps the entropy low at every step along the way to a low entropy result.

The way to do this is:

Avoid solving simultaneous equations.

Design-Oriented Analysis (D-OA) keeps the entropy low at every step along the way to a low entropy result.

The way to do this is:
Avoid solving simultaneous equations.

Instead, follow the signal path from input to output by Thevenin/Norton reduction and/or voltage/current dividers.

Design-Oriented Analysis (D-OA) keeps the entropy low at every step along the way to a low entropy result.

The way to do this is:
Avoid solving simultaneous equations.

Instead, follow the signal path from input to output by Thevenin/Norton reduction and/or voltage/current dividers.

There may be many such paths (algorithms), each of which gives a different Low Entropy Expression.

Design-Oriented Analysis (D-OA) keeps the entropy low at every step along the way to a low entropy result.

The way to do this is:
Avoid solving simultaneous equations.

Instead, follow the signal path from input to output by Thevenin/Norton reduction and/or voltage/current dividers.

There may be many such paths (algorithms), each of which gives a different Low Entropy Expression.

Avoid multiplying out the series/parallel combinations.

How can we overcome the negative results of the conventional approach?

How can we overcome the negative results of the conventional approach?

3. Recognize that we *don't want* an exact answer:
it would be too complicated to use, even if we could get it.

How can we overcome the negative results of the conventional approach?

3. Recognize that we *don't want* an exact answer:
it would be too complicated to use, even if we could get it.

> **Therefore, subsititute for the missing equations with: *inequalities, approximations, assumptions,* and *tradeoffs*.**

D-OA problem-solving approach: D-OA Rules

1. **Put only enough into the model to get the answer you need.**

2. **Make all the approximations you can, as soon as you can, justified or not. Plow through the problem leaving behind you a wake of assumptions and approximations.** ***You can't lose by trying*****.**

3. **Figure out in advance as many of the quantities as you can that you want to have in the answer, and put them into the statement of the problem as soon as possible –** ***even into the circuit model.***

4. **The less work you do, the more valuable the result.** ***You*** **control the algebra. You** ***make*** **the algebra come out in low entropy form by applying strategic mental energy before and during the math.**

5. **Every problem in not unique. There are problem solving strategies that apply to almost all engineering problems.**

This is a recipe for success!

The benefits of applying the D-OA Rules are:

1. You can *fend off* algebraic paralysis.

The benefits of applying the D-OA Rules are:

1. You can *fend off* algebraic paralysis.

2. Approximations are *good* things, not an admission of defeat.

The benefits of applying the D-OA Rules are:

1. You can *fend off* algebraic paralysis.

2. Approximations are *good* things, not an admission of defeat.

3. Algebra is *malleable*; you have *choices*.
You are *empowered* to exercise control: the math is your *slave*, not your master.

D-OA is the only kind of analysis worth doing!

TOPIC STRUCTURE

Getting Results:

Low Entropy Expressions
Ch 2

Presenting Results
Ch 3 Ch 4 Ch 5

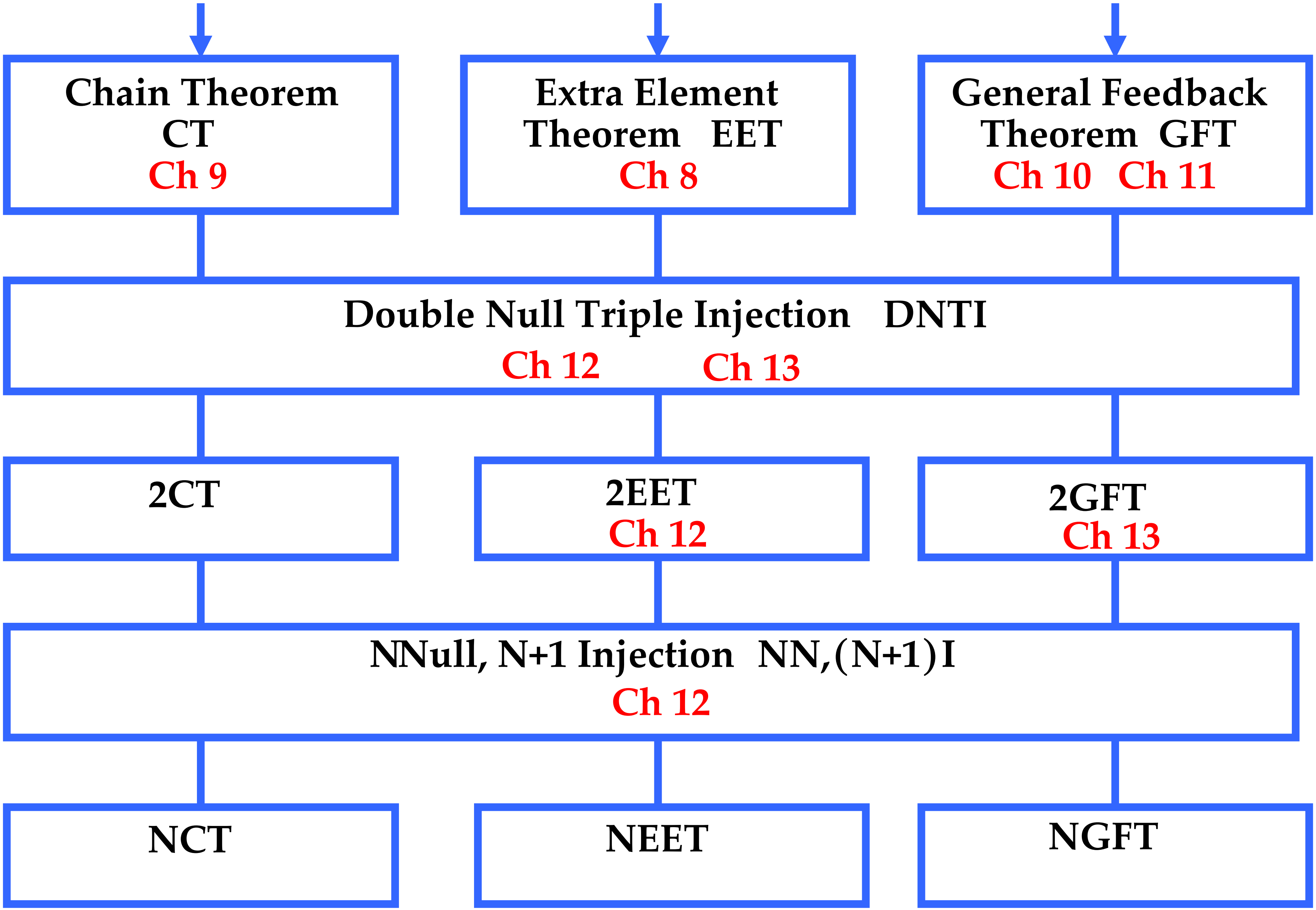
Combining Results
Ch 6

Extending Results:

Input/Output Impedance Theorem **I/O IT**
Ch 7

Null Double Injection **NDI**
Ch 8

Dissection Theorem **DT**
Ch 9



Chain Theorem CT Ch 9

Extra Element Theorem EET Ch 8

General Feedback Theorem GFT Ch 10 Ch 11

Double Null Triple Injection DNTI Ch 12 Ch 13

2CT

2EET Ch 12

2GFT Ch 13

NNull, N+1 Injection NN,(N+1)I Ch 12

NCT

NEET

NGFT

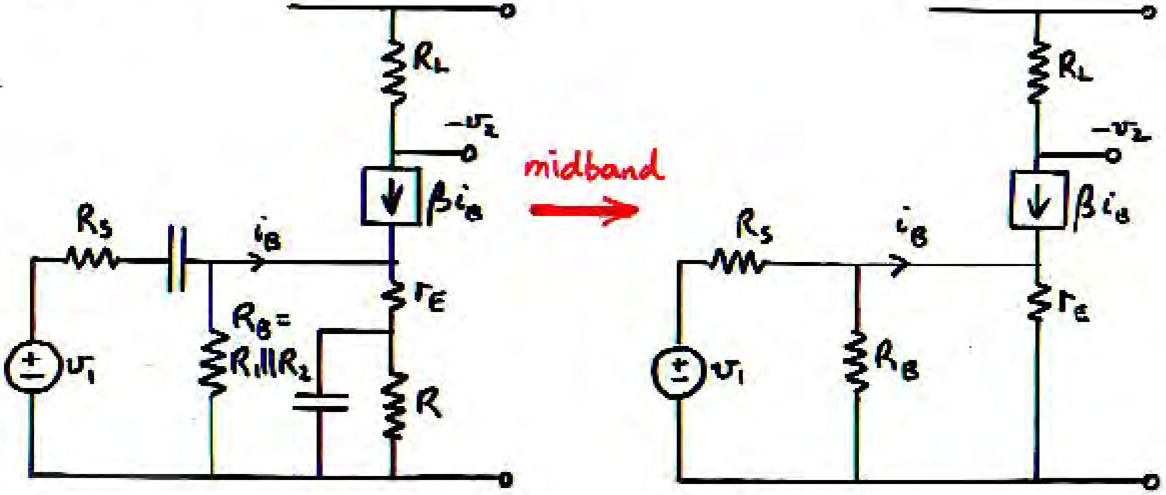
Examples: Chs 14 – 18

2. LOW ENTROPY EXPRESSIONS

The Key to D-OA

Conventional analysis

Gain of CE amplifier



"Midband" means frequencies at which reactive effects are negligible

The "brute-force" method: loop analysis

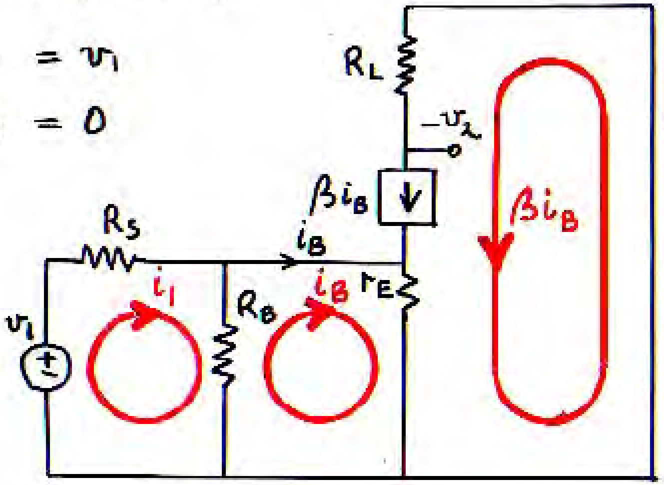
$$(R_s + R_B)\, i_1 \qquad - R_B\, i_B = v_1$$

$$-R_B\, i_1 + [R_B + (1+\beta) r_E]\, i_B = 0$$

$$i_B = \frac{\begin{vmatrix} R_s+R_B & v_1 \\ -R_B & 0 \end{vmatrix}}{\begin{vmatrix} R_s+R_B & -R_B \\ -R_B & R_B+(1+\beta)r_E \end{vmatrix}}$$

$$= \frac{R_B v_1}{(R_s+R_B)[R_B+(1+\beta)r_E] - R_B^2}$$

$$= \frac{R_B v_1}{R_s R_B + (1+\beta) r_E R_s + R_B^2 + (1+\beta) r_E R_B - R_B^2}$$



Finally, $v_2 = R_L \beta i_B$
which leads to:

$$A_m \equiv \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta) r_E R_s + (1+\beta) r_E R_B + R_s R_B}$$

This is a high entropy expression

Apply mental energy to:

Lower the Entropy of the result:

$$A_m = \frac{\beta R_B R_L}{(1+\beta) r_E R_s + (1+\beta) r_E R_B + R_s R_B}$$

$$= \frac{\beta R_B R_L}{(1+\beta) r_E (R_s + R_B) + R_s R_B}$$

$$= \frac{R_B}{R_s + R_B} \cdot \frac{\beta R_L}{(1+\beta) r_E + R_s \| R_B}$$

$$= \frac{R_B}{R_s + R_B} \cdot \frac{\alpha R_L}{\underbrace{r_E}_{(1)} + \underbrace{(R_s \| R_B)/(1+\beta)}_{(2)}}$$

The Low Entropy result exposes the following additional information, not apparent from the High Entropy version:

(a) The $R_B/(R_S+R_B)$ factor is identified as a voltage divider;

(b) Resistances appear in series/parallel combinations, so it is clear which ones are dominant;

(c) The relative values of the two terms labeled (1) and (2) determine the sensitivity of the gain A to variations of β.

The additional information makes possible a much better informed choice of element values.

Disadvantages of the "brute-force" method:

1. No direct physical interpretation of the result.
2. Obscures relationships as to how element values affect the result.
3. Difficult to use for design: given A_m (the Specification), how do you choose element values?
4. Purely algebraic derivation increases likelihood of mistakes.

Advantages of the Low-Entropy form of the result:

1. Direct physical interpretation of the result.
2. Clarifies relationships as to how element values affect the result.
3. Easy to use for design: given A_m (the Specification), how do you choose element values?
4. ?

It is easier to keep the Entropy low from the start of the analysis than it is to lower the Entropy once it has increased.

The "brute-force" method: loop analysis

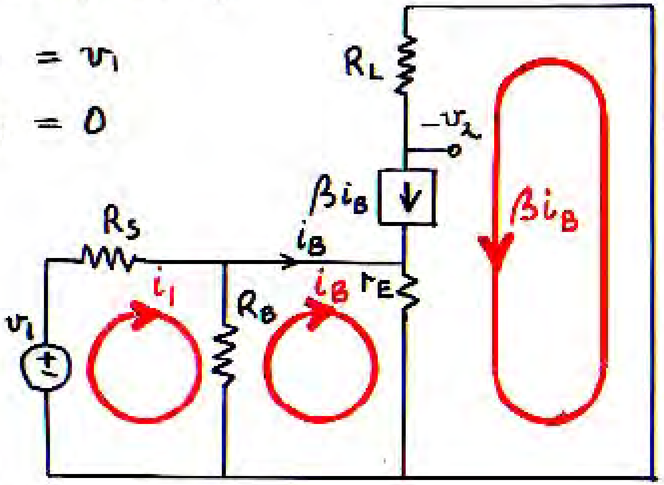
$$(R_s + R_B)\, i_1 \qquad - R_B \cdot i_B = v_1$$

$$-R_B\, i_1 + [R_B + (1+\beta) r_E]\, i_B = 0$$

$$i_B = \frac{\begin{vmatrix} R_s + R_B & v_1 \\ -R_B & 0 \end{vmatrix}}{\begin{vmatrix} R_s + R_B & -R_B \\ -R_B & R_B + (1+\beta) r_E \end{vmatrix}}$$

$$= \frac{R_B v_1}{(R_s + R_B)[R_B + (1+\beta) r_E] - R_B^2}$$

$$= \frac{R_B v_1}{R_s R_B + (1+\beta) r_E R_s + R_B^2 + (1+\beta) r_E R_B - R_B^2}$$



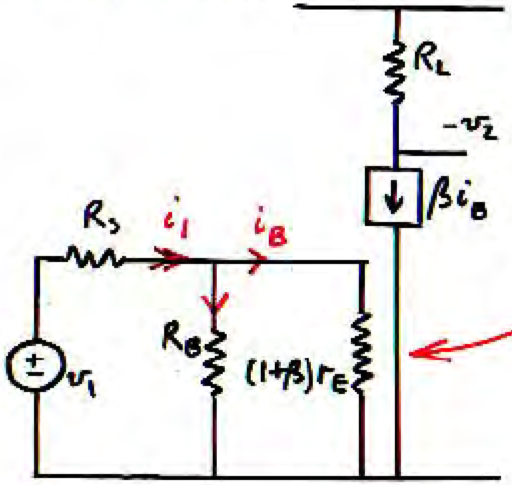
Finally, $v_2 = R_L \beta i_B$

which leads to:

$$A_m \equiv \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta) r_E R_s + (1+\beta) r_E R_B + R_s R_B}$$

Reflection of impedances

Better method #1:



"reflect" emitter impedance to the base

$$i_1 = \frac{v_1}{R_S + R_B \| (1+\beta) r_E}$$

Current divider:

$$\frac{i_B}{i_1} = \frac{\text{opposite branch impedance}}{\text{sum of branch impedances}}$$

$$v_2 = \beta R_L i_B$$

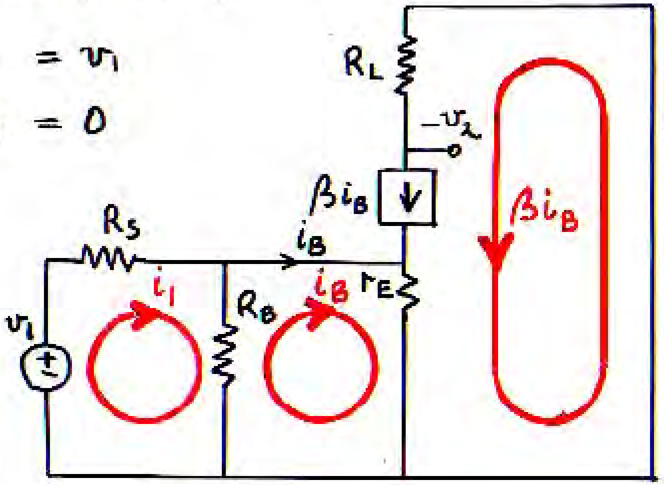
$$= \frac{R_B}{R_B + (1+\beta) r_E}$$

$$A_m \equiv \frac{v_2}{v_1} = \frac{R_B}{R_B + (1+\beta) r_E} \cdot \frac{\beta R_L}{R_S + R_B \| (1+\beta) r_E}$$

The "brute-force" method: loop analysis

$$(R_s + R_B)\, i_1 \qquad - R_B\, i_B = v_1$$

$$-R_B\, i_1 + [R_B + (1+\beta) r_E]\, i_B = 0$$



$$i_B = \frac{\begin{vmatrix} R_s + R_B & v_1 \\ -R_B & 0 \end{vmatrix}}{\begin{vmatrix} R_s + R_B & -R_B \\ -R_B & R_B + (1+\beta) r_E \end{vmatrix}}$$

$$= \frac{R_B v_1}{(R_s + R_B)[R_B + (1+\beta) r_E] - R_B^2}$$

$$= \frac{R_B v_1}{R_s R_B + (1+\beta) r_E R_s + R_B^2 + (1+\beta) r_E R_B - R_B^2}$$

Finally, $v_2 = R_L \beta i_B$

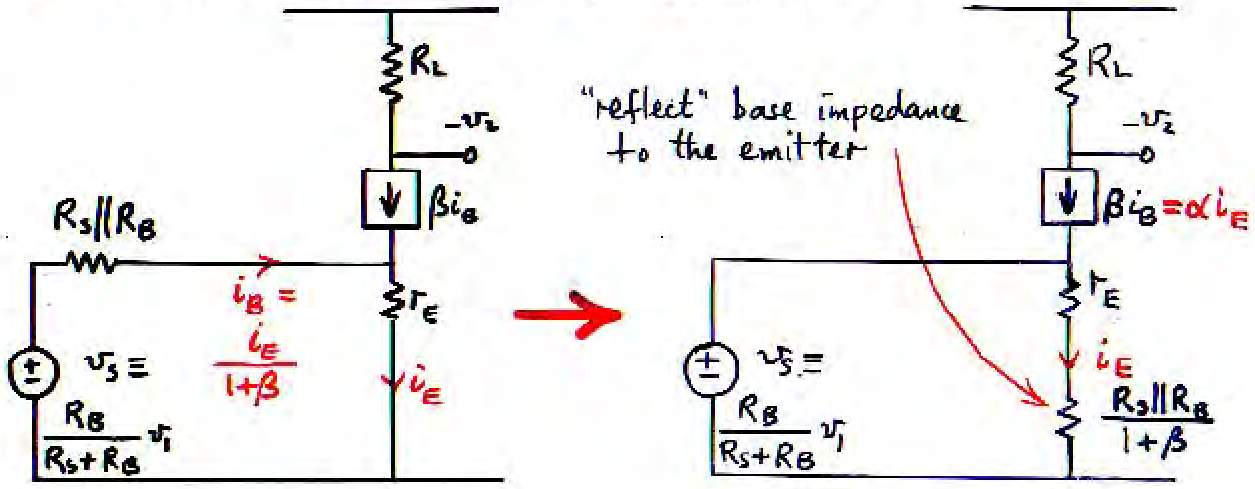
which leads to:

$$A_m \equiv \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta) r_E R_s + (1+\beta) r_E R_B + R_s R_B}$$

Thevenin/Norton

Doing the algebra on the circuit diagram

Better method: Use Thevenin's Theorem at the start



$$i_E = v_s \frac{1}{r_E + (R_s||R_B)/(1+\beta)} \qquad v_2 = R_L i_C = \alpha R_L i_E$$

$$\frac{v_2}{v_s} = \alpha \frac{R_L}{r_E + (R_s||R_B)/(1+\beta)} = \alpha \frac{\text{total collector load}}{\text{total emitter load including reflected base impedance}}$$

$$A_m \equiv \frac{v_2}{v_1} = \frac{R_B}{R_s+R_B} \cdot \frac{\alpha R_L}{r_E + (R_s||R_B)/(1+\beta)}$$

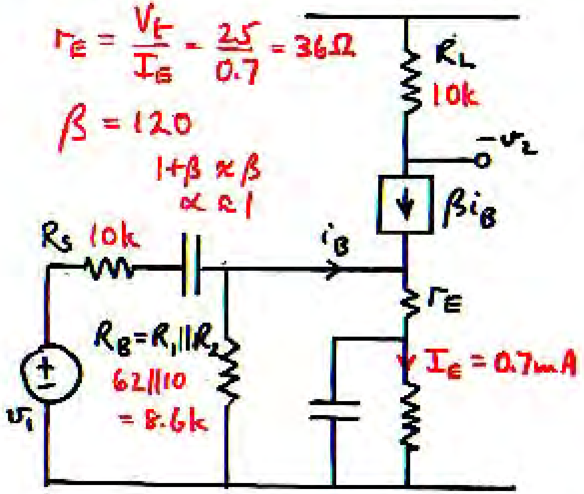
Example: Previous designed circuit

$$r_E = \frac{V_T}{I_E} = \frac{25}{0.7} = 36\Omega$$

$$\beta = 120$$

$$1+\beta \approx \beta$$

$$\alpha \approx 1$$



$$A_m = \frac{R_B}{R_s+R_B}\cdot\frac{\alpha R_L}{r_E + (R_s\|R_B)/(1+\beta)}$$

$$= \frac{8.6}{10+8.6}\cdot\frac{10}{0.036+\underbrace{(10\|8.6)/120}_{0.039}}$$

$$= 62 \Rightarrow 36\,dB$$

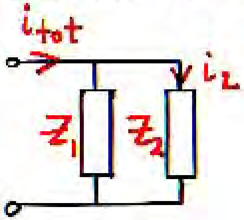
Don't leave out the penultimate line, because this is where the relative importance of the various element contributions is exposed!

The results for A_m by the two methods are of course the same, but the element contributions are grouped differently.

Any grouping contains more useful information about the relative contributions of the various elements than does the multiplied-out result obtained by the "brute-force" solution of simultaneous loop or node equations.

Generalization: Current and Voltage Dividers

Current divider

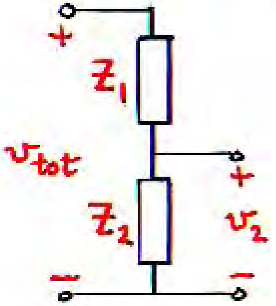


$$\frac{i_2}{i_{tot}} = \frac{Z_1}{Z_1 + Z_2}$$

$$\frac{\text{current in one branch}}{\text{total current}} = \frac{\text{opposite branch impedance}}{\text{sum of branch impedances}}$$

This is the dual of the:

Voltage divider

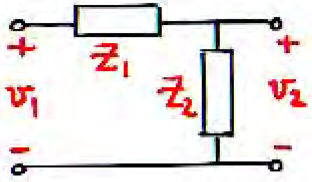


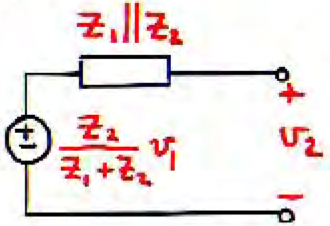
$$\frac{v_2}{v_{tot}} = \frac{Z_2}{Z_1 + Z_2}$$

$$\frac{\text{voltage at tap}}{\text{total voltage}} = \frac{\text{tap impedance to ground}}{\text{sum of impedances to ground}}$$

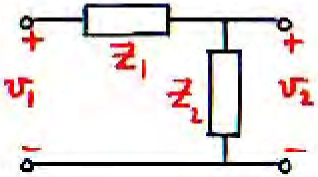
Generalization: Loop and Node Removal

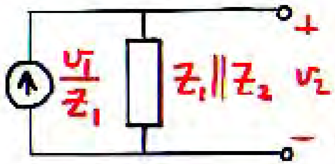
Every time Thevenin's theorem is used, one loop is removed from the circuit:





Every time Norton's theorem is used, one node is removed from the circuit:



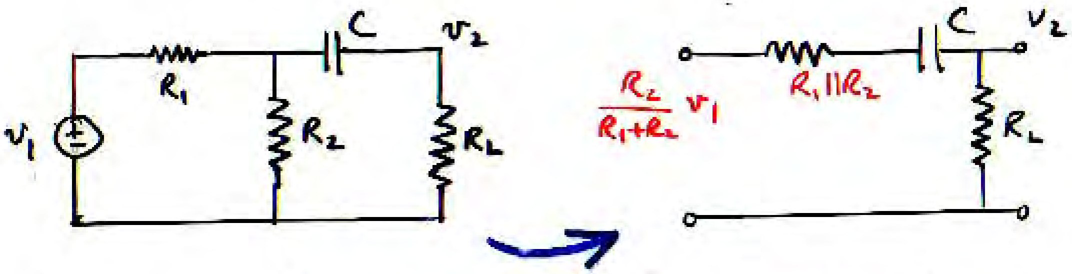


Generalization: Loop and node removal by Thevenin and Norton reduction

By successive use of the Thevenin and Norton theorems, a multi-loop, multi-node circuit can be reduced to a simple form from which the analytical results can be written by inspection.

This is an example of the powerful technique of doing the algebra on the circuit diagram

Another example:



This is how <u>element groupings</u> arise <u>naturally</u>, by circuit reduction through successive loop and node removal.

Generalization: Advantages of Doing the Algebra on the Circuit Diagram

1. Simultaneous solution of multiple loop or node equations is replaced by sequential, simple, semigraphical steps.
2. The element values in the successively reduced models automatically appear in usefully grouped combinations (to facilitate tradeoffs).
3. Less likelihood of making algebraic mistakes.
4. Because the physical origin of all terms in the analytic results remain explicit, the results are in optimum form for design: element values can be chosen so that the results meet the specifications.

BOTTOM LINE:

AVOID solving simultaneous equations.

Instead, follow the signal path from input to output by Thevenin/Norton reduction, voltage/current dividers, and reflection of impedances.

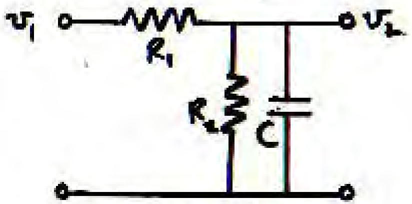
This automatically generates Low Entropy Expressions; AVOID multiplying out the series/parallel expressions.

There may be many such paths (algorithms), each of which gives a different Low Entropy Expression.

3. NORMAL AND INVERTED POLES AND ZEROS

How to choose the gain at any frequency as the Reference Gain

"Flat gain"

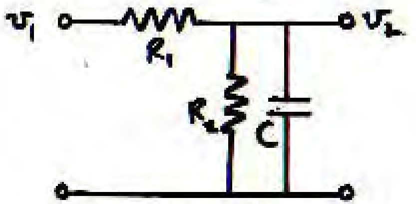


The hard way:

$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

"Flat gain"



The hard way:

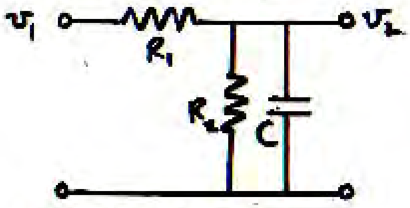
$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

This format is commonly considered to be "the answer."

However, it is much better to extract the constant term from both the numerator and denominator polynomials in *s*:

"Flat gain"



The hard way:

$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

This *normalizes* the polynomials, and exposes a zero-frequency gain and a corner frequency.

This is a special case of the general result as a ratio of polynomials in complex frequency s:

$$A = \frac{b_0 + b_1 s + b_2 s^2 + b_3 s^3 + \ldots}{a_0 + a_1 s + a_2 s^2 + a_3 s^3 + \ldots}$$

Extraction of the constant term from numerator and denominator defines the zero-frequency reference gain A_{ref} and normalizes the polynomials :

$$A = A_{ref} \frac{1 + \frac{b_1}{b_0} s + \frac{b_2}{b_0} s^2 + \frac{b_3}{b_0} s^3 + \ldots}{1 + \frac{a_1}{a_0} s + \frac{a_2}{a_0} s^2 + \frac{a_3}{a_0} s^3 + \ldots}$$

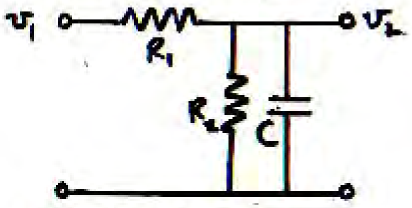
Factorization of the polynomials defines the poles and zeros, and hence the final (preferred) "factored pole-zero" form:

$$A = A_{ref}\frac{\left(1+\frac{s}{\omega_{z1}}\right)\left(1+\frac{s}{\omega_{z2}}\right)\left(1+\frac{s}{\omega_{z3}}\right)...}{\left(1+\frac{s}{\omega_{p1}}\right)\left(1+\frac{s}{\omega_{p2}}\right)\left(1+\frac{s}{\omega_{p3}}\right)...}$$

The reference gain and the poles and zeros should, of course, be low entropy expressions in terms of the circuit elements.

Return to the example:

"Flat gain"



The hard way:

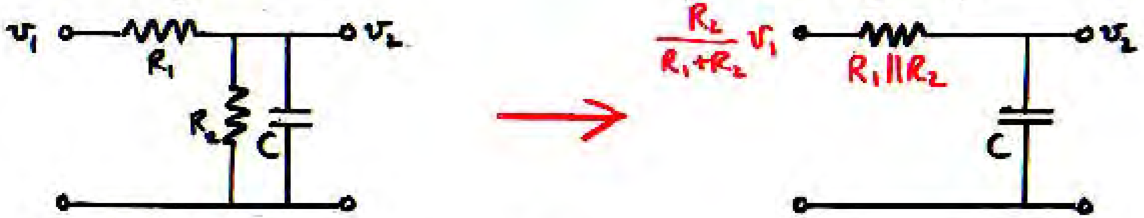
$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

"Flat gain"

The easy way:



The hard way:

$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

$$\frac{v_2}{v_1} = \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

"Flat gain"

The easy way:

The hard way:

$$\frac{v_2}{v_1} = \frac{R_2 \| \frac{1}{sC}}{R_1 + R_2 \| \frac{1}{sC}} = \frac{\frac{R_2}{1+sCR_2}}{R_1 + \frac{R_2}{1+sCR_2}}$$

$$= \frac{R_2}{R_1 + R_2 + sCR_1R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

$$\frac{v_2}{v_1} = \frac{R_2}{R_1+R_2} \cdot \frac{1}{1+sC(R_1 \| R_2)}$$

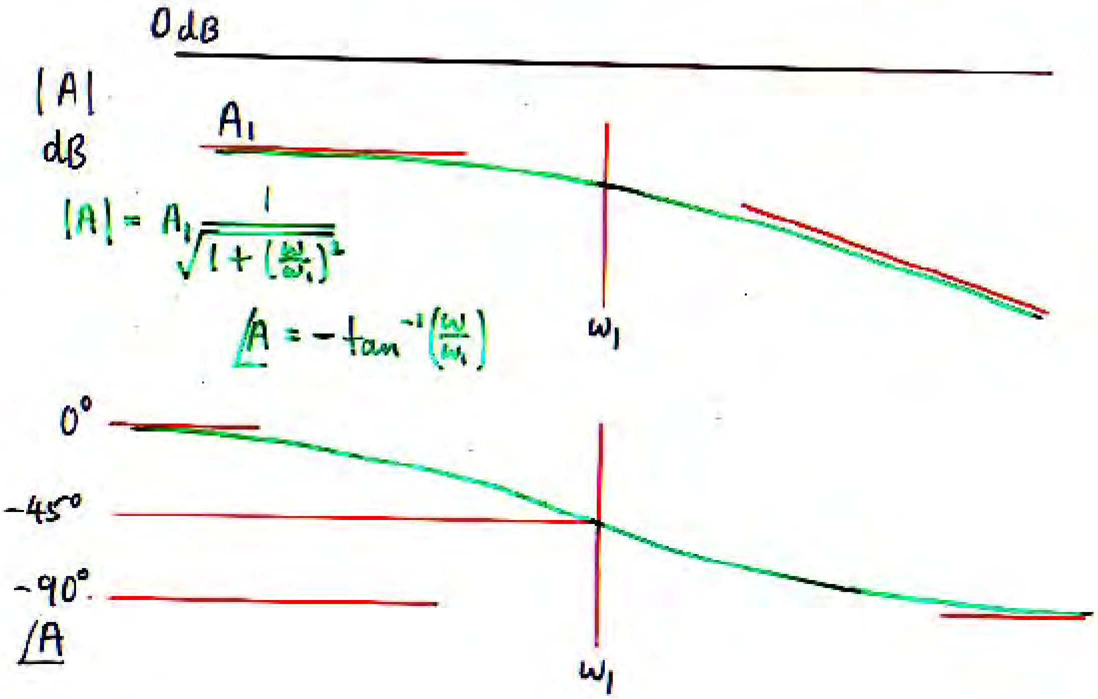
Result:

$$\frac{v_2}{v_1} \equiv A = A_1 \frac{1}{1+\frac{s}{\omega_1}} \quad \text{where} \quad A_1 \equiv \frac{R_2}{R_1+R_2} \qquad \omega_1 \equiv \frac{1}{C(R_1 \| R_2)}$$

Single-pole response:

$$A = A_1 \frac{1}{1 + \frac{s}{\omega_1}}$$

flat gain ↗ ↖ normal pole



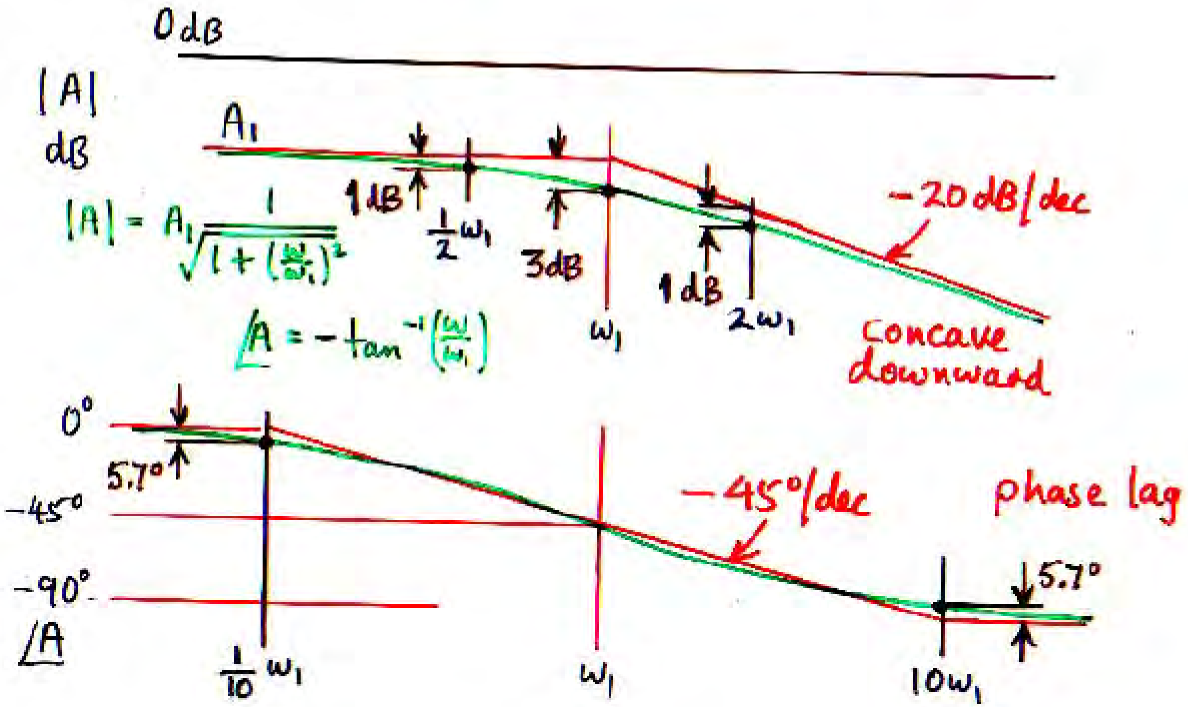
$$|A| = A_1 \frac{1}{\sqrt{1 + \left(\frac{\omega}{\omega_1}\right)^2}}$$

$$\angle A = -\tan^{-1}\left(\frac{\omega}{\omega_1}\right)$$

Single-pole response:

$$A = A_1 \frac{1}{1+\frac{s}{\omega_1}}$$

flatgain ↗ ⇖ normal pole



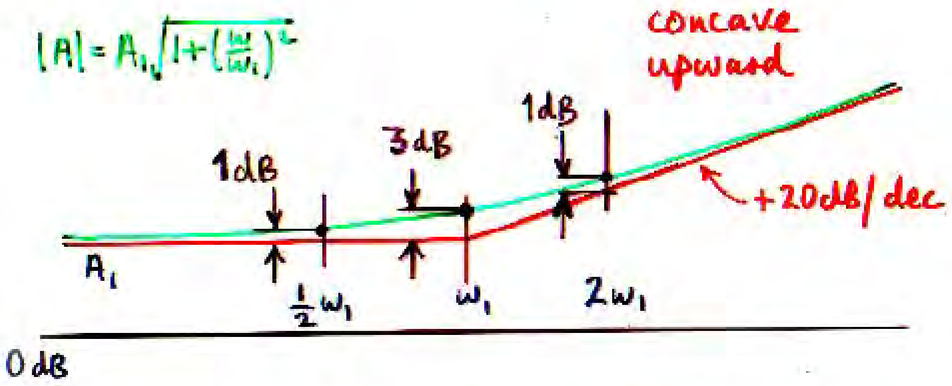
$$|A| = A_1 \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}}$$

$$\angle A = -\tan^{-1}\left(\frac{\omega}{\omega_1}\right)$$

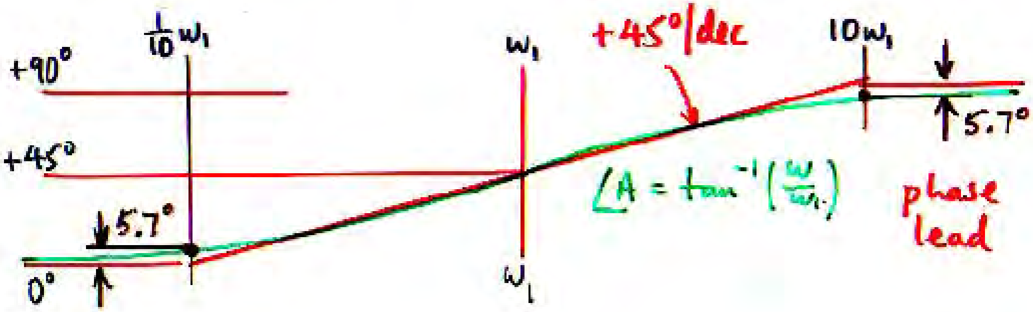
Single-zero response:

$$A = A_1\left(1+\frac{s}{\omega_1}\right)$$

flat gain ↑ ↖ normal zero



$$|A| = A_1\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2}$$



$$\angle A = \tan^{-1}\left(\frac{\omega}{\omega_1}\right)$$

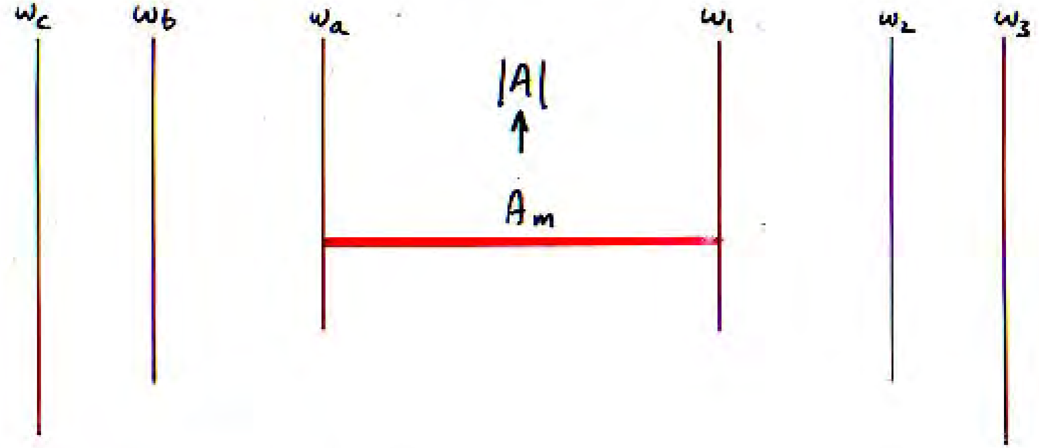
Generalization: Property of Magnitude and Phase Graphs

A corner can be "seen" from further away on the phase graph than on the magnitude graph.

OR:

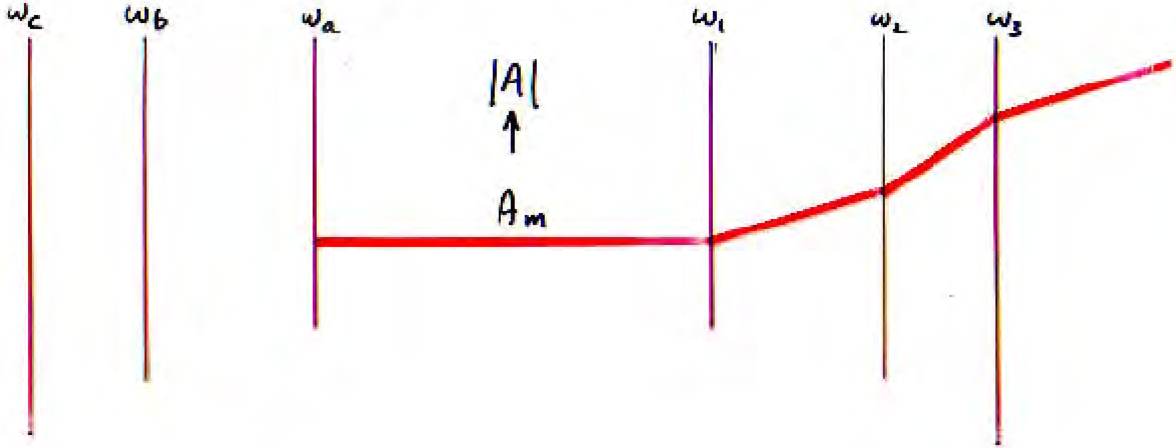
The phase gives a more accurate value of a nearby corner frequency than does the magnitude.

Normal and Inverted poles and zeros



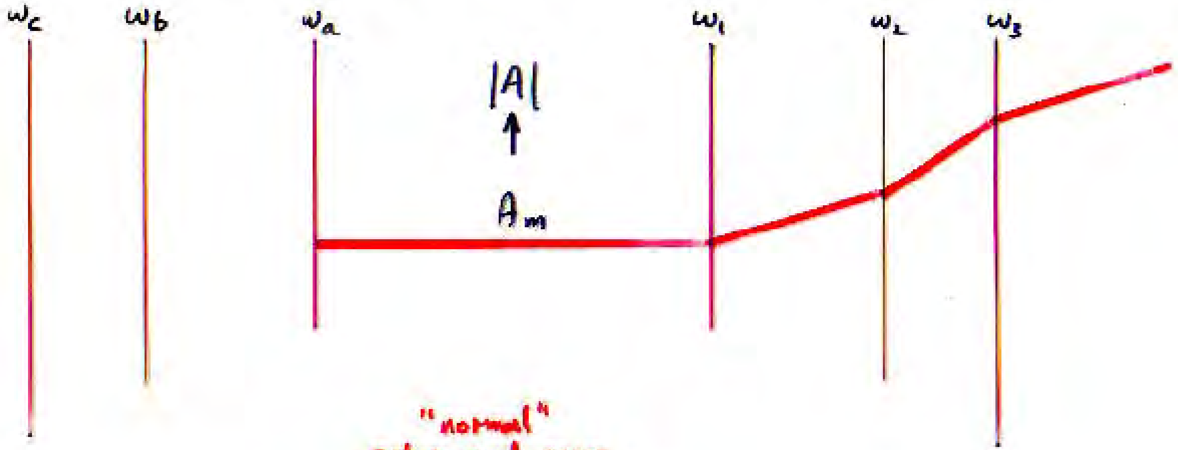
$$A = A_m$$

Normal and Inverted poles and zeros



$A = A_m$

Normal and Inverted poles and zeros



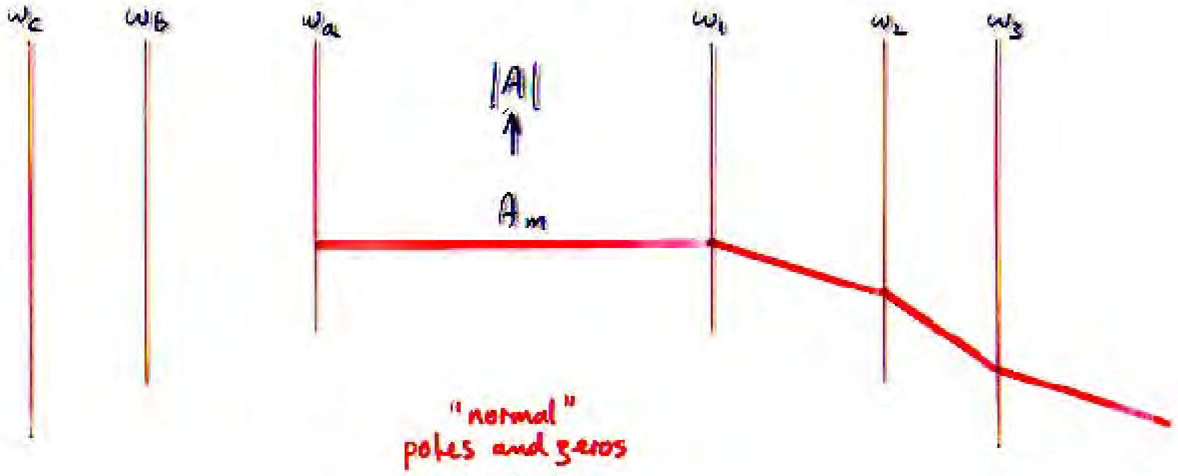
"normal"
poles and zeros

$$A = A_m \frac{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_3}\right)}$$

Normal and Inverted poles and zeros

$$A = A_m$$

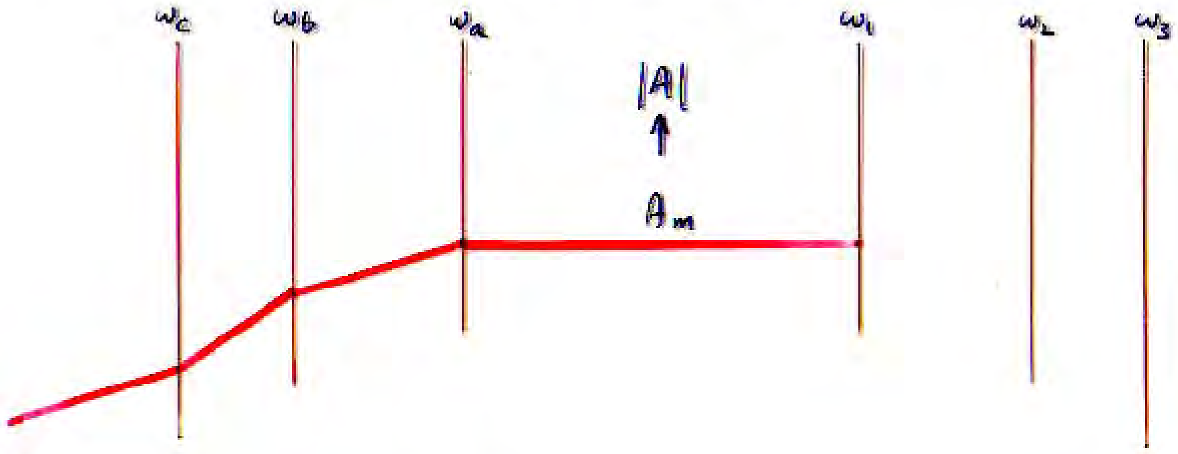
Normal and Inverted poles and zeros



$$A = A_m \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

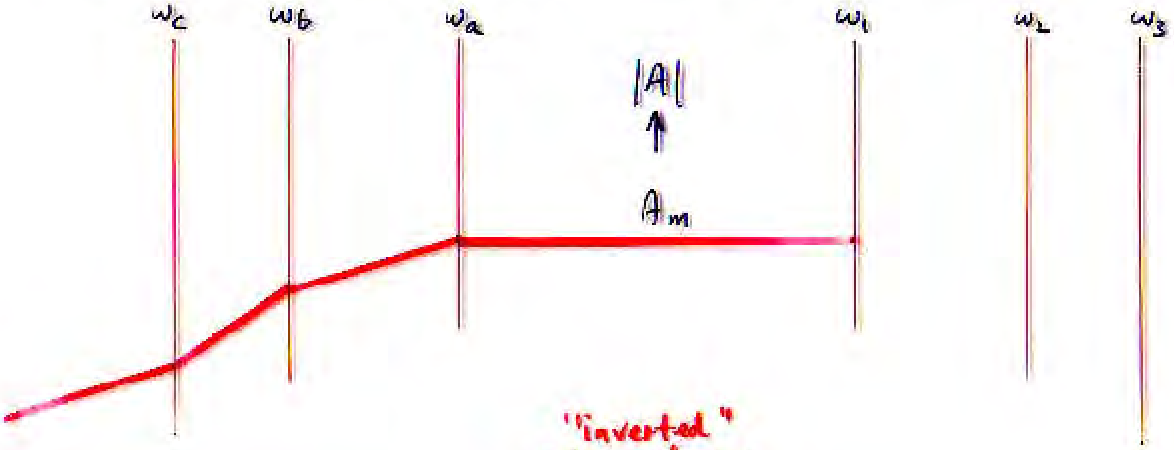
Inversion of pole-zero factors ⟺ vertical inversion of magnitude graph

Normal and Inverted poles and zeros



$A = A_m$

Normal and Inverted poles and zeros

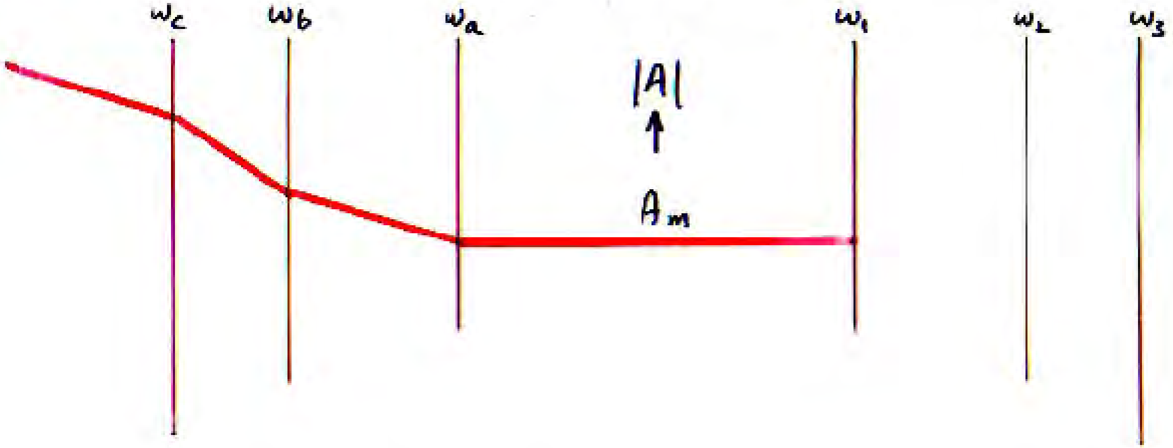


"inverted" poles and zeros

$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)}$$

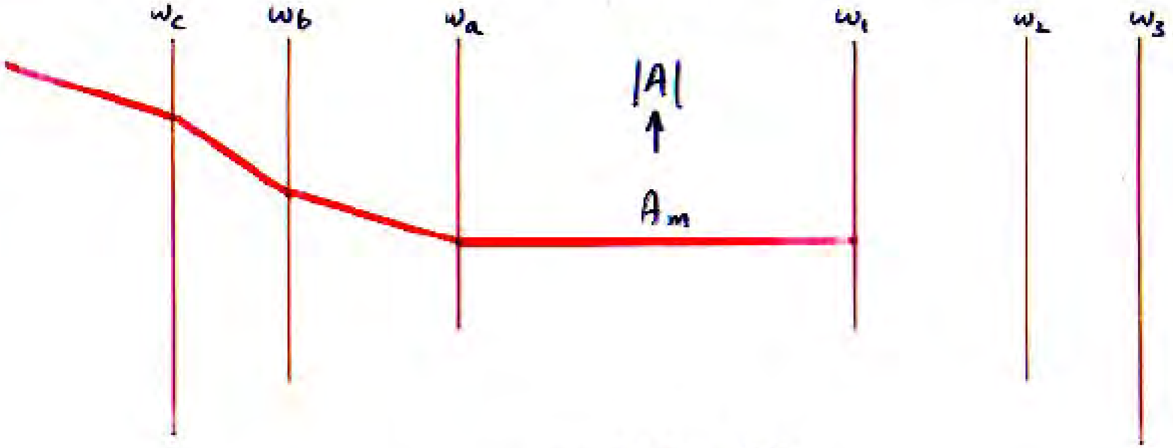
Inversion of frequency terms $\Leftrightarrow$ horizontal reversal of magnitude graph

Normal and Inverted poles and zeros



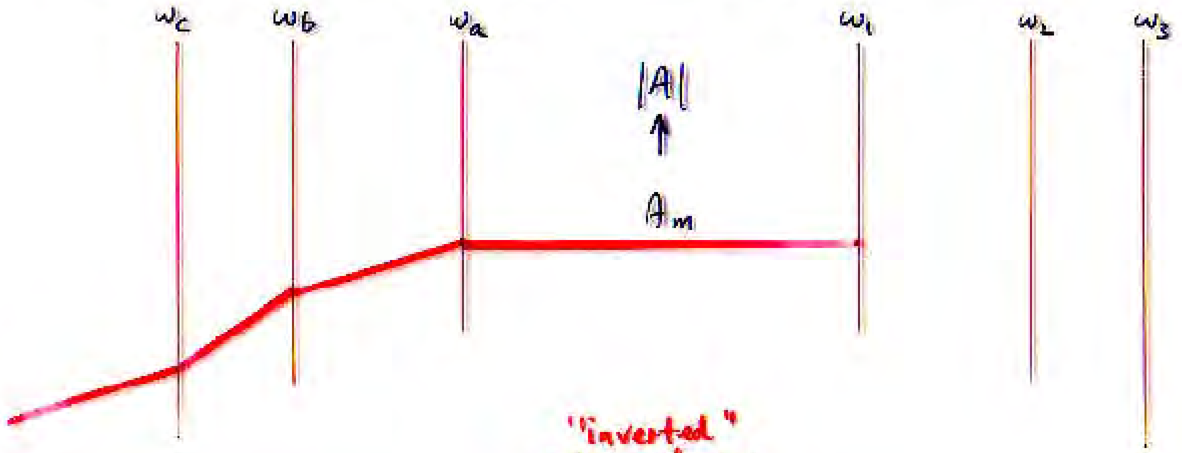
$$A = A_m$$

Normal and Inverted poles and zeros



$$A = A_m \frac{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)}{1+\frac{\omega_c}{s}}$$

Normal and Inverted poles and zeros

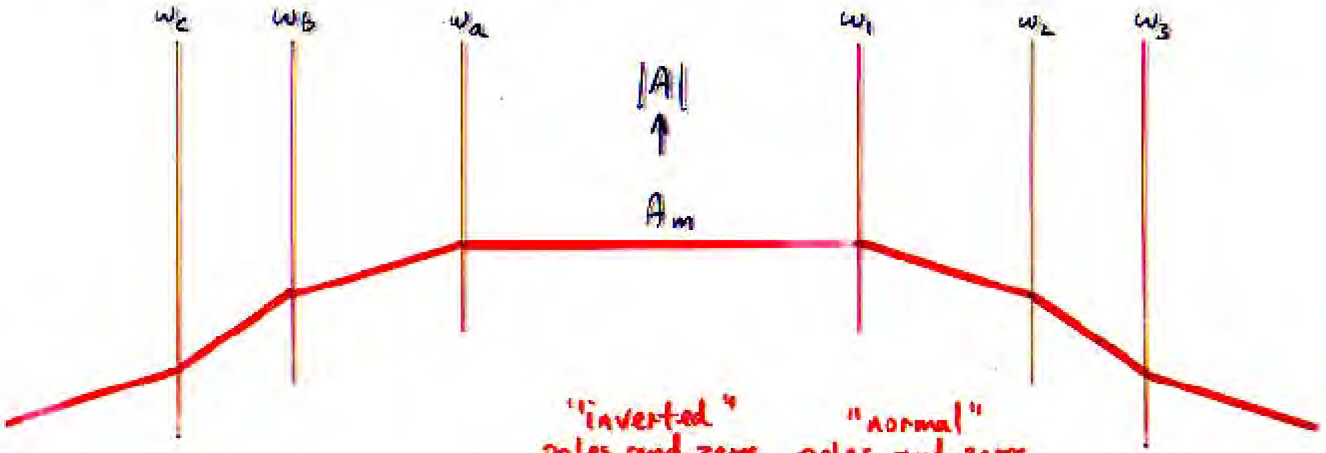


"inverted" poles and zeros

$$A = A_m \frac{\left(1 + \frac{\omega_c}{s}\right)}{\left(1 + \frac{\omega_a}{s}\right)\left(1 + \frac{\omega_b}{s}\right)}$$

Inversion of frequency terms ⇔ horizontal reversal of magnitude graph

Normal and Inverted poles and zeros



"inverted" poles and zeros

"normal" poles and zeros

$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)} \cdot \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

Inversion of frequency terms ⇔ horizontal reversal of magnitude graph

Relationships to conventional forms:

$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)} = A_m \frac{\frac{\omega_c}{s}}{\frac{\omega_a}{s}\,\frac{\omega_b}{s}} \frac{\left(\frac{s}{\omega_c}+1\right)}{\left(\frac{s}{\omega_a}+1\right)\left(\frac{s}{\omega_b}+1\right)}$$

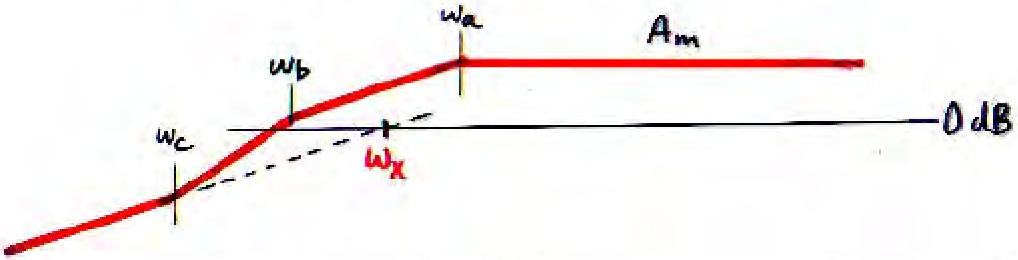
$$= \frac{A_m \omega_c s}{\omega_a \omega_b} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)} = \frac{s}{\omega_x} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)}$$

conventional form (normal poles and zeros)

Where is ω_x on the graph? Where is A_m in the formula?

ω_x is not a useful parameter.

Relationships to conventional forms:



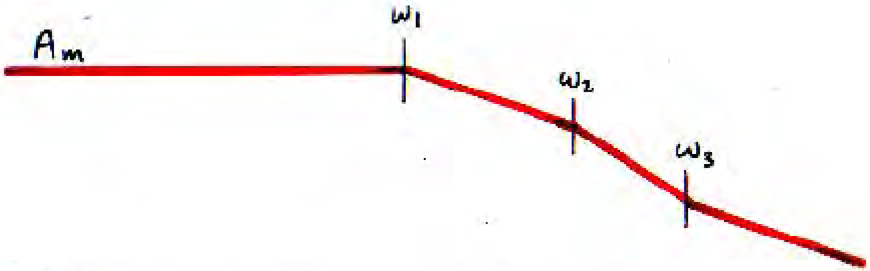
$$A = A_m \frac{\left(1+\frac{\omega_c}{s}\right)}{\left(1+\frac{\omega_a}{s}\right)\left(1+\frac{\omega_b}{s}\right)} = A_m \frac{\frac{\omega_c}{s}}{\frac{\omega_a}{s}\,\frac{\omega_b}{s}} \frac{\left(\frac{s}{\omega_c}+1\right)}{\left(\frac{s}{\omega_a}+1\right)\left(\frac{s}{\omega_b}+1\right)}$$

$$= \frac{A_m \omega_c s}{\omega_a \omega_b} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)} = \frac{s}{\omega_x} \frac{\left(1+\frac{s}{\omega_c}\right)}{\left(1+\frac{s}{\omega_a}\right)\left(1+\frac{s}{\omega_b}\right)}$$

conventional form (normal poles and zeros)

Where is ω_x on the graph? Where is A_m in the formula?

ω_x is not a useful parameter.



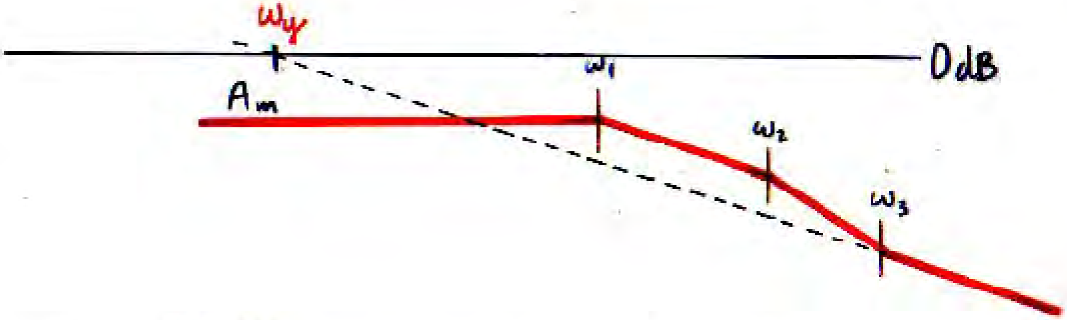
$$A = A_m \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)} = A_m \frac{\frac{1}{\omega_3}}{\frac{1}{\omega_1}\frac{1}{\omega_2}} \frac{(\omega_3+s)}{(\omega_1+s)(\omega_2+s)}$$

$$= \frac{A_m \omega_1 \omega_2}{\omega_3} \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)} = \omega_y \cdot \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)}$$

conventional form

Where is ω_y on the graph? Where is A_m in the formula?

ω_y is not a useful parameter.



$$A = A_m \frac{\left(1+\frac{s}{\omega_3}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)} = A_m \frac{\frac{1}{\omega_3}}{\frac{1}{\omega_1}\frac{1}{\omega_2}} \frac{(\omega_3+s)}{(\omega_1+s)(\omega_2+s)}$$

$$= \frac{A_m \omega_1 \omega_2}{\omega_3} \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)} = \omega_y \frac{(s+\omega_3)}{(s+\omega_1)(s+\omega_2)}$$

conventional form

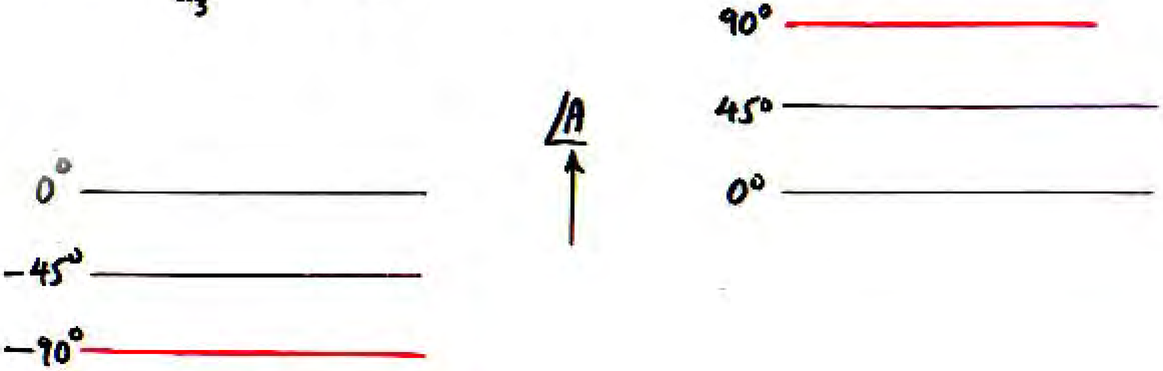
Where is ω_y on the graph? Where is A_m in the formula?

ω_y is not a useful parameter.

If there is no "flat gain", use a reference value:

$$A = A_3 \frac{1}{\frac{s}{\omega_3}} = A_3 \frac{\omega_3}{s}$$

$$A = A_4 \frac{s}{\omega_4}$$

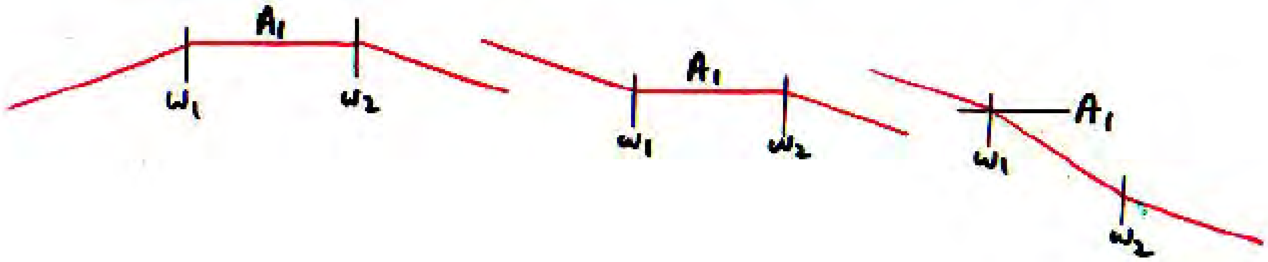


Exercise 3.1

Write factored pole-zero forms from asymptotes

Exercises

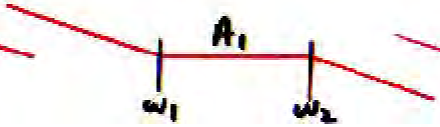
Express the gains in factored pole-zero form

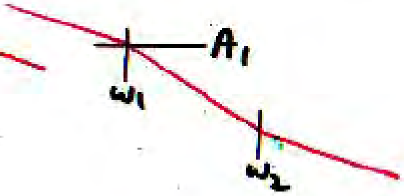


Exercise 3.1 - Solution

Exercises

Express the gains in factored pole-zero form



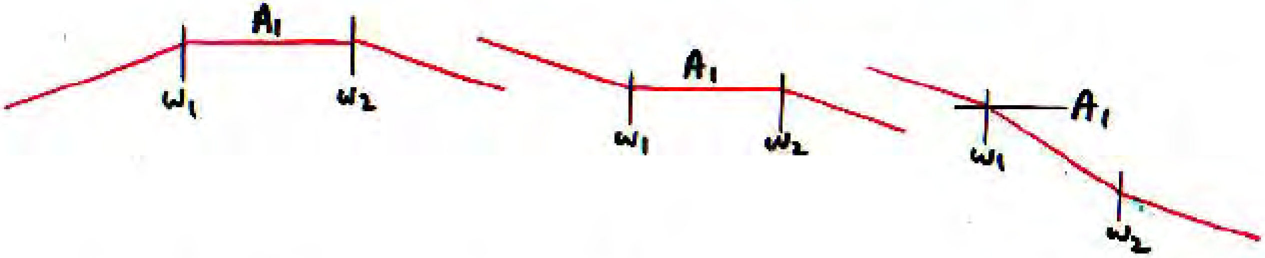


$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

Exercise 3.1 - Solution

Exercises

Express the gains in factored pole-zero form



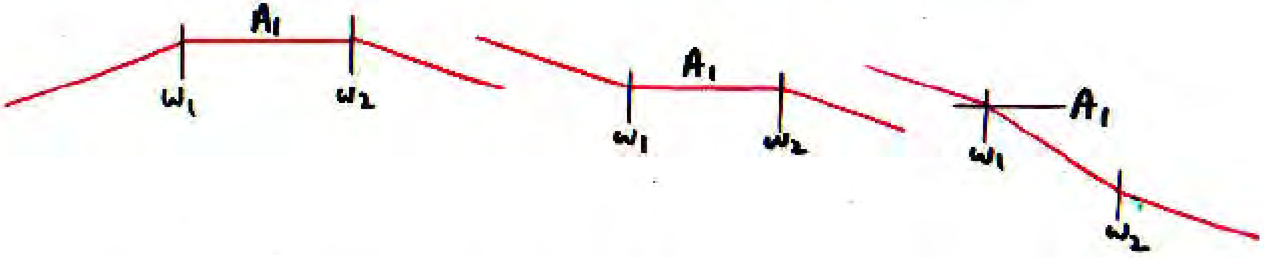
$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A = A_1 \frac{1+\frac{\omega_1}{s}}{1+\frac{s}{\omega_2}}$$

Exercise 3.1 - Solution

Exercises

Express the gains in factored pole-zero form



$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

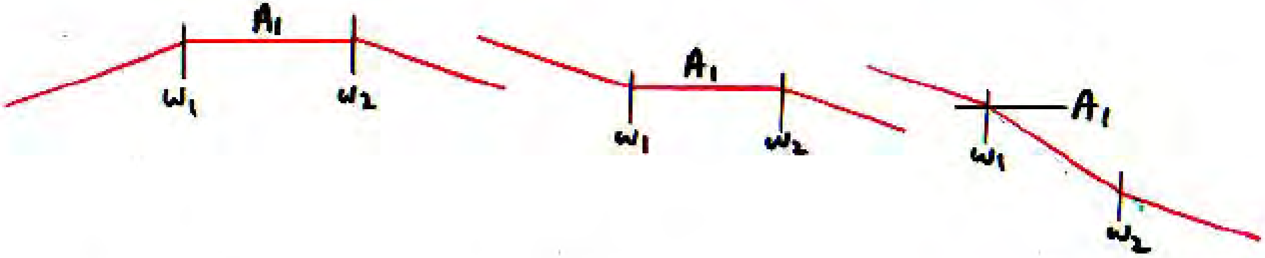
$$A = A_1 \frac{1+\frac{\omega_1}{s}}{1+\frac{s}{\omega_2}}$$

$$A = A_1\left(\frac{\omega_1}{s}\right) \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

Exercise 3.1 - Solution

Exercises

Express the gains in factored pole-zero form



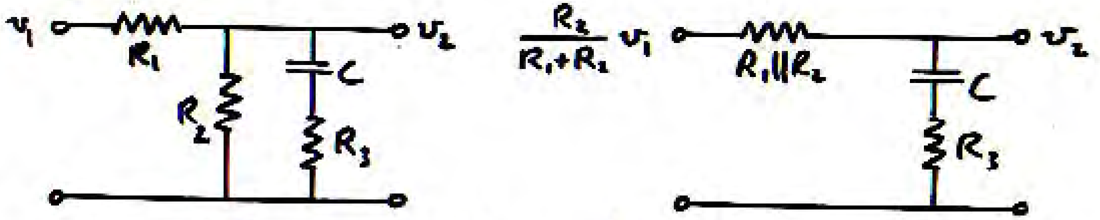
$$A = A_1 \frac{1}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A = A_1 \frac{1+\frac{\omega_1}{s}}{1+\frac{s}{\omega_2}}$$

$$A = A_1\left(\frac{\omega_1}{s}\right)\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

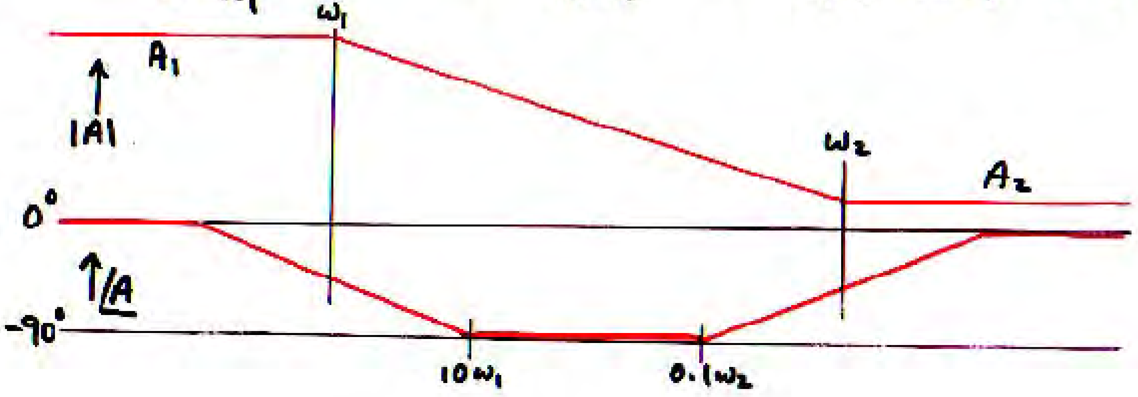
$$A = A_1\left(\frac{\omega_1}{s}\right)^2\frac{1+\frac{s}{\omega_2}}{1+\frac{\omega_1}{s}}$$

Lag-lead network



$$\frac{v_2}{v_1} \equiv A = \frac{R_2}{R_1+R_2} \cdot \frac{\frac{1}{sC} + R_3}{\frac{1}{sC} + R_3 + R_1 \| R_2}$$

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} \quad \text{where} \quad A_1 \equiv \frac{R_2}{R_1+R_2} \quad \omega_1 \equiv \frac{1}{C(R_3+R_1 \| R_2)} \quad \omega_2 \equiv \frac{1}{CR_3}$$



In this case, there are two flat gains. As derived, the low-frequency flat gain A_1 appears as coefficient, together with normal pole and zero:

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

Equally well, directly from the $|A|$ asymptotes, the result could be written with the high-frequency flat gain A_2 as coefficient, together with inverted zero and pole:

$$A = A_2 \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

What is the relation between A_1 and A_2? One form of the result can be derived from the other algebraically:

$$A = \underset{\uparrow}{\boxed{A_1}} \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_1 \frac{\frac{s}{\omega_2}}{\frac{s}{\omega_1}} \frac{\frac{\omega_2}{s}+1}{\frac{\omega_1}{s}+1} = A_1 \frac{\omega_1}{\omega_2} \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

This is $A|_{s\to 0}$

In this case, there are two flat gains. As derived, the low-frequency flat gain A_1 appears as coefficient, together with normal pole and zero:

$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}}$$

Equally well, directly from the $|A|$ asymptotes, the result could be written with the high-frequency flat gain A_2 as coefficient, together with inverted zero and pole:

$$A = \boxed{A_2} \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

What is the relation between A_1 and A_2? One form of the result can be derived from the other algebraically:

$$A = \boxed{A_1} \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_1 \frac{\frac{s}{\omega_2}}{\frac{s}{\omega_1}} \frac{\frac{\omega_2}{s}+1}{\frac{\omega_1}{s}+1} = \boxed{A_1 \frac{\omega_1}{\omega_2}} \frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

↑ This is $A|_{s\to 0}$

↑ This is $A|_{s\to\infty}$, so must be A_2

Result:

$$\frac{A_2}{A_1} = \frac{\omega_1}{\omega_2}$$

For the lag-lead network:

$$A_2 = A_1 \frac{\omega_1}{\omega_2} = \frac{R_2}{R_1 + R_2} \frac{\cancel{C} R_3}{\cancel{C}(R_3 + R_1 \| R_2)}$$

which is obvious from the reduced model.

Generalization: Gain-Bandwidth Trade-Off

For a single-slope (± 20dB/dec)

Ratio of flat gains = Ratio of corner frequencies that separate them

This is a form of gain-bandwidth trade-off.

More than one flat gain

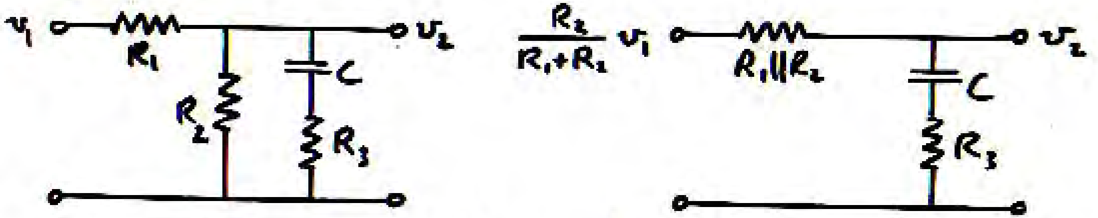
$$A = A_1 \frac{1+\frac{s}{\omega_1}}{1+\frac{s}{\omega_2}} = A_1 \frac{\omega_2}{\omega_1} \frac{1+\frac{\omega_1}{s}}{1+\frac{\omega_2}{s}} = A_2 \frac{1+\frac{\omega_1}{s}}{1+\frac{\omega_2}{s}}$$

Hence: "gain-bandwidth tradeoff":

$$\frac{A_2}{A_1} = \frac{\omega_2}{\omega_1}$$

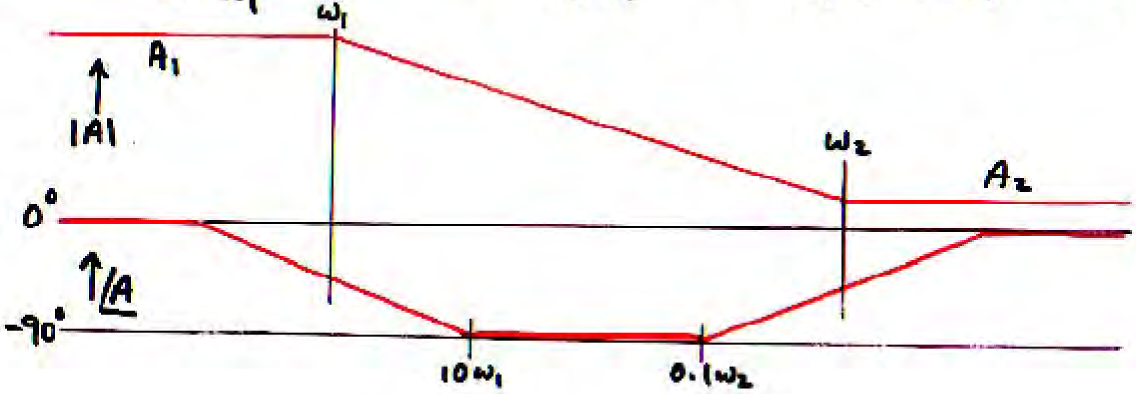
Either flat gain can be used as "reference" gain.

Lag-lead network



$$\frac{v_2}{v_1} \equiv A = \frac{R_2}{R_1+R_2} \cdot \frac{\frac{1}{sC}+R_3}{\frac{1}{sC}+R_3+R_1\|R_2}$$

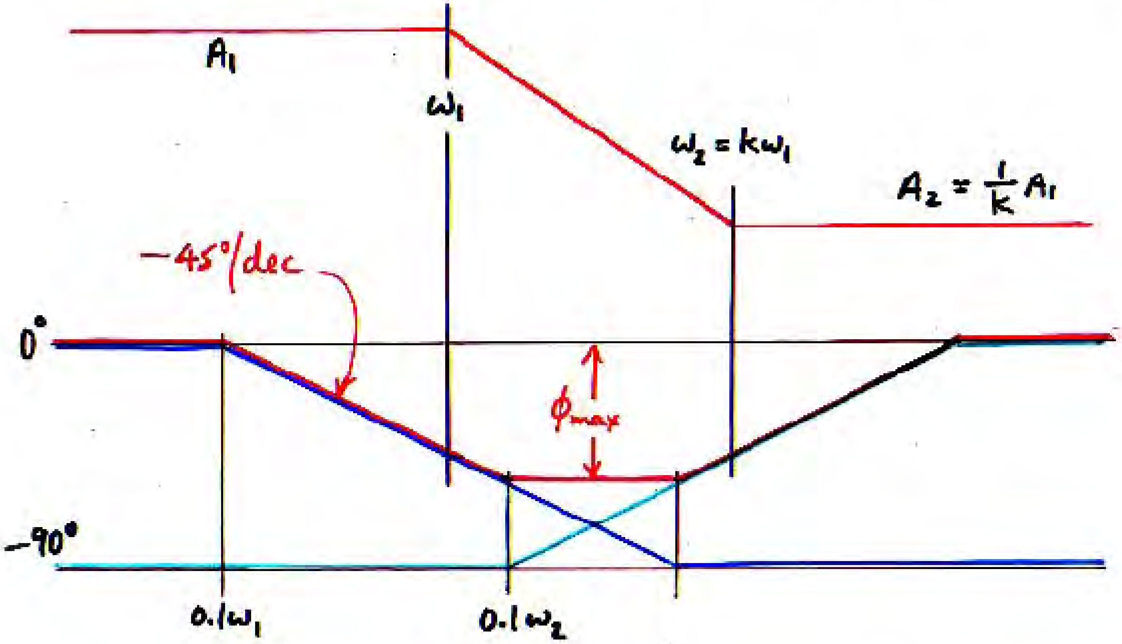
$$A = A_1 \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} \quad \text{where} \quad A_1 \equiv \frac{R_2}{R_1+R_2} \qquad \omega_1 \equiv \frac{1}{C(R_3+R_1\|R_2)} \qquad \omega_2 \equiv \frac{1}{CR_3}$$



If $\omega_2 > 100\omega_1$, phase asymptotes do not overlap and the phase lag reaches 90° before returning to zero.

If $\omega_2 < 100\omega_1$, the phase asymptotes do overlap, and the phase lag reaches a maximum, less than 90°, which is a function of the ratio of the flat gains.

Find the maximum phase lag ϕ_{max} as a function of the gain ratio $k \equiv A_1/A_2 = \omega_2/\omega_1$



$$\phi_{max} = -45° \log \frac{0.1\omega_2}{0.1\omega_1} = -45° \log k \qquad (k < 100)$$

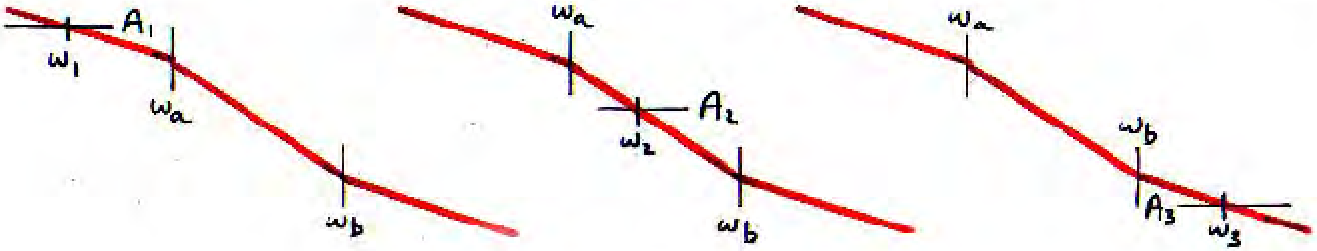
Exercise 3.2

Write factored pole-zero forms for different Reference Gains, and write A_2 and A_3 in terms of A_1.

Exercise:

No flat gain

Identify the gain at any chosen frequency as "reference" gain

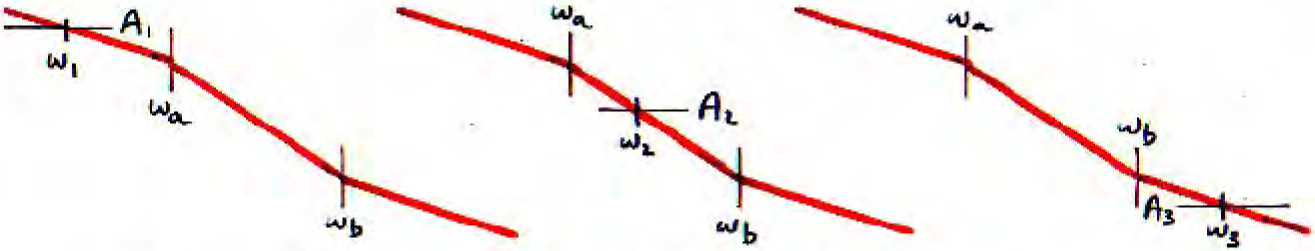


Exercise 3.2 - Solution

Exercise:

No flat gain

Identify the gain at any chosen frequency as "reference" gain



$$A = A_1 \frac{\omega_1}{s} \frac{1+\frac{s}{\omega_b}}{1+\frac{s}{\omega_a}}$$

$$A = A_2\left(\frac{\omega_2}{s}\right)^2 \frac{1+\frac{s}{\omega_b}}{1+\frac{\omega_a}{s}}$$

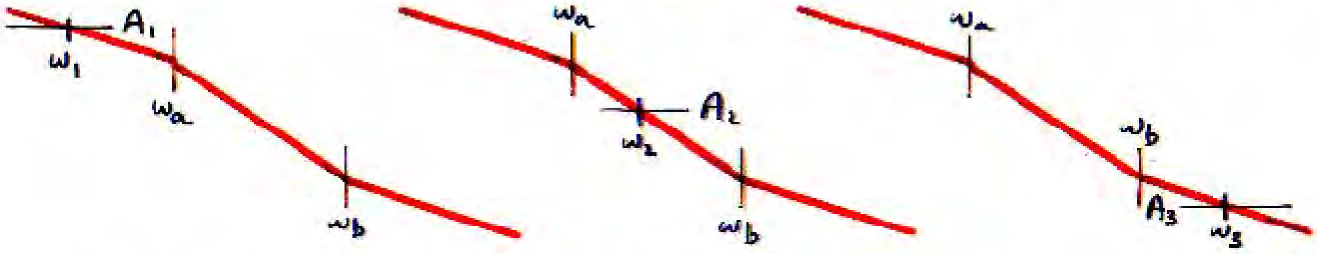
$$A = A_3 \frac{\omega_3}{s} \frac{1+\frac{\omega_b}{s}}{1+\frac{\omega_a}{s}}$$

Exercise 3.2 - Solution

Exercise:

No flat gain

Identify the gain at any chosen frequency as "reference" gain



$$A = A_1 \frac{\omega_1}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \qquad A = A_2 \left(\frac{\omega_2}{s}\right)^2 \frac{1 + \frac{s}{\omega_b}}{1 + \frac{\omega_a}{s}} \qquad A = A_3 \frac{\omega_3}{s} \frac{1 + \frac{\omega_b}{s}}{1 + \frac{\omega_a}{s}}$$

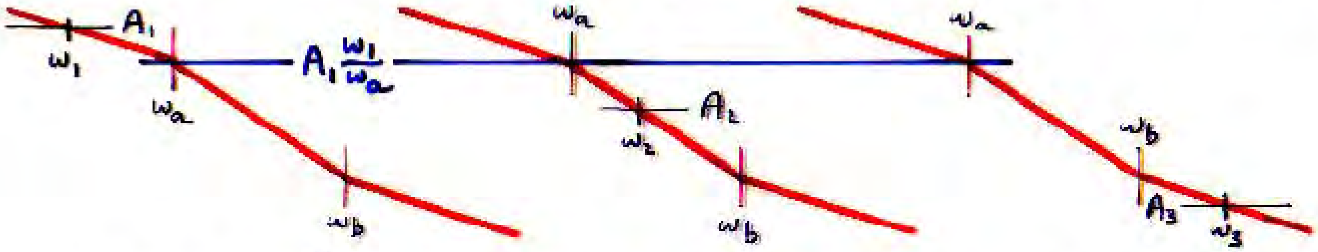
Exercise: Express A_2 and A_3 in terms of A_1.

Exercise 3.2 - Solution

Exercise:

No flat gain

Identify the gain at any chosen frequency as "reference" gain



$$A = A_1 \frac{\omega_1}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \qquad A = A_2 \left(\frac{\omega_2}{s}\right)^2 \frac{1 + \frac{s}{\omega_b}}{1 + \frac{\omega_a}{s}} \qquad A = A_3 \frac{\omega_3}{s} \frac{1 + \frac{\omega_b}{s}}{1 + \frac{\omega_a}{s}}$$

Exercise: Express A_2 and A_3 in terms of A_1.

$$A_2 = \left(A_1 \frac{\omega_1}{\omega_a}\right)\left(\frac{\omega_a}{\omega_2}\right)^2$$

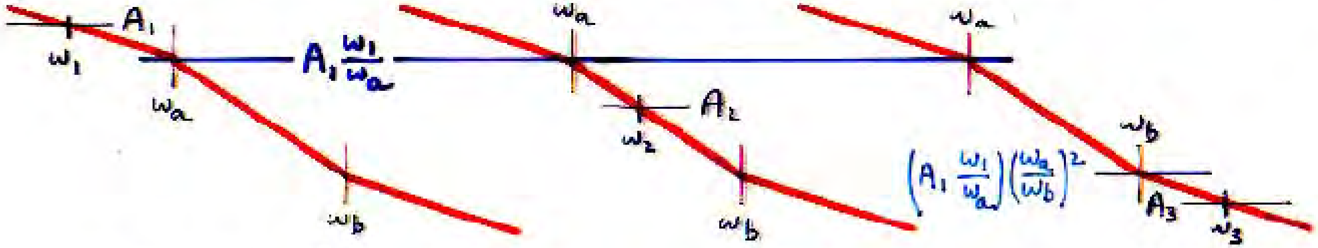
$$= A_1 \frac{\omega_1 \omega_a}{\omega_2{}^2}$$

Exercise 3.2 - Solution

Exercise:

No flat gain

Identify the gain at any chosen frequency as "reference" gain



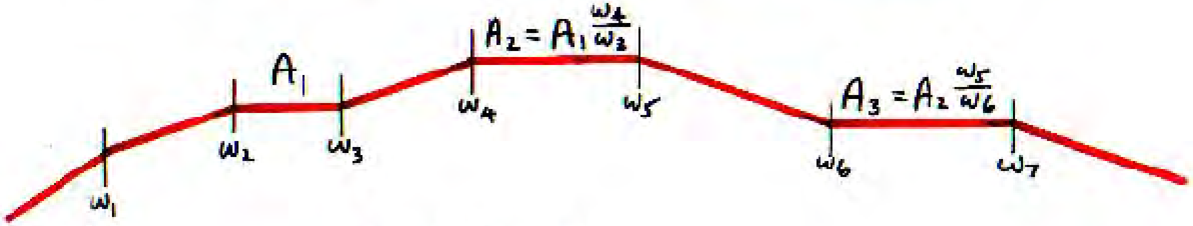
$$A = A_1 \frac{\omega_1}{s} \frac{1 + \frac{s}{\omega_b}}{1 + \frac{s}{\omega_a}} \qquad A = A_2 \left(\frac{\omega_2}{s}\right)^2 \frac{1 + \frac{s}{\omega_b}}{1 + \frac{\omega_a}{s}} \qquad A = A_3 \frac{\omega_3}{s} \frac{1 + \frac{\omega_b}{s}}{1 + \frac{\omega_a}{s}}$$

Exercise: Express A_2 and A_3 in terms of A_1.

$$A_2 = \left(A_1 \frac{\omega_1}{\omega_a}\right)\left(\frac{\omega_a}{\omega_2}\right)^2 \qquad A_3 = \left[\left(A_1 \frac{\omega_1}{\omega_a}\right)\left(\frac{\omega_a}{\omega_b}\right)^2\right] \frac{\omega_b}{\omega_3}$$

$$= A_1 \frac{\omega_1 \omega_a}{\omega_2^{\,2}} \qquad = A_1 \frac{\omega_1 \omega_a}{\omega_3 \omega_b}$$

Any flat gain can be used as "reference" gain A_{ref}. With respect to A_{ref}, poles and zeros above A_{ref} are normal, those below A_{ref} are inverted.



$$A = A_1 \frac{\left(1+\frac{s}{\omega_3}\right)\left(1+\frac{s}{\omega_6}\right)}{\left(1+\frac{\omega_2}{s}\right)\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_4}\right)\left(1+\frac{s}{\omega_5}\right)\left(1+\frac{s}{\omega_7}\right)}$$

$$A = A_2 \frac{\left(1+\frac{\omega_3}{s}\right)\left(1+\frac{s}{\omega_6}\right)}{\left(1+\frac{\omega_4}{s}\right)\left(1+\frac{\omega_2}{s}\right)\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_5}\right)\left(1+\frac{s}{\omega_7}\right)}$$

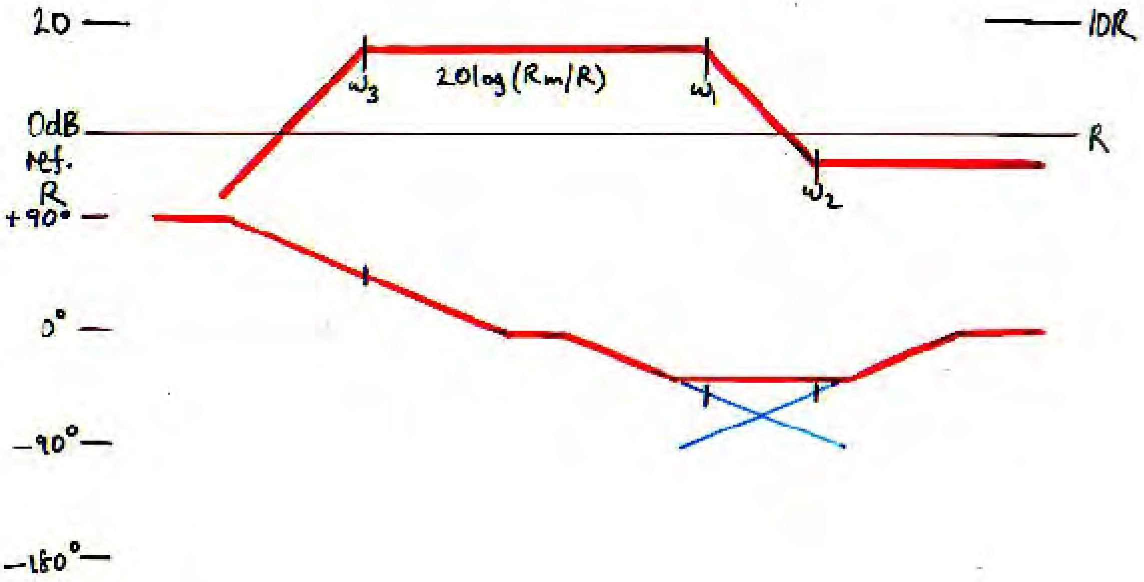
$$A = A_3 \frac{\left(1+\frac{\omega_6}{s}\right)\left(1+\frac{\omega_3}{s}\right)}{\left(1+\frac{\omega_5}{s}\right)\left(1+\frac{\omega_4}{s}\right)\left(1+\frac{\omega_2}{s}\right)\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_7}\right)}$$

If you don't use inverted poles and zeros, you are stuck with the zero-frequency gain as the reference gain.

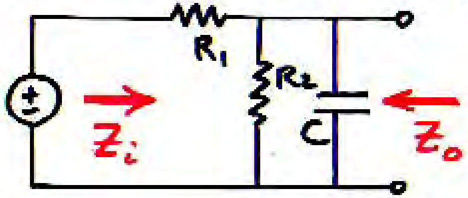
The principal benefit of using inverted poles and zeros is that you can choose the gain at *any* frequency as the reference gain.

Impedance asymptotes

$$Z = R_m \frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} \frac{1}{1+\frac{\omega_3}{s}}$$



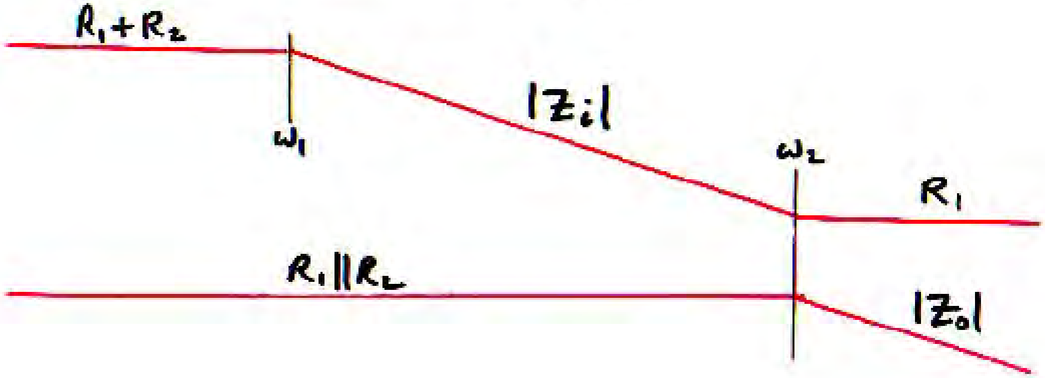
Input and output impedances



$$\omega_1 \equiv \frac{1}{CR_2} \qquad \omega_2 \equiv \frac{1}{C(R_1 \| R_2)}$$

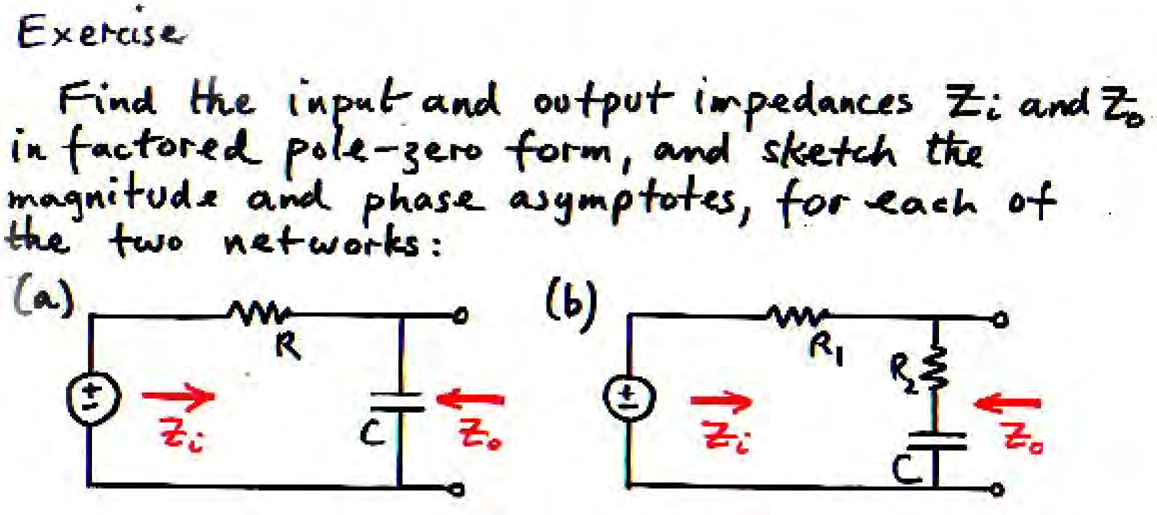
$$Z_i = R_1 + \frac{R_2}{1+sCR_2} = (R_1+R_2)\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = R_1\frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

$$Z_o = R_1 \| R_2 \| \frac{1}{sC} = R_1 \| R_2 \frac{1}{1+\frac{s}{\omega_2}}$$



Exercise 3.3

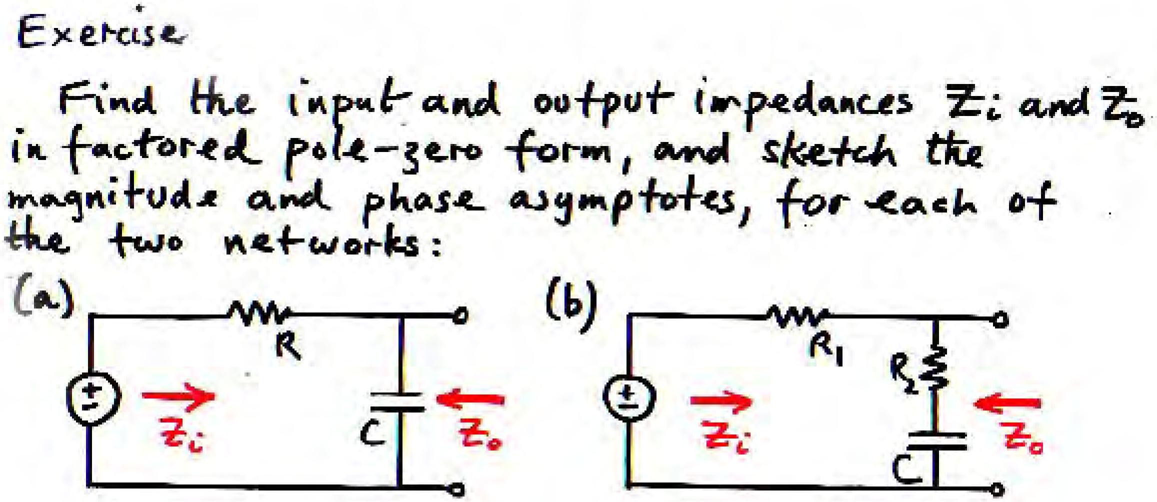
Write input and output impedances Z_i and Z_o in factored pole-zero forms.



Exercise

Find the input and output impedances Z_i and Z_o in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

Exercise 3.3 - Solution



Exercise

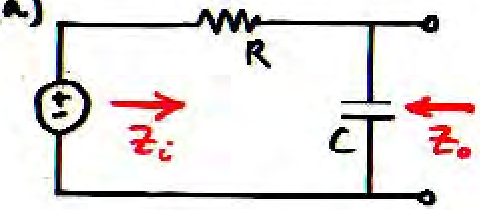
Find the input and output impedances Z_i and Z_o in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

Exercise 3.3 - Solution

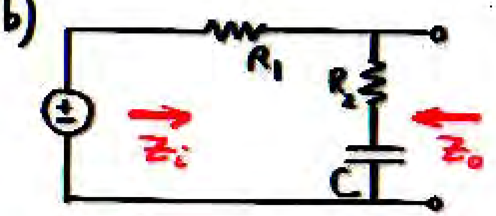
Exercise

Find the input and output impedances Z_i and Z_o in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

(a)



(b)



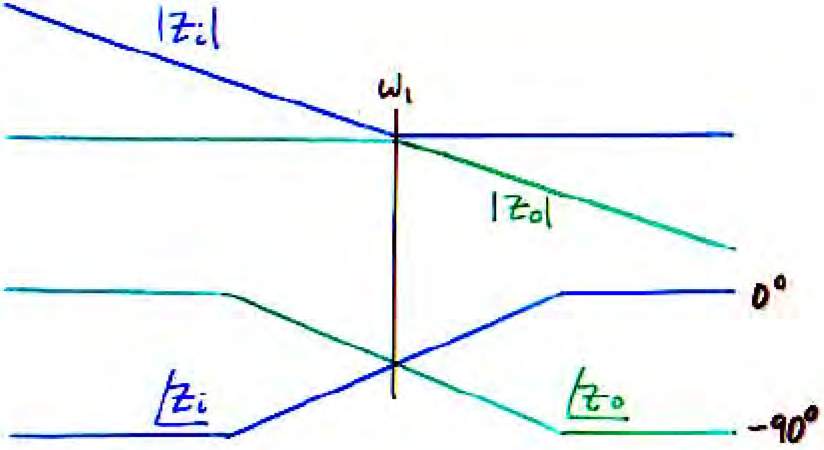
$$Z_i = R + \frac{1}{sC}$$

$$= R\left(1 + \frac{\omega_1}{s}\right) \qquad \omega_1 \equiv \frac{1}{CR}$$

$$Z_o = R \| \frac{1}{sC}$$

$$= R\,\frac{1}{1 + \frac{s}{\omega_1}}$$

Exercise 3.3 - Solution



$$Z_i = R + \frac{1}{sC}$$

$$= R\left(1 + \frac{\omega_1}{s}\right) \quad \omega_1 \equiv \frac{1}{CR}$$

$$Z_o = R \| \frac{1}{sC}$$

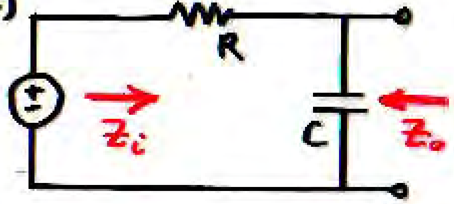
$$= R \frac{1}{1 + \frac{s}{\omega_1}}$$

Exercise 3.3 - Solution

Exercise

Find the input and output impedances Z_i and Z_o in factored pole-zero form, and sketch the magnitude and phase asymptotes, for each of the two networks:

(a)



(b)

$$Z_i = R_1 + R_2 + \frac{1}{sC}$$

$$= (R_1 + R_2)\left(1 + \frac{\omega_3}{s}\right)$$

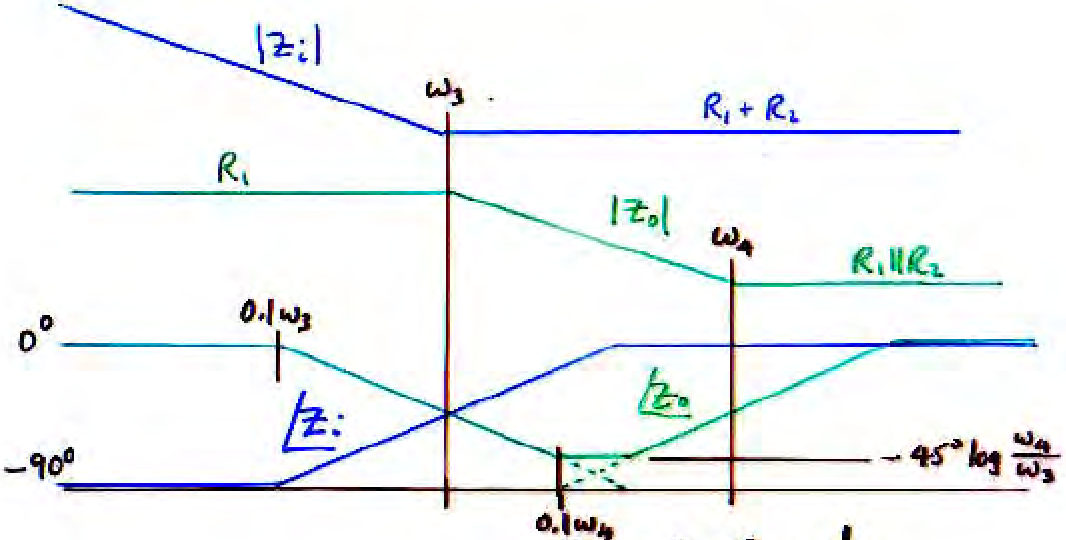
$$\omega_3 \equiv \frac{1}{C(R_1 + R_2)}$$

$$Z_o = R_1 \| \left(R_2 + \frac{1}{sC}\right)$$

$$= (R_1 \| R_2)\,\frac{1 + \frac{\omega_4}{s}}{1 + \frac{\omega_3}{s}}$$

$$\omega_4 \equiv \frac{1}{CR_2}$$

Exercise 3.3 - Solution



$$Z_i = R_1 + R_2 + \frac{1}{sC}$$

$$= (R_1 + R_2)\left(1 + \frac{\omega_3}{s}\right)$$

$$\omega_3 = \frac{1}{C(R_1 + R_2)}$$

$$Z_0 = R_1 \| \left(R_2 + \frac{1}{sC}\right)$$

$$= (R_1 \| R_2)\frac{1 + \frac{\omega_4}{s}}{1 + \frac{\omega_3}{s}}$$

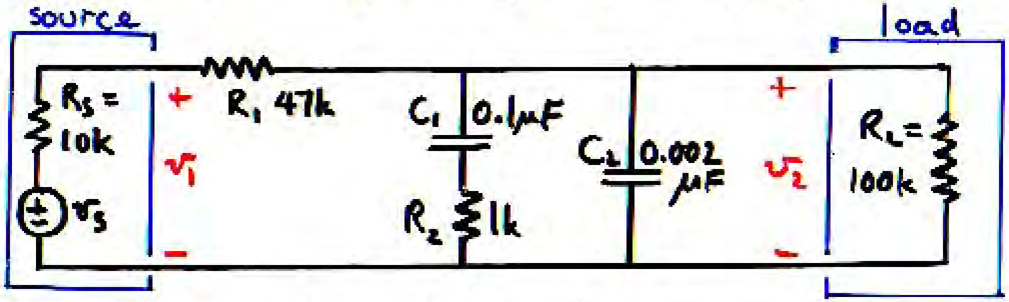
$$\omega_4 = \frac{1}{CR_2}$$

4. AN IMPROVED FORMULA FOR QUADRATIC ROOTS

The Conventional Formula suffers from two congenital defects

Example

Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod (1+s/\omega_x)}{\prod (1+s/\omega_y)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.

$$A = \frac{\dfrac{\left(\dfrac{R_L}{1+sC_2R_L}\right)\left(R_2+\dfrac{1}{sC_1}\right)}{\dfrac{R_L}{1+sC_2R_L}+R_2+\dfrac{1}{sC_1}}}{\dfrac{\left(\dfrac{R_L}{1+sC_2R_L}\right)\left(R_2+\dfrac{1}{sC_1}\right)}{\dfrac{R_L}{1+sC_2R_L}+R_2+\dfrac{1}{sC_1}}+R_1}$$

$$\text{a lot } \Big\downarrow \text{ of algebra}$$

$$= \frac{R_L + sC_1R_2R_L}{[R_1+R_L]+s[C_1(R_1R_2+R_LR_2+R_1R_L)+C_2R_1R_L]+s^2[C_1C_2R_1R_2R_L]}$$

This is a high-entropy expression. To lower the entropy, write the polynomials in s with a leading term of unity:

$$A = \frac{R_L}{R_1+R_L}\,\frac{1+sC_1R_2}{1+s\left[C_1\left(\frac{R_1R_2+R_LR_2+R_1R_L}{R_1+R_L}\right)+C_2\left(\frac{R_1R_L}{R_1+R_L}\right)\right]+s^2\left[C_1C_2\left(\frac{R_1R_2R_L}{R_1+R_L}\right)\right]}$$

Now, recognize series/parallel resistance combinations:

$$\begin{array}{ccc}\downarrow & \downarrow & \downarrow \\ (R_2+R_1\|R_L) & (R_1\|R_L) & R_2(R_1\|R_L)\end{array}$$

$$A = \frac{R_L}{R_1+R_L}\,\frac{1+sC_1R_2}{1+s\left[C_1\left(\frac{R_1R_2+R_LR_2+R_1R_L}{R_1+R_L}\right)+C_2\left(\frac{R_1R_L}{R_1+R_L}\right)\right]+s^2\left[C_1C_2\left(\frac{R_1R_2R_L}{R_1+R_L}\right)\right]}$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

Now, recognize series/parallel resistance combinations:

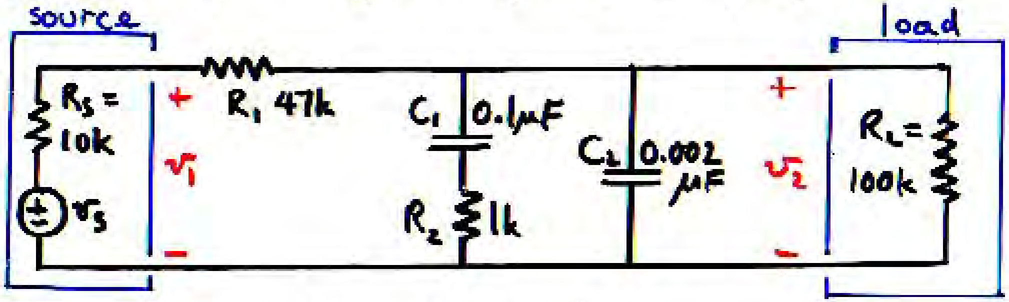
$$(R_2 + R_1\|R_L) \qquad\qquad (R_1\|R_L) \qquad\qquad R_2(R_1\|R_L)$$

The same result, *including* the series/parallel resistance grouping, could have been obtained with *less* algebra by elimination, first, of one of the loops of the original circuit.

Circuit with R_1 and R_L absorbed into a Thevenin equivalent:

Example

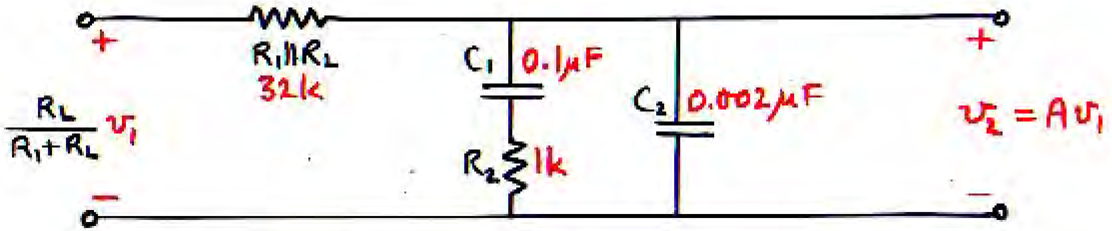
Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:



Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod (1+s/\omega_x)}{\prod (1+s/\omega_y)}$$

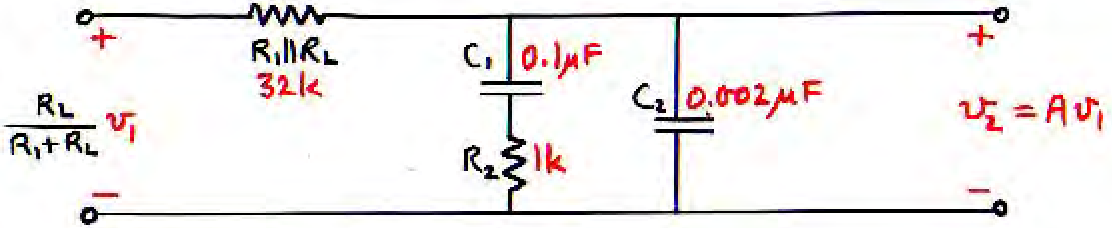
Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)} + R_1 \| R_L} \cdot \frac{R_L}{R_1+R_L}$$

less ↓ algebra

$$= \frac{R_L}{R_1+R_L} \cdot \frac{1+sC_1R_2}{1+s\left[C_1(R_2+R_1\|R_L)+C_2(R_1\|R_L)\right]+s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}+R_1\|R_L}\cdot\frac{R_L}{R_1+R_L}$$

less ↓ algebra

$$= \frac{R_L}{R_1+R_L}\cdot\frac{1+sC_1R_2}{1+s\left[C_1(R_2+R_1\|R_L)+\cancel{C_2(R_1\|R_L)}\right]+s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$

↑

Use of numerical values to justify analytic approximation

Generalization: Use of Numerical Values to Justify Analytic Approximations

Use numbers to justify leaving out a term, but continue the analysis with the symbols.

This way, the analysis result can be used for design, because the numbers can be changed so that the answer has the desired value. (The approximation must be checked to ensure that it is not invalidated by the new numbers.)

You can't lose by trying!

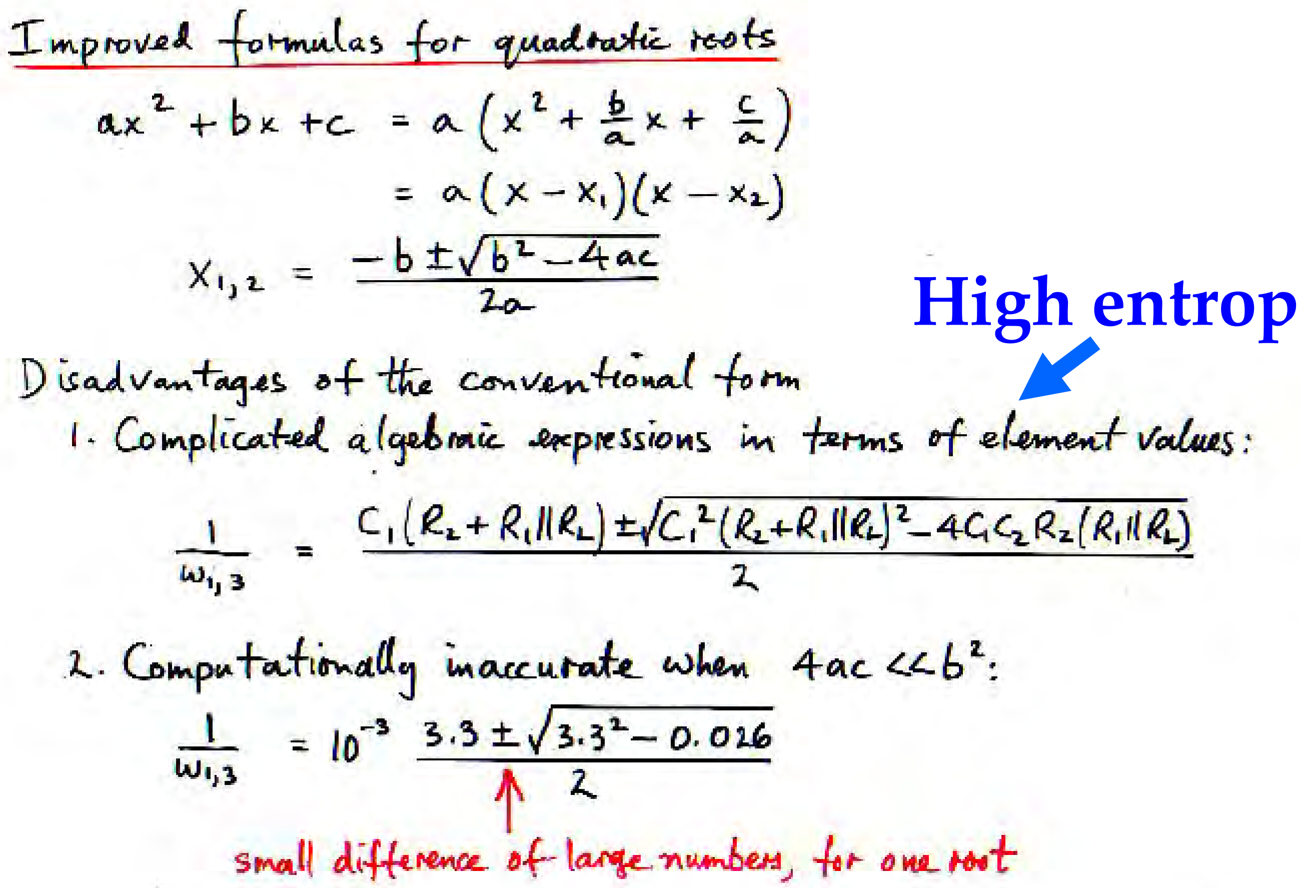
$$A = \frac{R_L}{R_1 + R_L} \frac{1 + sC_1R_2}{1 + s[C_1(R_2 + (R_1 \| R_L)] + s^2[C_1C_2R_2(R_1 \| R_L)]}$$

$$= A_0 \frac{\left(1 + \frac{s}{\omega_2}\right)}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_3}\right)}$$

where

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2 \overset{3.3\times10^{-3}}{\overset{\downarrow}{+}} R_1\|R_L) \pm \sqrt{C_1^2(R_2 \overset{10\times10^{-6}}{\overset{\downarrow}{+}} R_1\|R_L)^2 - 4C_1\overset{0.026\times10^{-6}}{\overset{\downarrow}{C_2}}R_2(R_1\|R_L)}}{2}$$

This is useless for <u>design</u>, and in any case is inaccurate numerically.



Improved formulas for quadratic roots

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$

$$= a(x-x_1)(x-x_2)$$

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Disadvantages of the conventional form

1. Complicated algebraic expressions in terms of element values:

High entropy

$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2+R_1\|R_L)\pm\sqrt{C_1^2(R_2+R_1\|R_L)^2-4C_1C_2R_2(R_1\|R_L)}}{2}$$

2. Computationally inaccurate when $4ac \ll b^2$:

$$\frac{1}{\omega_{1,3}} = 10^{-3}\,\frac{3.3\pm\sqrt{3.3^2-0.026}}{2}$$

small difference of large numbers, for one root

These are congenital defects!

Better method:

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right] = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}, \qquad Q^2 \equiv \frac{ac}{b^2}$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]$$

$$= -\frac{b}{a}\frac{\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]} = -\frac{b}{a}\frac{\frac{1}{4}-\frac{1}{4}\left(1-4Q^2\right)}{F}$$

$$= -\frac{b}{a}\frac{Q^2}{F} = -\frac{c}{b}\frac{1}{F}$$

Better method:

$$x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right] = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2}, \qquad Q^2 \equiv \frac{ac}{b^2}$$

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} = -\frac{b}{a}\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]$$

$$= -\frac{b}{a}\,\frac{\left[\frac{1}{2}-\frac{1}{2}\sqrt{1-4Q^2}\right]\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}\right]} = -\frac{b}{a}\,\frac{\frac{1}{4}-\frac{1}{4}(1-4Q^2)}{F}$$

$$= -\frac{b}{a}\,\frac{Q^2}{F} = -\frac{c}{b}\,\frac{1}{F}$$

Crucial step: Large numbers are subtracted exactly, leaving the small difference in analytic form.

Hence, both roots can be computed with equal accuracy:

$$x_1 = -\frac{c}{b}\,\frac{1}{F} \qquad x_2 = -\frac{b}{a}F$$

Rewrite the two roots:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

x_2 is acceptable for all values; x_1 is unacceptable for $4ac \ll b^2$.

Rewrite x_2:

$$x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Now, instead of using the formula for x_1 directly, use the property of the quadratic that $x_1 x_2 = \frac{c}{a}$:

$$x_1 = \frac{c}{a}\,\frac{1}{x_2} = -\frac{c}{a}\,\frac{a}{b}\,\frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}}$$

Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\,\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Rewrite the two roots:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

x_2 is acceptable for all values; x_1 is unacceptable for $4ac \ll b^2$.

Rewrite x_2:

$$x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Now, instead of using the formula for x_1 directly, use the property of the quadratic that $x_1 x_2 = \frac{c}{a}$:

$$x_1 = \frac{c}{a}\,\frac{1}{x_2} = -\frac{c}{a}\,\frac{a}{b}\,\frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}}$$

Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\,\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a(x-x_1)(x-x_2) \end{aligned}$$

$x_1 x_2 = \frac{c}{a}$

Rewrite the two roots:

$$x_1 = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad x_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

x_2 is acceptable for all values; x_1 is unacceptable for $4ac \ll b^2$.

Rewrite x_2:

$$x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Now, instead of using the formula for x_1 directly, use the property of the quadratic that $x_1x_2 = \frac{c}{a}$:

$$x_1 = \frac{c}{a}\frac{1}{x_2} = -\frac{c}{a}\frac{a}{b}\frac{1}{\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}}$$

Thus, the improved formulas for the quadratic roots are:

$$x_1 = -\frac{c}{b}\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

More elegant form:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

root ratio

More elegant form:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a(x - x_1)(x - x_2)$$

root ratio

More elegant form:

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

This is exact for all values.

If $Q > 0.5$, F is complex $\Rightarrow$ complex roots

If $Q < 0.5$, F is real $\Rightarrow$ real roots

If $Q \ll 0.5$, $F \approx 1$

Note how simple the analytic roots, and therefore the quadratic factorization, become if $F \approx 1$.

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

root ratio

More elegant form:

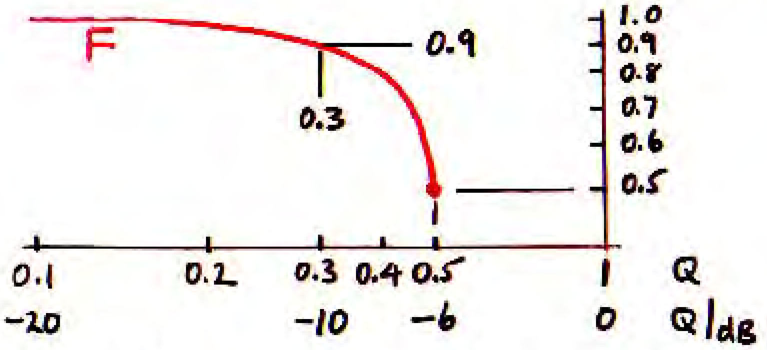
$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad \frac{x_1}{x_2} = \frac{Q^2}{F^2} \qquad x_2 = -\frac{b}{a}F$$

simple ratios of the original quadratic coefficients

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2} \quad \text{in which} \quad Q^2 \equiv \frac{ac}{b^2}$$

$F \to 1$ very rapidly as Q drops below 0.5:



$F \approx 1$ with 10% error for $Q \leq 0.3$

Remember this graph!

One more time!

$$\begin{aligned} ax^2+bx+c &= a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right) \\ &= a(x-x_1)(x-x_2) \end{aligned}$$

Bad!

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Good!

$$x_1 = -\frac{c}{b}\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

One more time!

$$ax^2+bx+c = a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$$
$$= a(x-x_1)(x-x_2)$$

Bad!

$$x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

Good!

$$x_1 = -\frac{c}{b}\frac{1}{\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]} \qquad x_2 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

Further information:

TT DVD Ch 3

Paper

SCIENTIFIC AMERICAN
BOOK CLUB

P.O. Box 6301
Camp Hill, PA 17012-6301

CHANGE SERVICE REQUESTED

www.sciambookclub.com

July 2005

PRSRT STD
U.S. POSTAGE
PAID
BOOK CLUB
ASSOCIATES

Math in action!

Apply mathematical tools to real-world problems...
see inside for details.

IMPORTANT: Dated Materials Enclosed

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

SCIENTIFIC AMERICAN
BOOK CLUB

P.O. Box 6301
Camp Hill, PA 17012-6301

July 2005

CHANGE SERVICE REQUESTED

www.sciambookclub.com

Math in action!

Apply mathematical tools to real-world problems...
see inside for details.

$$x_1 = -\frac{b}{a}\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

$$x_2 = -\frac{c}{b}\Big/\left[\frac{1}{2}+\frac{1}{2}\sqrt{1-\frac{4ac}{b^2}}\right]$$

IMPORTANT: Dated Materials Enclosed

General result:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$= a\left(x + \frac{c}{b}\frac{1}{F}\right)\left(x + \frac{b}{a}F\right)$$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}$$

$$x_1 = -\frac{c}{b}\frac{1}{F}$$

$$x_2 = -\frac{b}{a}F$$

General result:

$$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a(x - x_1)(x - x_2)$$

$$= a\left(x + \frac{c}{b}\frac{1}{F}\right)\left(x + \frac{b}{a}F\right) \overset{F=1}{\approx} a\left(x + \frac{c}{b}\right)\left(x + \frac{b}{a}\right)$$

Good approximation for real roots, $Q \equiv \sqrt{\frac{ac}{b^2}} \leq 0.5$: $F \approx 1$

where

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4ac}{b^2}}$$

$$x_1 = -\frac{c}{b}\frac{1}{F} \approx -\frac{c}{b}$$

$$x_2 = -\frac{b}{a}F \approx -\frac{b}{a}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\frac{b}{c}x+\frac{a}{c}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\overset{\overset{a_1}{\downarrow}}{\frac{b}{c}}x+\overset{\overset{a_2}{\downarrow}}{\frac{a}{c}}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Redefine coefficients:

$$1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= \left(1+a_1Fx\right)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{a_2}{a_1^2}$$

Alternative format:

$$ax^2+bx+c = c\left(1+\overset{a_1}{\frac{b}{c}}x+\overset{a_2}{\frac{a}{c}}x^2\right)$$

$$= c\left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= c\left(1+\frac{b}{c}Fx\right)\left(1+\frac{a}{b}\frac{1}{F}x\right) \overset{F=1}{\approx} c\left(1+\frac{b}{c}x\right)\left(1+\frac{a}{b}x\right)$$

Good approximation for real roots, $Q \leq 0.5$: $F \approx 1$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2}$$

Redefine coefficients:

$$1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$= \left(1+a_1Fx\right)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right) \overset{F=1}{\approx} \left(1+a_1x\right)\left(1+\frac{a_2}{a_1}x\right)$$

where

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{a_2}{a_1^2}$$

Generalization: Improved Formulas for Roots of a Quadratic

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2}$$

$$Q^2 \equiv \frac{ac}{b^2} \qquad\qquad Q^2 \equiv \frac{a_2}{a_1^2}$$

$$ax^2+bx+c = a(x-x_1)(x-x_2) \qquad\qquad 1+a_1x+a_2x^2 = \left(1-\frac{x}{x_1}\right)\left(1-\frac{x}{x_2}\right)$$

$$x_1 = -\frac{c}{b}\frac{1}{F} \qquad\qquad x_1 = -\frac{1}{a_1F}$$

$$\frac{x_1}{x_2} = \frac{Q^2}{F^2}$$

$$x_2 = -\frac{b}{a}F \qquad\qquad x_2 = -\frac{a_1}{a_2}F$$

$$ax^2+bx+c = a\left(x+\frac{c}{b}\frac{1}{F}\right)\left(x+\frac{b}{a}F\right) \qquad\qquad 1+a_1x+a_2x^2 = (1+a_1Fx)\left(1+\frac{a_2}{a_1}\frac{1}{F}x\right)$$

For real roots, $Q \le 0.5$ and $F \approx 1$:

$$x_1 \approx -\frac{c}{b} \qquad\qquad x_1 \approx -\frac{1}{a_1}$$

$$\frac{x_1}{x_2} \approx Q^2$$

$$x_2 \approx -\frac{b}{a} \qquad\qquad x_2 \approx -\frac{a_1}{a_2}$$

$$\underset{\underset{\text{①}}{\uparrow}}{a}x^2+\underset{\underset{\text{②}}{\uparrow}}{b}x+\underset{\underset{\text{③}}{\uparrow}}{c} \approx a\left(x+\underset{\underset{\text{③}/\text{②}}{\uparrow}}{\frac{c}{b}}\right)\left(x+\underset{\underset{\text{②}/\text{①}}{\uparrow}}{\frac{b}{a}}\right) \qquad \underset{\underset{\text{①}}{\uparrow}}{1}+\underset{\underset{\text{②}}{\uparrow}}{a_1x}+\underset{\underset{\text{③}}{\uparrow}}{a_2x^2} \approx \left(1+\underset{\underset{\text{②}/\text{①}}{\uparrow}}{a_1x}\right)\left(1+\underset{\underset{\text{③}/\text{②}}{\uparrow}}{\frac{a_2}{a_1}}x\right)$$

Advantages over the conventional formulas

1. Both roots can be computed with equal accuracy (avoids small difference of large numbers).
2. For real roots, to a very good approximation, there is no $\sqrt{\quad}$ anywhere in the results, and each root is a simple ratio of coefficients of the original quadratic.

Useful format of quadratic $1+a_1 s+a_2 s^2$

Define: $Q \equiv \frac{\sqrt{a_2}}{a_1}$

If $Q>0.5$ (F complex), roots are complex.

Leave in quadratic form:

$$1+a_1 s+a_2 s^2 = 1+\underbrace{\frac{a_1}{\sqrt{a_2}}}_{\frac{1}{Q}}\left(\sqrt{a_2}\,s\right)+\underbrace{\left(\sqrt{a_2}\,s\right)}_{\text{normalized frequency}}{}^2$$

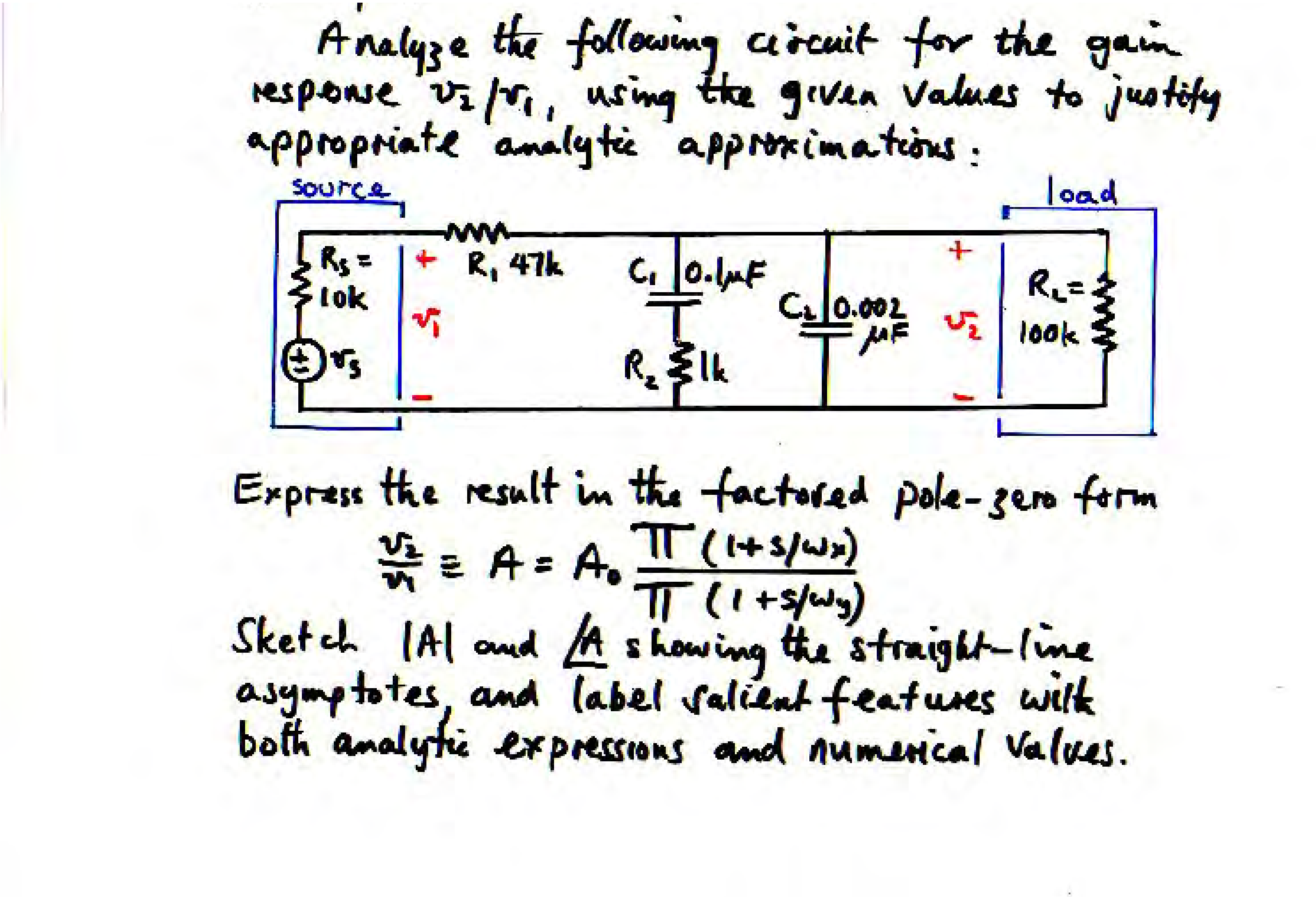
If $Q<0.5$ $(F \approx 1)$, roots are real.

Factor into two real roots: real corner frequencies

$$1+a_1 s+a_2 s^2 \approx (1+a_1 s)\left(1+\frac{a_2}{a_1}s\right)$$

$$= \left[1+\underbrace{\frac{a_1}{\sqrt{a_2}}}_{\frac{1}{Q}}\left(\sqrt{a_2}\,s\right)\right]\left[1+\underbrace{\frac{\sqrt{a_2}}{a_1}}_{Q}\underbrace{\left(\sqrt{a_2}\,s\right)}_{\text{normalized frequency}}\right]$$

Return to the circuit example:

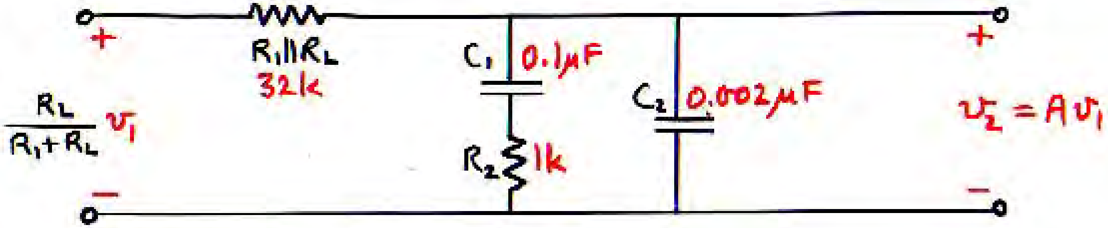


Analyze the following circuit for the gain response v_2/v_1, using the given values to justify appropriate analytic approximations:

Express the result in the factored pole-zero form

$$\frac{v_2}{v_1} \equiv A = A_0 \frac{\prod(1+s/\omega_x)}{\prod(1+s/\omega_y)}$$

Sketch $|A|$ and $\angle A$ showing the straight-line asymptotes, and label salient features with both analytic expressions and numerical values.



$$A = \frac{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}}{\dfrac{\frac{1}{sC_2}\left(R_2+\frac{1}{sC_1}\right)}{R_2+\frac{1}{s}\left(\frac{1}{C_1}+\frac{1}{C_2}\right)}+R_1\|R_L}\cdot\frac{R_L}{R_1+R_L}$$

less $\downarrow$ algebra

$$= \frac{R_L}{R_1+R_L}\cdot\frac{1+sC_1R_2}{1+s\left[C_1(R_2+R_1\|R_L)+\cancel{C_2(R_1\|R_L)}\right]+s^2\left[C_1C_2R_2(R_1\|R_L)\right]}$$

↑

Use of numerical values to justify analytic approximation

Find, both analytically and numerically, the Q and hence the roots ω_1 and ω_3 of the quadratic:

$$1 + C_1[R_2 + R_1 \| R_L]s + [C_1 C_2 R_2 (R_1 \| R_L)]s^2$$

where $C_1 = 0.1\mu F$, $C_2 = 0.002\mu F$, $R_1 = 47k$, $R_2 = 1k$, $R_L = 100k$. Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.

Find, both analytically and numerically, the Q and hence the roots w_1 and w_3 of the quadratic:

$$1 + \underbrace{C_1[R_2 + R_1\|R_L]}_{a_1}s + \underbrace{[C_1C_2R_2(R_1\|R_L)]}_{a_2}s^2$$

where $C_1 = 0.1\mu F$, $C_2 = 0.002\mu F$, $R_1 = 47k$, $R_2 = 1k$, $R_L = 100k$. Express the analytic results in terms of series/parallel element combinations, and express the numerical results in Hz or kHz.

$$Q^2 = \frac{a_2}{a_1^2} = \frac{C_1C_2R_2(R_1\|R_L)}{C_1^2(R_2 + R_1\|R_L)^2} = \frac{C_2}{C_1}\frac{R_2\|\cancel{R_1\|R_L}}{\cancel{R_2}+R_1\|R_L} \approx \frac{C_2}{C_1}\frac{R_2}{R_1\|R_L} = \frac{1}{50}\frac{1}{47\|100} = \frac{1}{1{,}600}$$

Hence, $F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1-4Q^2} \approx 1$, so the roots are real: $(F = 0.9994)$ $\quad Q = \frac{1}{40} << 0.5$

$$w_1 = \frac{1}{a_1} = \frac{1}{C_1(R_2 + R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})}\,Hz = 48\,Hz$$

$$w_3 = \frac{a_1}{a_2} = \frac{C_1(R_2 + R_1\|R_L)}{C_1C_2R_2(R_1\|R_L)}$$

$$= \frac{1}{C_2(R_2\|R_1\|R_L)} \qquad f_3 = \frac{159}{0.002(\underbrace{1\|32}_{0.97})}\,Hz = 82\,kHz$$

Hence

$$A \approx \frac{R_L}{R_L+R_1} \frac{1+sC_1R_2}{[1+C_1(R_2+R_1||R_L)s][1+C_2(R_1||R_2||R_L)s]}$$

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$A_0 \equiv \frac{R_L}{R_L+R_1}$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1||R_L)}$$

$$\omega_2 \equiv \frac{1}{C_1R_2}$$

$$\omega_3 \equiv \frac{1}{C_2(R_1||R_2||R_L)}$$

Hence

$$A \approx \frac{R_L}{R_L+R_1}\,\frac{1+sC_1R_2}{[1+C_1(R_2+R_1\|R_L)s][1+C_2(R_1\|R_2\|R_L)s]}$$

$$= A_0\frac{(1+\frac{s}{\omega_2})}{(1+\frac{s}{\omega_1})(1+\frac{s}{\omega_3})}$$

where

$$A_0 \equiv \frac{R_L}{R_L+R_1} \qquad = \frac{100}{100+47} = 0.68 \Rightarrow -3.4dB$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})} = 48Hz$$

$$\omega_2 \equiv \frac{1}{C_1R_2} \qquad f_2 = \frac{159}{0.1\times 1} = 1.6kHz$$

$$\omega_3 \equiv \frac{1}{C_2(R_1\|R_2\|R_L)} \qquad f_3 = \frac{159}{0.002(47\|1\|100)} = 82kHz$$

The conventional quadratic formula for the two poles ω_1 and ω_3 is much higher entropy (gives much less useful information) than does the modified formula.

Conventional:

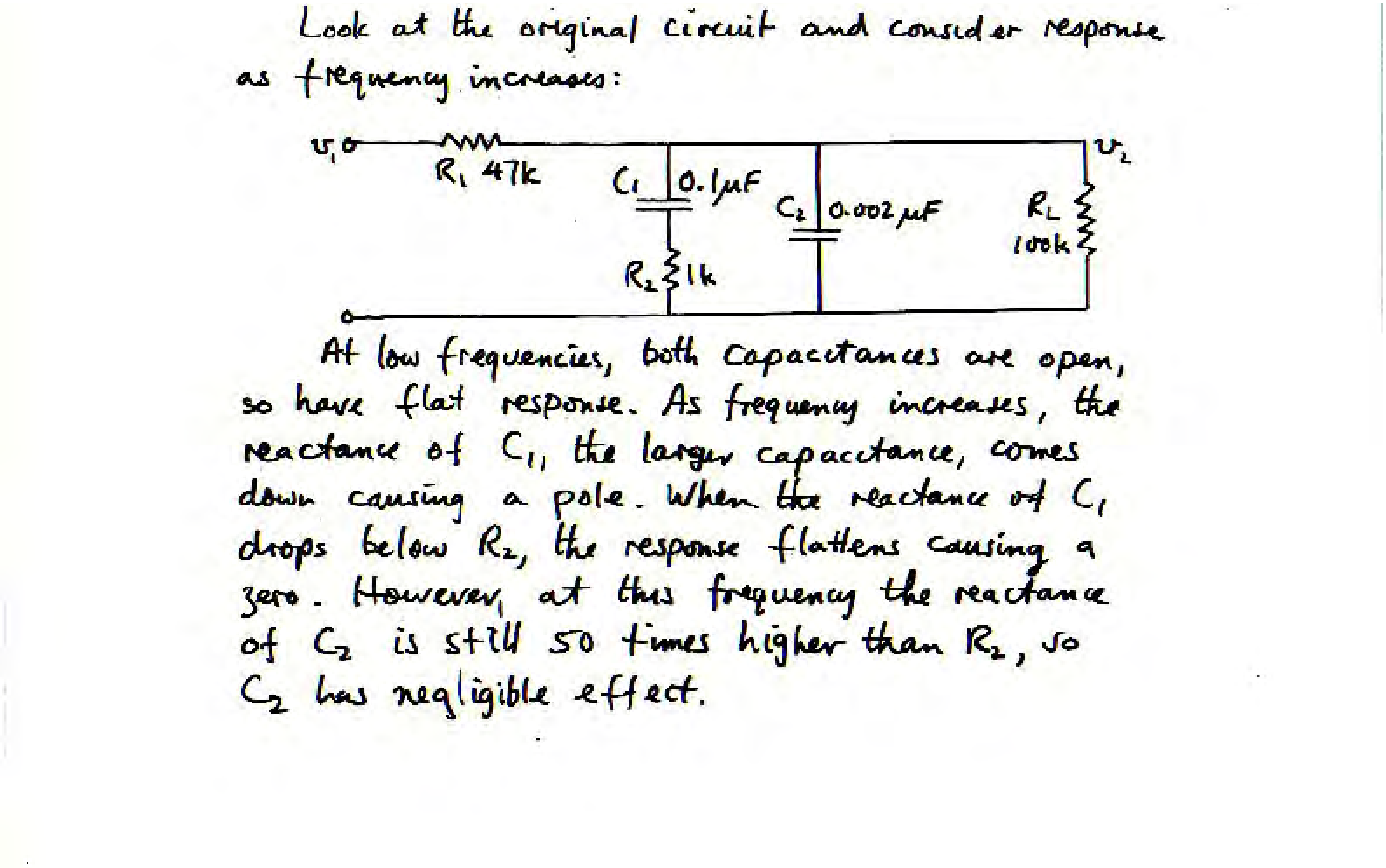
$$\frac{1}{\omega_{1,3}} = \frac{C_1(R_2+R_1\|R_L) \pm \sqrt{C_1^2(R_2+R_1\|R_L)^2 - 4C_1C_2R_2(R_1\|R_L)}}{2}$$

Modified:

$$\omega_1 = \frac{1}{C_1(R_2+R_1\|R_L)} \qquad \omega_3 = \frac{1}{C_2(R_1\|R_2\|R_L)}$$

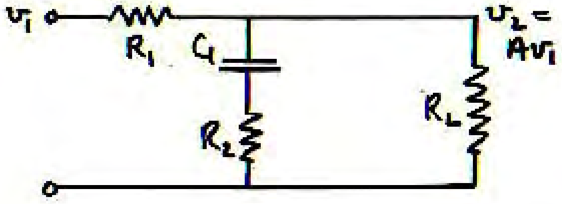
Note, in particular, (when the two roots are real and well-separated) that the modified formula is much lower entropy and not only gives both roots with equal numerical accuracy, but also exposes the fact that C_1 affects only ω_1 and C_2 affects only ω_3 — which is useful information for design purposes.

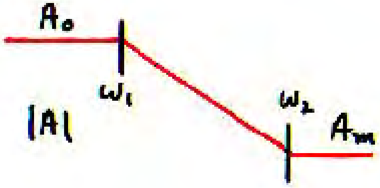
A still better solution:
Apply the mental frequency sweep

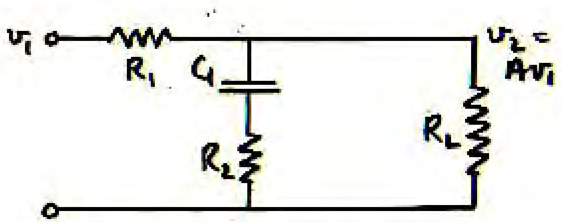


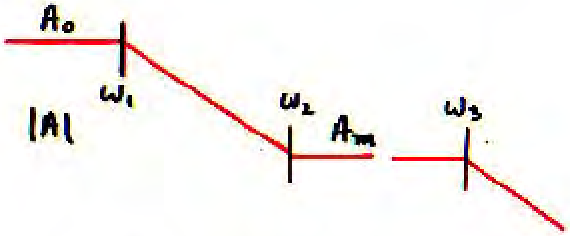
Look at the original circuit and consider response as frequency increases:

At low frequencies, both capacitances are open, so have flat response. As frequency increases, the reactance of C_1, the larger capacitance, comes down causing a pole. When the reactance of C_1 drops below R_2, the response flattens causing a zero. However, at this frequency the reactance of C_2 is still 50 times higher than R_2, so C_2 has negligible effect.

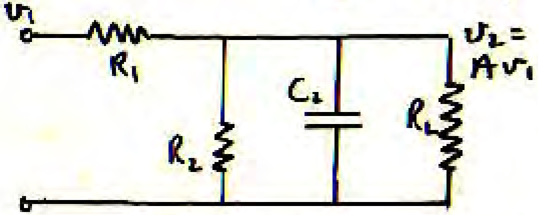

v_1
R_1
C_1
R_2
$v_2 = Av_1$
R_L


A_0
ω_1
$|A|$
ω_2
A_m



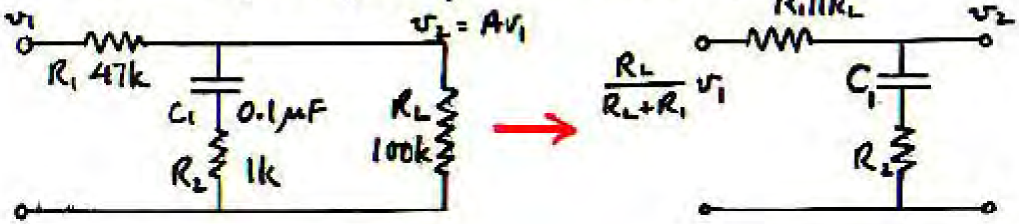


At still higher frequencies, the reactance of C_2 drops below R_2, causing a second pole



Hence, the solution can be obtained in two parts, each containing only one reactance (one pole).

Low-frequency range (C_2 open)



$$A = \frac{R_L}{R_L+R_1}\,\frac{R_2+\frac{1}{sC_1}}{R_2+\frac{1}{sC_1}+R_1\|R_L} = A_0\,\frac{1+\frac{s}{\omega_2}}{1+\frac{s}{\omega_1}} = A_m\frac{1+\frac{\omega_2}{s}}{1+\frac{\omega_1}{s}}$$

where

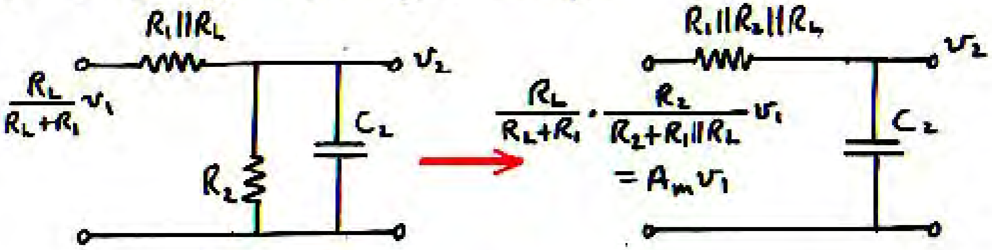
$$A_0 \equiv \frac{R_L}{R_L+R_1} = \frac{100}{100+47} = 0.68 \Rightarrow -3.4\,dB$$

$$\omega_1 \equiv \frac{1}{C_1(R_2+R_1\|R_L)} \qquad f_1 = \frac{159}{0.1(1+\underbrace{47\|100}_{32})} = 48\,Hz$$

$$\omega_2 \equiv \frac{1}{C_1R_2} \qquad f_2 = \frac{159}{0.1\times 1} = 1.6\,kHz$$

$$A_m \equiv A_0\frac{\omega_1}{\omega_2} = \frac{R_L}{R_L+R_1}\cdot\frac{R_2}{R_2+R_1\|R_L} = 0.68\,\frac{0.048}{1.6} = 0.02 \Rightarrow -34\,dB$$

High-frequency range (C_1 short)

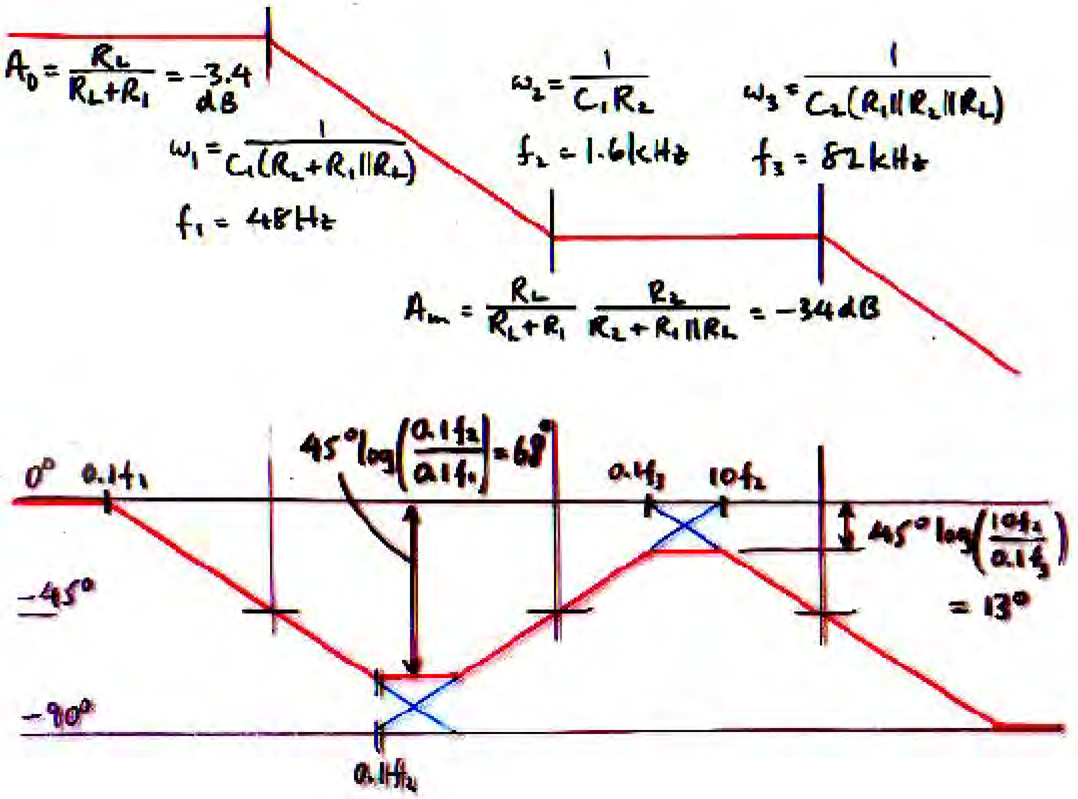


$$A = A_m \frac{1}{1+\frac{s}{\omega_3}} \quad \text{where} \quad \omega_3 \equiv \frac{1}{C_2(R_1 \| R_2 \| R_L)}$$

$$f_3 = \frac{159}{0.002(47 \| 1 \| 100)} = 82\,\text{kHz}$$

Hence, overall response is

$$A = A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)} = A_m \frac{\left(1+\frac{\omega_2}{s}\right)}{\left(1+\frac{\omega_1}{s}\right)\left(1+\frac{s}{\omega_3}\right)}$$


A0 = RL/(RL+R1) = -3.4 dB
ω1 = 1/(C1(R2+R1||RL))
f1 = 48Hz
ω2 = 1/(C1R2)
f2 = 1.6kHz
ω3 = 1/(C2(R1||R2||RL))
f3 = 82kHz
Am = RL/(RL+R1) · R2/(R2+R1||RL) = -34dB
45° log(0.1f2/0.1f1) = 68°
0°
0.1f1
0.1f3
10f2
-45°
-90°
45° log(10f2/0.1f3) = 13°
0.1f2

$$= A_0 \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_3}\right)}$$

where

$$\frac{1}{\omega_{1,3}} = \frac{\overset{\overset{3.3\times10^{-3}}{\downarrow}}{C_1(R_2+R_1\|R_L)} \pm \sqrt{\overset{\overset{10\times10^{-6}}{\downarrow}}{C_1^2(R_2+R_1\|R_L)^2} - \overset{\overset{0.026\times10^{-6}}{\downarrow}}{4C_1C_2R_2(R_1\|R_L)}}}{2}$$

This is useless for <u>design</u>, and in any case is inaccurate numerically.

Generalization: Presentation of Results

Sketch magnitude and phase by straight-line asymptotes, and label salient features (flat gains, corner frequencies, Q's, etc.) with both analytic expressions and numerical values.

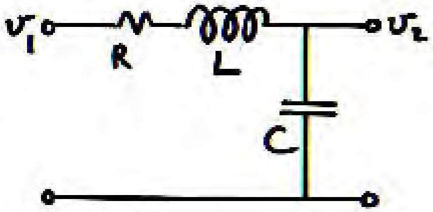
This is a compact summary so that both the analytic and numerical results can be interpreted at a glance, which is especially useful for reports, design reviews, etc. so that managers can easily and quickly see and understand the results obtained by others.

For design, the element values that must be changed to give different numerical results can easily be seen.

5. APPROXIMATIONS AND ASSUMPTIONS

How to build Low Entropy Expressions with minimum work

Double-pole low-pass LC filter



$$\frac{v_2}{v_1} = \frac{1}{1 + sRC + s^2LC}$$

$$= \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$= \frac{1}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

roots

in which

$\omega_0 \equiv \frac{1}{\sqrt{LC}}$ ← corner (resonant) frequency

$Q \equiv \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR} = \frac{R_0}{R}$ where $R_0 \equiv \sqrt{\frac{L}{C}}$

characteristic resistance

$Q < 0.5$: roots ω_1 and ω_2 are real

$Q > 0.5$: roots ω_1 and ω_2 are complex

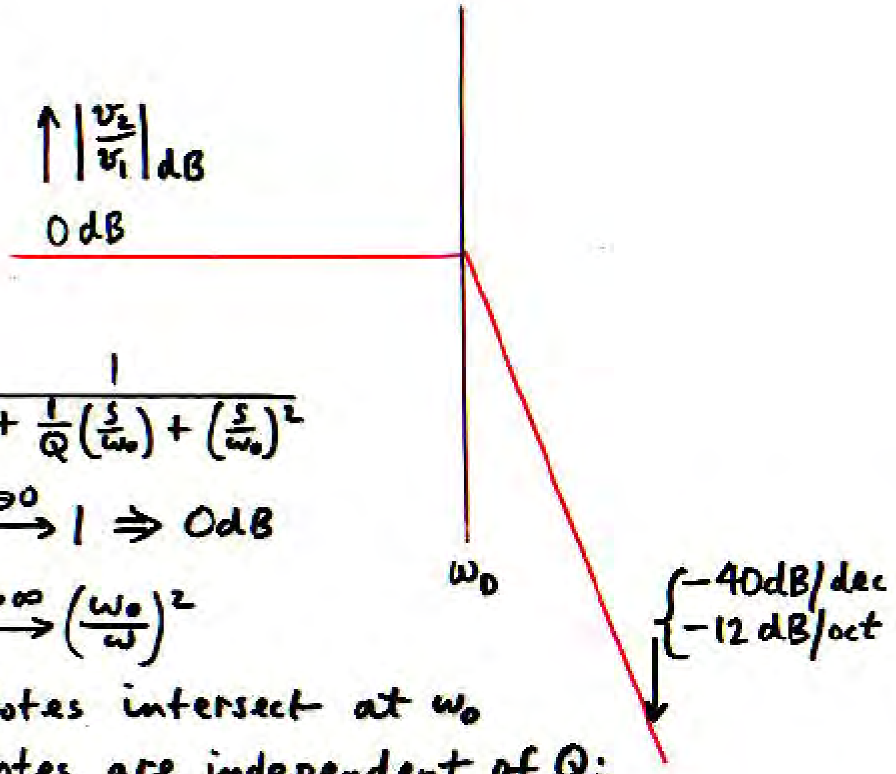
$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| \xrightarrow{\omega \to 0} 1 \Rightarrow 0\text{dB}$$

$$\xrightarrow{\omega \to \infty} \left(\frac{\omega_0}{\omega}\right)^2$$

Asymptotes intersect at ω_0

Asymptotes are independent of Q;
Q affects shape only in neighborhood of ω_0



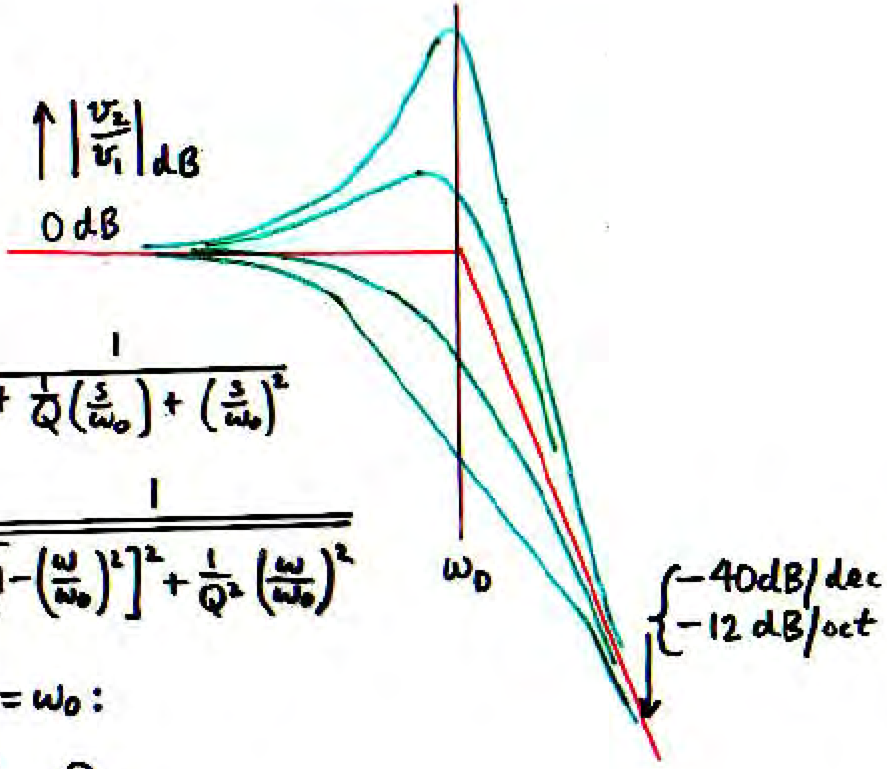
$$\frac{v_2}{v_1} = \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| \xrightarrow{\omega\to 0} 1 \Rightarrow 0\,\text{dB}$$

$$\xrightarrow{\omega\to\infty} \left(\frac{\omega_0}{\omega}\right)^2$$

Asymptotes intersect at ω_0

Asymptotes are independent of Q;
Q affects shape only in neighborhood of ω_0

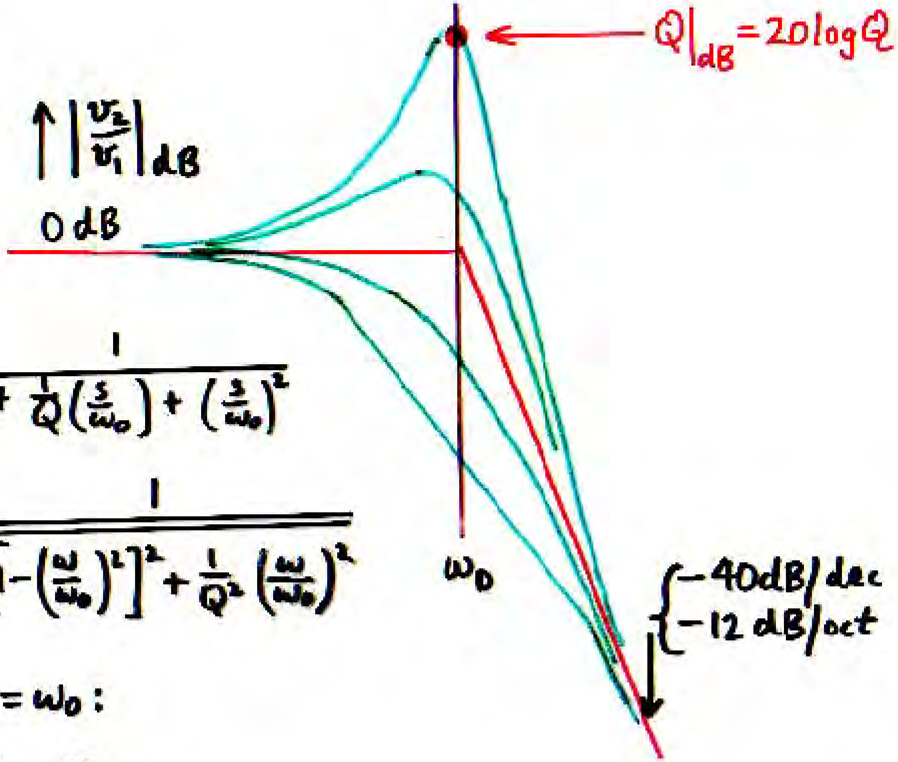


$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$

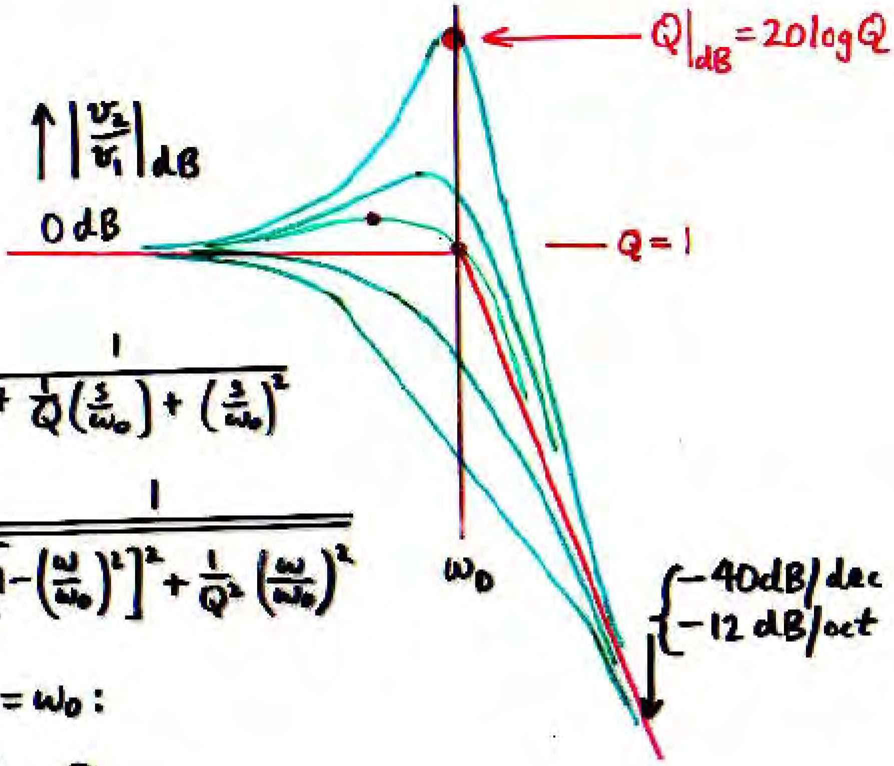


$$\frac{v_2}{v_1} = \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2+\frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$

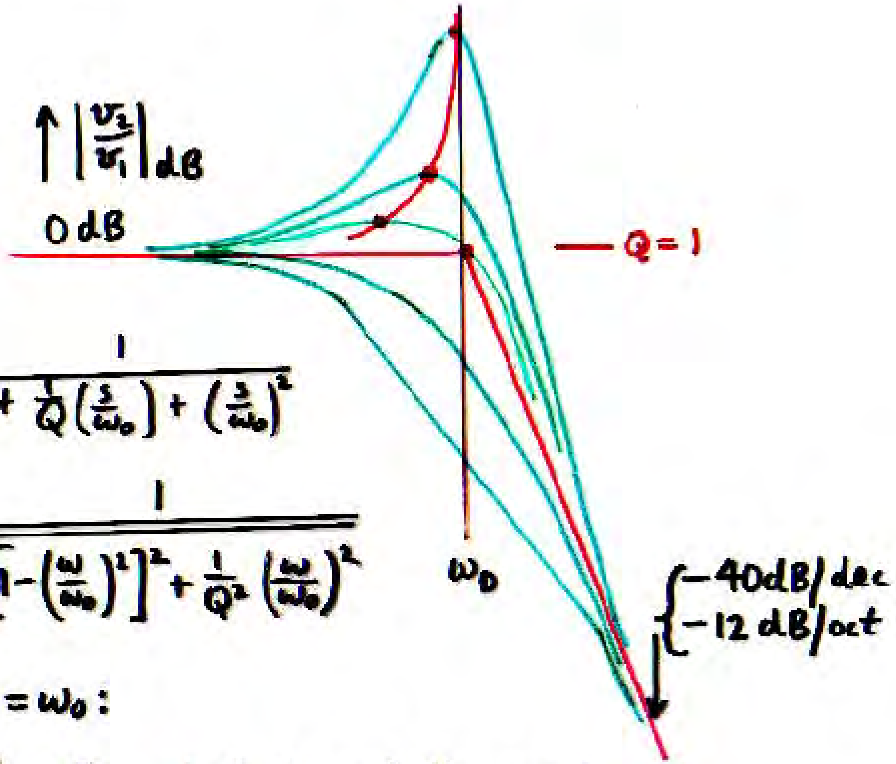


$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$

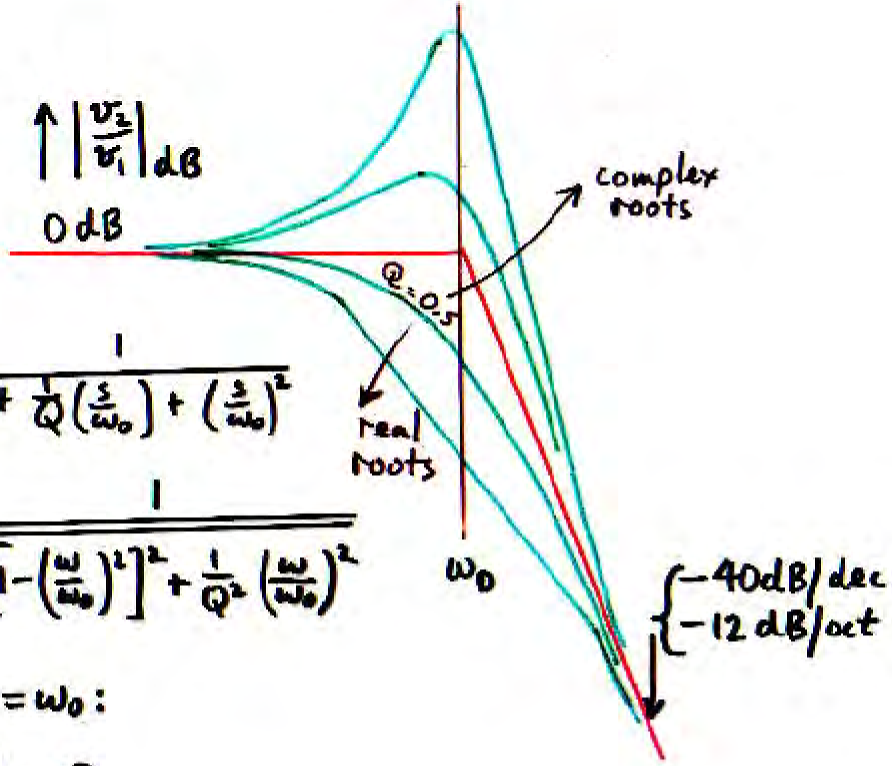


$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

At $\omega = \omega_0$:

$\left|\frac{v_2}{v_1}\right| = Q$ which is not the maximum; the maximum moves to the left for lower Q.



$$\frac{v_2}{v_1} = \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$\left|\frac{v_2}{v_1}\right| = \frac{1}{\sqrt{\left[1-\left(\frac{\omega}{\omega_0}\right)^2\right]^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2}}$$

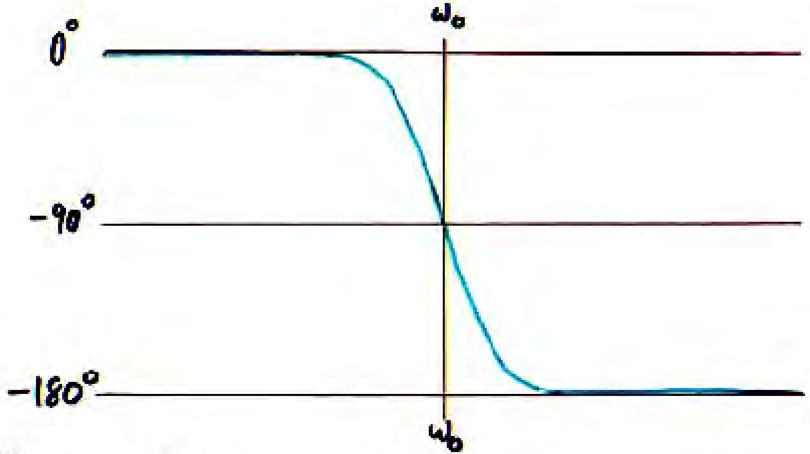
At $\omega = \omega_0$:

$$\left|\frac{v_2}{v_1}\right| = Q$$

Phase shape:

$$\angle \frac{v_2}{v_1} = -\tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

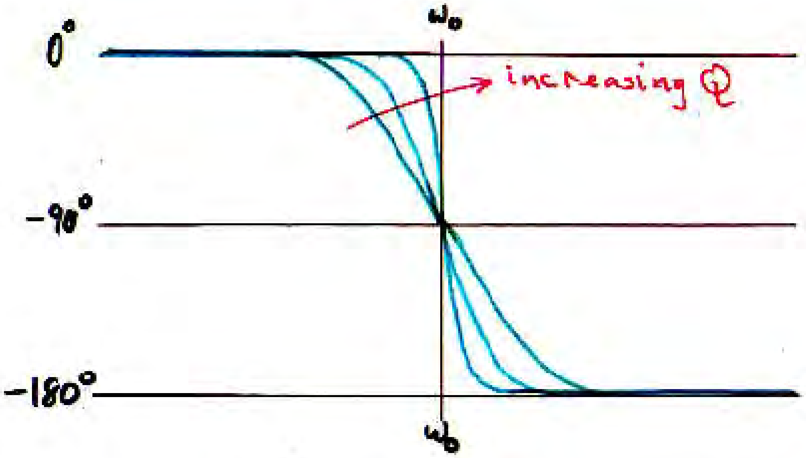
$$\left.\begin{aligned} &\xrightarrow{\omega\to 0} 0^\circ \\ &\xrightarrow{\omega=\omega_0} -90^\circ \\ &\xrightarrow{\omega=\infty} -180^\circ \end{aligned}\right\} \begin{gathered}\text{independent} \\ \text{of } Q\end{gathered}$$



Phase shape:

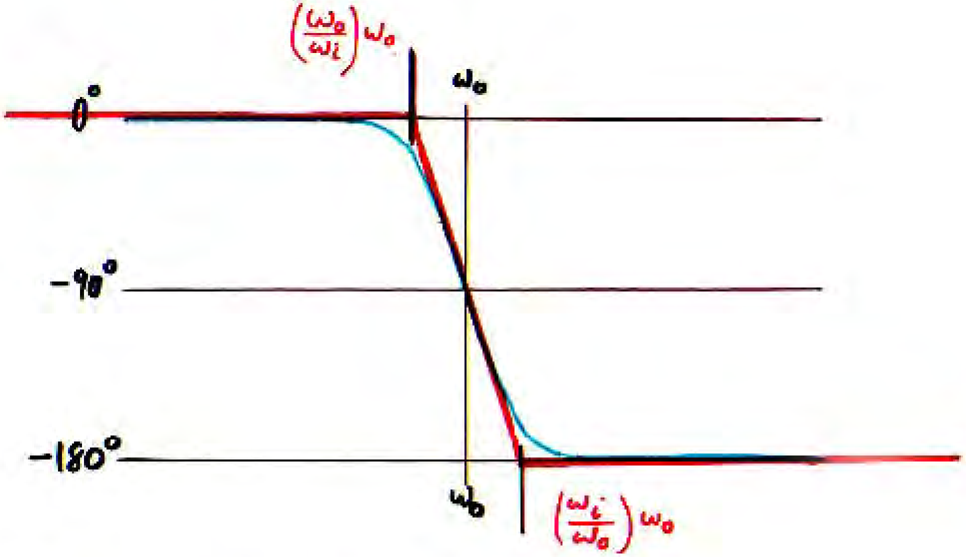
$$\angle\frac{v_2}{v_1} = -\tan^{-1}\left[\frac{\frac{1}{Q}\left(\frac{\omega}{\omega_0}\right)}{1-\left(\frac{\omega}{\omega_0}\right)^2}\right]$$

$$\left.\begin{array}{l} \xrightarrow{\omega\to 0} 0^\circ \\ \xrightarrow{\omega=\omega_0} -90^\circ \\ \xrightarrow{\omega=\infty} -180^\circ \end{array}\right\} \text{independent of } Q$$



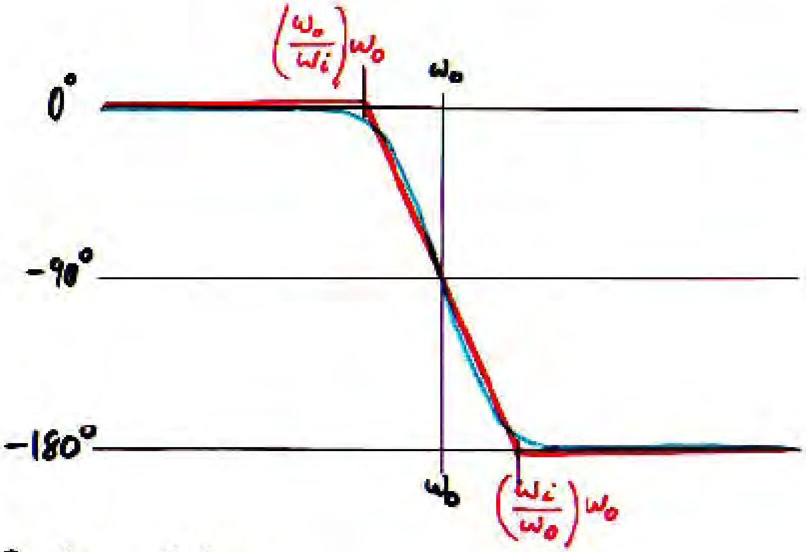
Increased Q causes sharper phase change between the 0° and − 180° asymptotes.

Need: a straight-line approximation.



Choose same slope at $\omega = \omega_o$:

$$\frac{\omega_i}{\omega_o} = \left(e^{\frac{\pi}{2}}\right)^{\frac{1}{2Q}} = (4.81)^{\frac{1}{2Q}}$$



Better choice :

$$\frac{\omega_i}{\omega_o} \approx 5^{\frac{1}{2Q}}$$

An even better choice is

$$\frac{\omega_i}{\omega_o} = 10^{\frac{1}{2Q}}$$

because for Q = 0.5 (two equal real roots)

$$\frac{\omega_i}{\omega_o} = 10$$

and the slope is $-90° / \text{dec}$, the same as twice the $-45° / \text{dec}$ slope for a single pole.

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} = A_1 \frac{1}{1 + \frac{a_1}{\sqrt{a_2}}\left(\sqrt{a_2}x\right) + \left(\sqrt{a_2}x\right)^2}$$

$x = \frac{s}{\omega_0}$

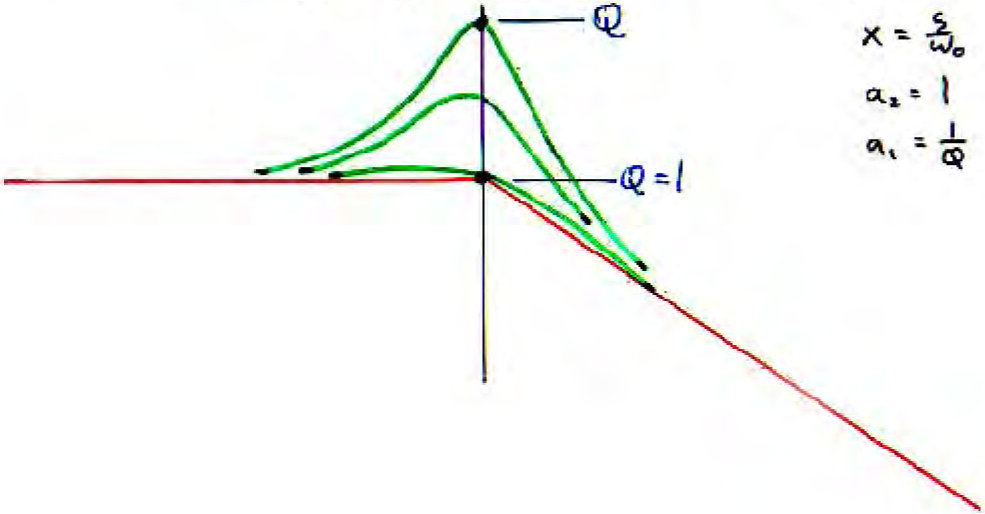
$a_2 = 1$

$a_1 = \frac{1}{Q}$

$x = s$

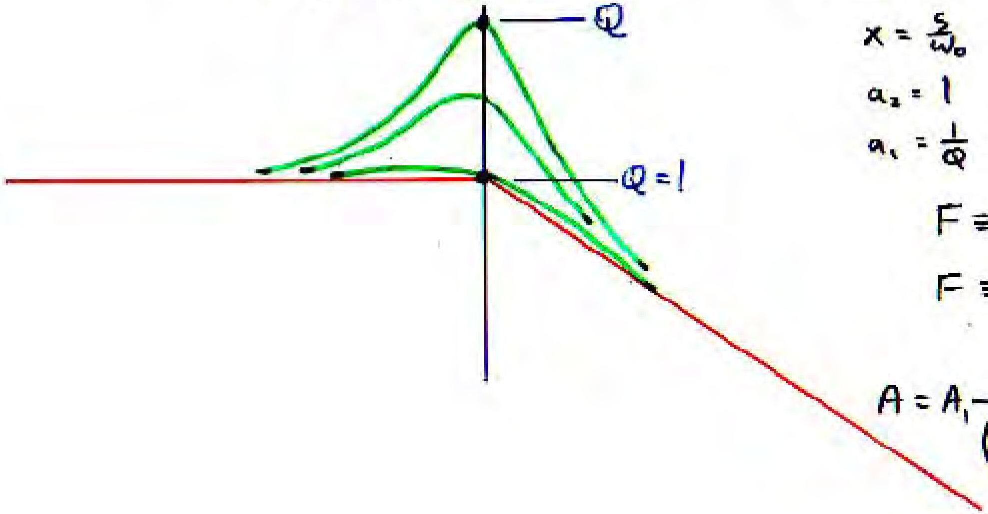
$a_2 = \frac{1}{\omega_0^2}$

$a_1 = \frac{1}{\omega_0 Q}$



Second-order response:

$$A = A_1 \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}$$



$$A = A_1 \frac{1}{1+a_1(x)+a_2(x)^2} = A_1 \frac{1}{1+\frac{a_1}{\sqrt{a_2}}(\sqrt{a_2}x)+(\sqrt{a_2}x)^2}$$

$x = \frac{s}{\omega_0}$

$a_2 = 1$

$a_1 = \frac{1}{Q}$

$x = s$

$a_2 = \frac{1}{\omega_0^2}$

$a_1 = \frac{1}{\omega_0 Q}$

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4a_2/a_1^2}$$

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$A = A_1 \frac{1}{(1+a_1 F x)\left(1+\frac{a_2}{a_1 F}x\right)}$$

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

$$A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} = A_1 \frac{1}{1 + \frac{a_1}{\sqrt{a_2}}(\sqrt{a_2}x) + (\sqrt{a_2}x)^2}$$

$x = \frac{s}{\omega_0}$

$a_2 = 1$

$a_1 = \frac{1}{Q}$

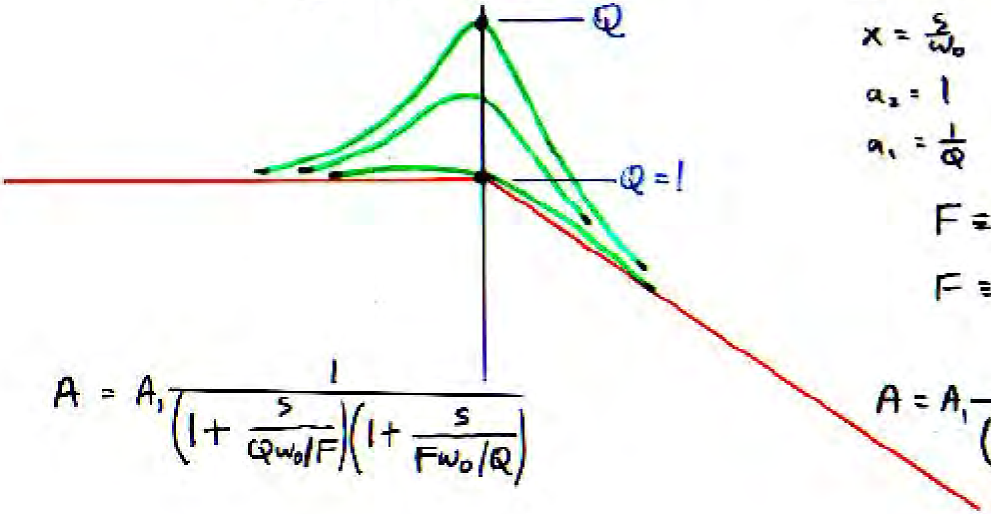
$x = s$

$a_2 = \frac{1}{\omega_0^2}$

$a_1 = \frac{1}{\omega_0 Q}$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4a_2/a_1^2}$$

$$F = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}$$



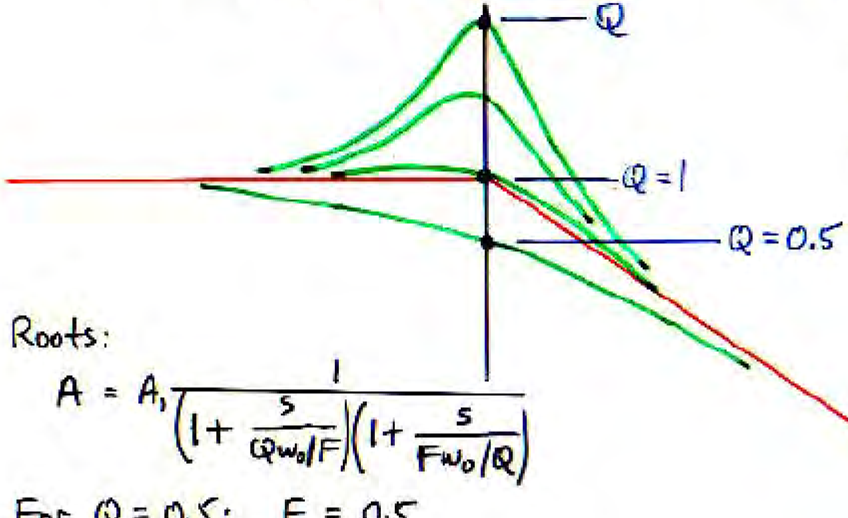
$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_0/F}\right)\left(1 + \frac{s}{F\omega_0/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F} x\right)}$$

Second-order response:

$$A = A_1 \frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2}$$

$$A = A_1 \frac{1}{1+a_1(x)+a_2(x)^2} = A_1 \frac{1}{1+\frac{a_1}{\sqrt{a_2}}(\sqrt{a_2}x)+(\sqrt{a_2}x)^2}$$



$x = \frac{s}{\omega_o}$

$a_2 = 1$

$a_1 = \frac{1}{Q}$

$x = s$

$a_2 = \frac{1}{\omega_o^2}$

$a_1 = \frac{1}{\omega_o Q}$

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4a_2/a_1^2}$$

$$F \equiv \frac{1}{2}+\frac{1}{2}\sqrt{1-4Q^2}$$

$$A = A_1 \frac{1}{(1+a_1Fx)\left(1+\frac{a_2}{a_1F}x\right)}$$

Roots:

$$A = A_1 \frac{1}{\left(1+\frac{s}{Q\omega_o/F}\right)\left(1+\frac{s}{F\omega_o/Q}\right)}$$

For $Q = 0.5$: $F = 0.5$

$$A = A_1 \frac{1}{\left(1+\frac{s}{\omega_o}\right)^2}$$

Second-order response:

$$A = A_1 \frac{1}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

$$A = A_1 \frac{1}{1 + a_1(x) + a_2(x)^2} = A_1 \frac{1}{1 + \frac{a_1}{\sqrt{a_2}}(\sqrt{a_2}x) + (\sqrt{a_2}x)^2}$$

$x = \frac{s}{\omega_o}$

$a_2 = 1$

$a_1 = \frac{1}{Q}$

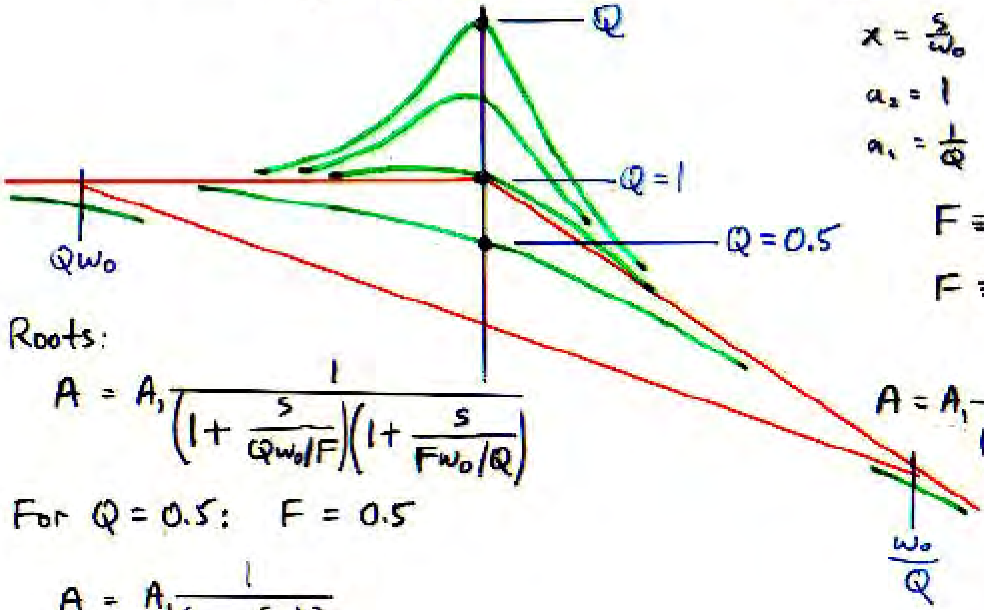
$x = s$

$a_2 = \frac{1}{\omega_o^2}$

$a_1 = \frac{1}{\omega_o Q}$

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4a_2/a_1^2}$$

$$F \equiv \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4Q^2}$$



Roots:

$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_o/F}\right)\left(1 + \frac{s}{F\omega_o/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 F x)\left(1 + \frac{a_2}{a_1 F} x\right)}$$

For $Q = 0.5$: $F = 0.5$

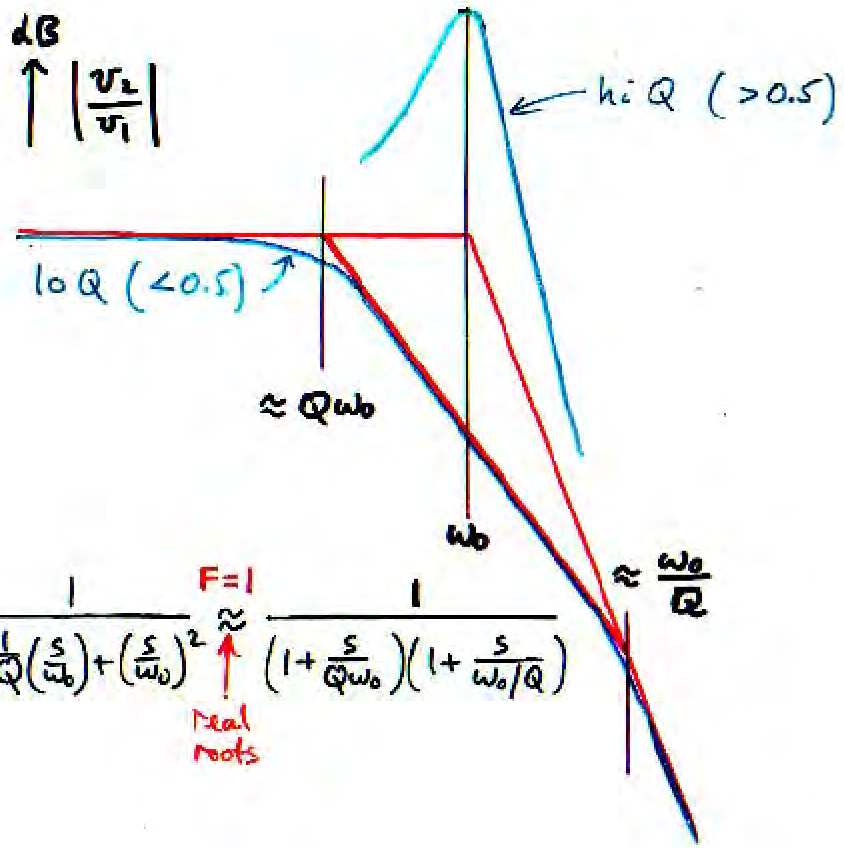
$$A = A_1 \frac{1}{\left(1 + \frac{s}{\omega_o}\right)^2}$$

For $Q << 0.5$: $F \approx 1$

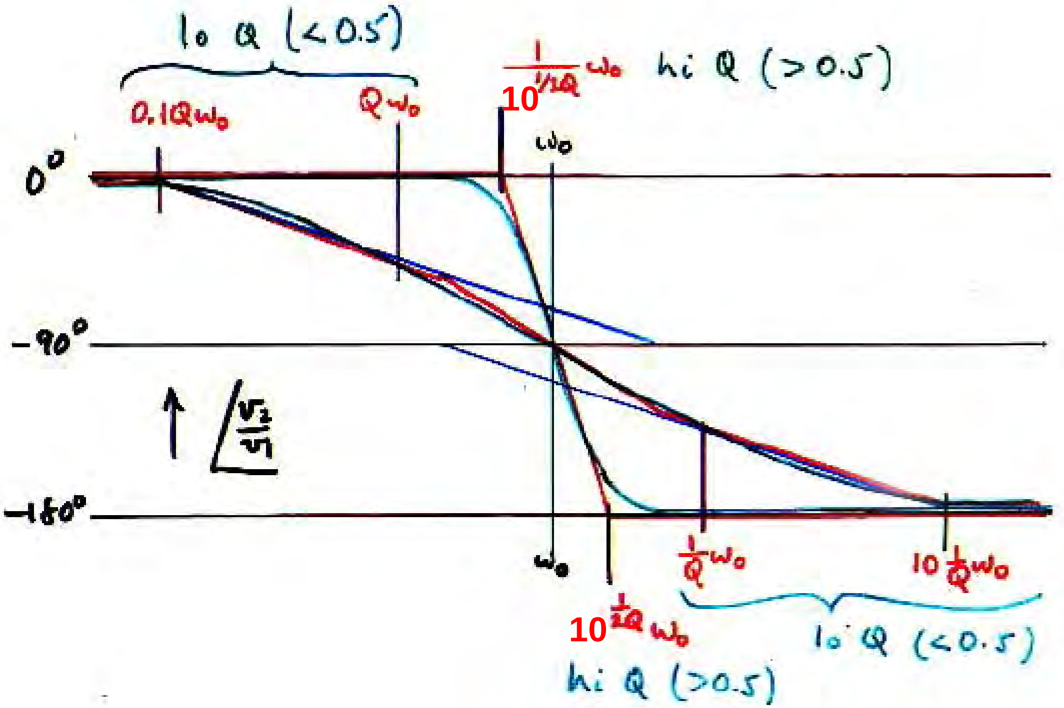
$$A = A_1 \frac{1}{\left(1 + \frac{s}{Q\omega_o}\right)\left(1 + \frac{s}{\omega_o/Q}\right)}$$

$$A = A_1 \frac{1}{(1 + a_1 x)\left(1 + \frac{a_2}{a_1} x\right)}$$

Low-pass 2-pole characteristic:



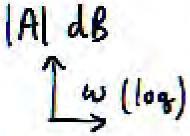
$$\frac{v_2}{v_1}=\frac{1}{1+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+\left(\frac{s}{\omega_0}\right)^2}\underset{\substack{\uparrow\\ \text{real}\\ \text{roots}}}{\overset{F=1}{\approx}}\frac{1}{\left(1+\frac{s}{Q\omega_0}\right)\left(1+\frac{s}{\omega_0/Q}\right)}$$

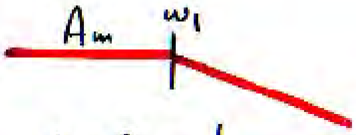

lo Q (<0.5)
0.1Qω₀
Qω₀
1 / 10^{1/2Q} ω₀
hi Q (>0.5)
ω₀
0°
-90°
-180°
∠V₂/V₁
ω₀
1/Q ω₀
10 1/Q ω₀
10^{1/2Q} ω₀
lo Q (<0.5)
hi Q (>0.5)

Normal and inverted poles and zeros:

$$A = A_m \frac{1}{(1+\frac{\omega_1}{s})}$$

inverted pole





$$A = A_m \frac{1}{(1+\frac{s}{\omega_1})}$$

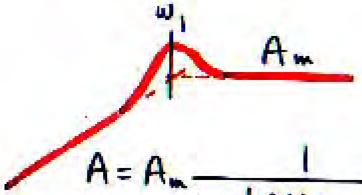
normal pole

$$A = A_m (1+\frac{\omega_1}{s})$$

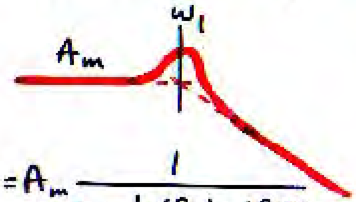
inverted zero

$$A = A_m (1+\frac{s}{\omega_1})$$

normal zero

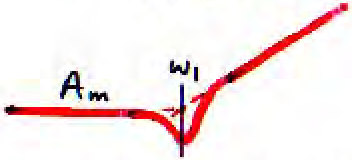


$$A = A_m \frac{1}{1+\frac{1}{Q}(\frac{\omega_1}{s})+(\frac{\omega_1}{s})^2}$$



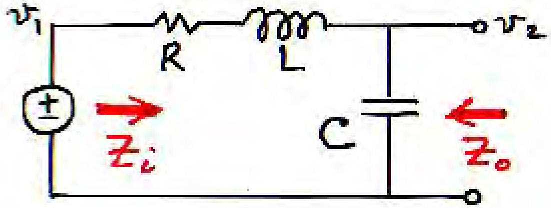
$$A = A_m \frac{1}{1+\frac{1}{Q}(\frac{s}{\omega_1})+(\frac{s}{\omega_1})^2}$$

$$A = A_m\left[1+\frac{1}{Q}\left(\frac{\omega_1}{s}\right)+\left(\frac{\omega_1}{s}\right)^2\right]$$



$$A = A_m\left[1+\frac{1}{Q}\left(\frac{s}{\omega_1}\right)+\left(\frac{s}{\omega_1}\right)^2\right]$$

Input and Output Impedances of low-pass filter



$$\omega_o \equiv \frac{1}{\sqrt{LC}}$$

$$R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q = \frac{R_o}{R} = \frac{1}{\omega_o CR} = \frac{\omega_o L}{R}$$

$$Z_i = \frac{1}{sC} + R + sL$$

$$\frac{1 + sCR + s^2LC}{sC}$$

$$Z_o = \frac{\frac{1}{sC}(R + sL)}{\frac{1}{sC} + R + sL}$$

$$= \frac{R + sL}{1 + sCR + s^2LC}$$

Express in terms of ω_o, Q, R_o :

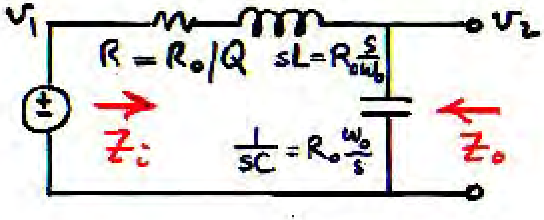
$$Z_i = \frac{1}{\omega_o C}\,\frac{1 + \omega_o CR\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$= R_o\,\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)}$$

$$Z_o = \omega_o L\,\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{R}{sL}\right)}{1 + \omega_o CR\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

$$= R_o\,\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Input and Output Impedances of low-pass filter



$$\omega_0 \equiv \frac{1}{\sqrt{LC}}$$

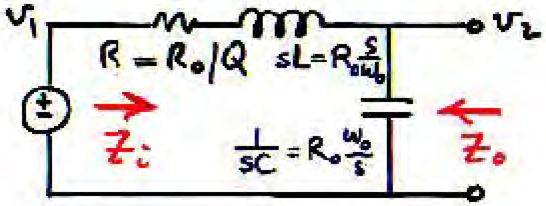
$$R_0 \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_0}{R}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s} \qquad Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

$$= R_0\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)} \qquad = R_0\frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

Note how the algebra is shortened when the analysis starts with the normalized element values.

Input and Output Impedances of low-pass filter



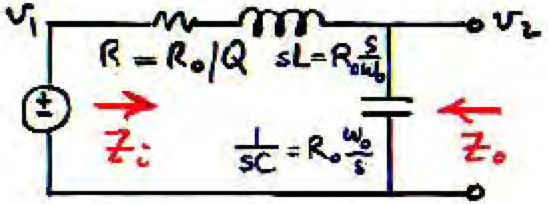
$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_o}{R}$$

$$Z_i = \frac{R_o}{Q} + R_o\frac{s}{\omega_o} + R_o\frac{\omega_o}{s} \qquad Z_o = \frac{\left(\frac{R_o}{Q} + R_o\frac{s}{\omega_o}\right)R_o\frac{\omega_o}{s}}{\frac{R_o}{Q} + R_o\frac{s}{\omega_o} + R_o\frac{\omega_o}{s}}$$

$$= R_o\frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}{\left(\frac{s}{\omega_o}\right)} \qquad = R_o\frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.

Input and Output Impedances of low-pass filter



$$\omega_0 \equiv \frac{1}{\sqrt{LC}}$$

$$R_0 \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_0}{R}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}$$

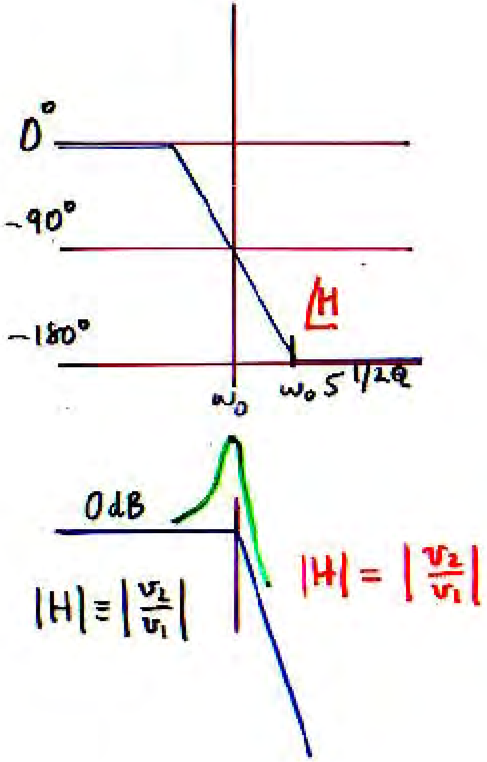
$$\boxed{= R_0 \frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)}}$$

$$Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

$$= R_0 \frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.

$$Z_i = \underset{\text{ref. value}}{R_o} \times \underset{\frac{1}{H}}{\left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right]} \times \underset{\text{single slope}}{\left[\frac{1}{\frac{s}{\omega_o}}\right]}$$

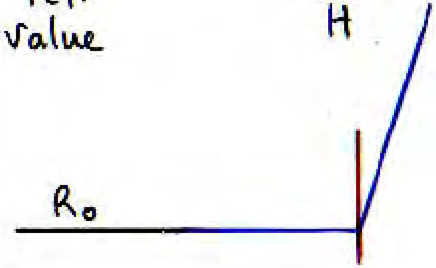

0°
−90°
−180°
∠H
ω₀
ω₀ 5^{1/2Q}
0dB
|H| ≡ |v₂/v₁|
|H| = |v₂/v₁|

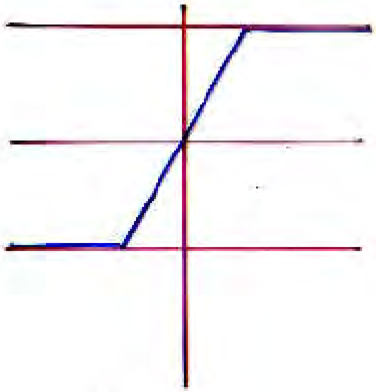
$$Z_i = R_o \times \left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

ref. value

$\frac{1}{H}$

single slope



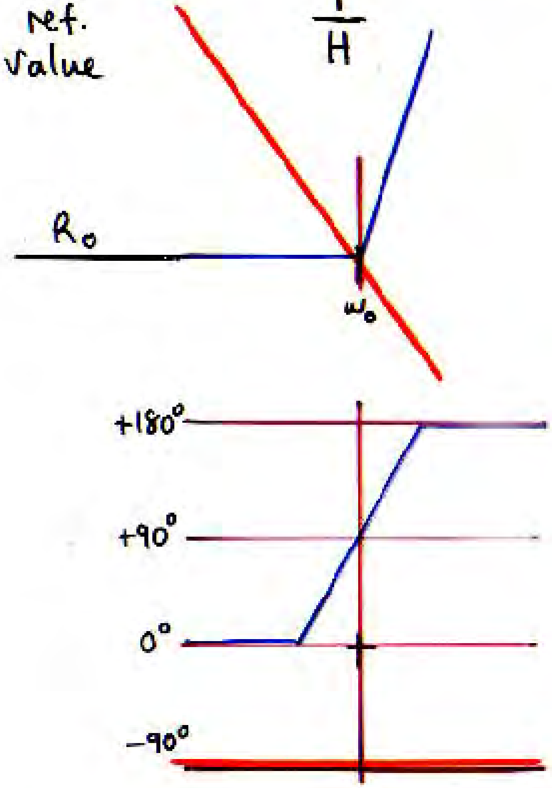


$$Z_i = R_o \times \left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

ref. value

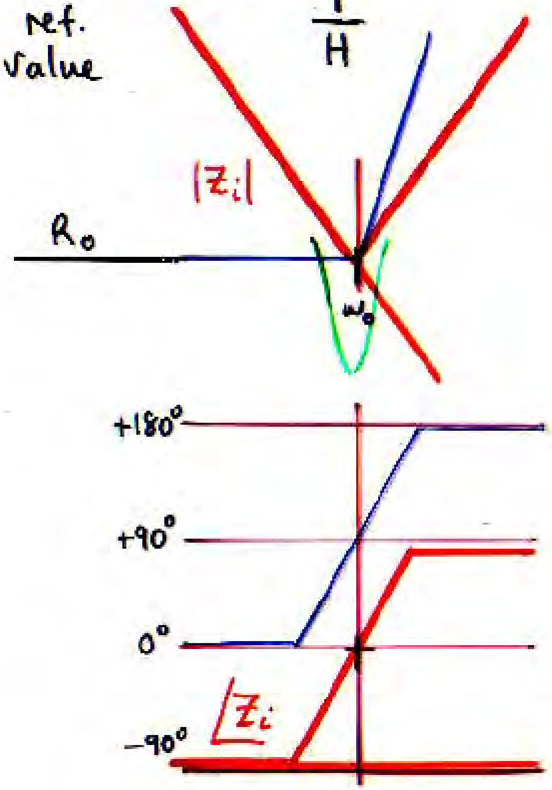
$\frac{1}{H}$

single slope

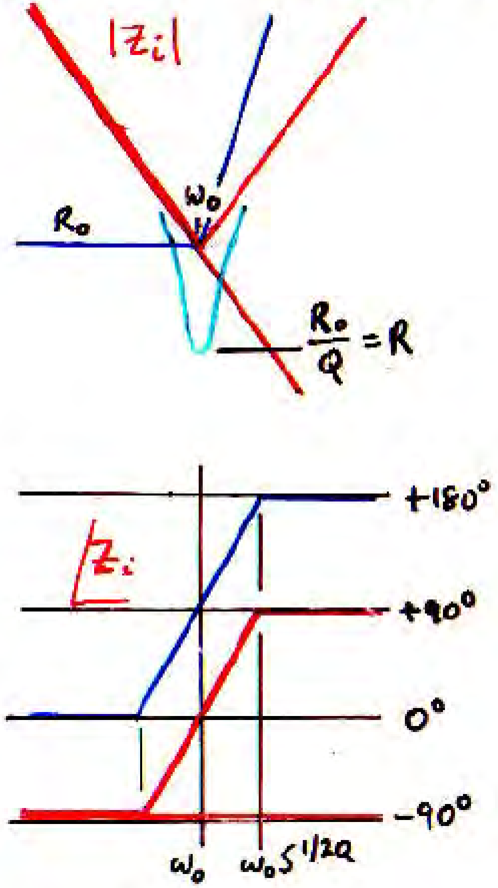


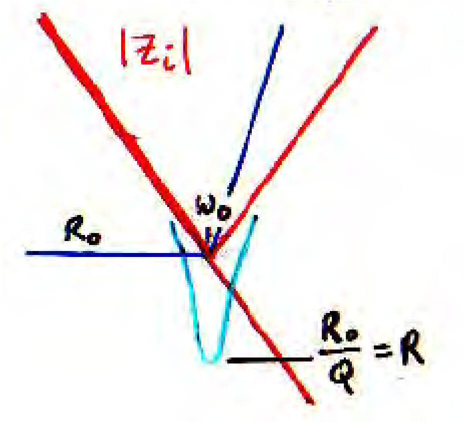
$$Z_i = R_o \times \left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right] \times \left[\frac{1}{\frac{s}{\omega_o}}\right]$$

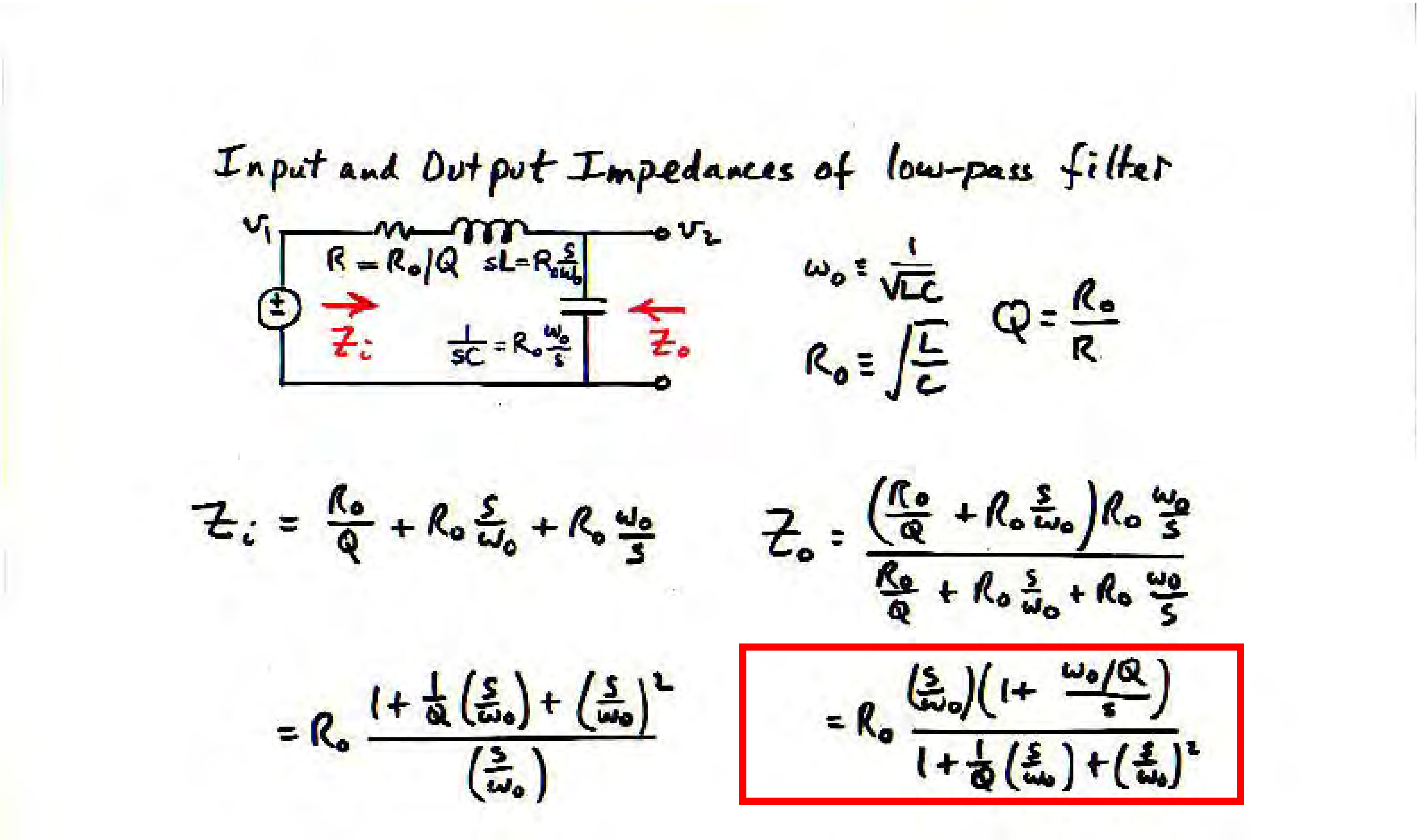
single slope



Asymptote sketches for high Q (>>0.5)




$|Z_i|$
ω_o
R_o
$\frac{R_o}{Q} = R$



Input and Output Impedances of low-pass filter

$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}} \qquad Q = \frac{R_0}{R}$$

$$Z_i = \frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}$$

$$= R_0 \frac{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)}$$

$$Z_o = \frac{\left(\frac{R_0}{Q} + R_0\frac{s}{\omega_0}\right)R_0\frac{\omega_0}{s}}{\frac{R_0}{Q} + R_0\frac{s}{\omega_0} + R_0\frac{\omega_0}{s}}$$

$$\boxed{= R_0 \frac{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q}{s}\right)}{1 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}}$$

Put the quantities you know you want in the answer into the statement of the problem as soon as possible, even into the circuit diagram.

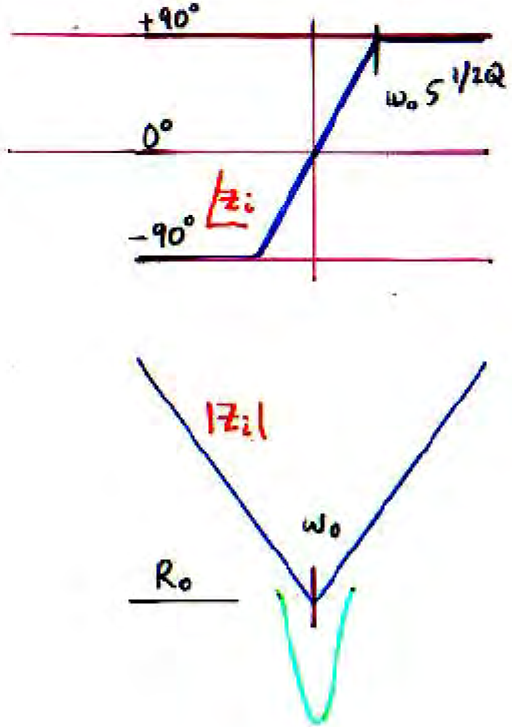
$$Z_o = R_o \times \underbrace{\left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}\right]}_{\frac{1}{Z_i}} \times \underbrace{\left(1 + \frac{\omega_o/Q}{s}\right)}_{\text{inverted zero}}$$

ref. value

$$\frac{1}{Z_i}$$

inverted zero

$$\underline{R_o}$$

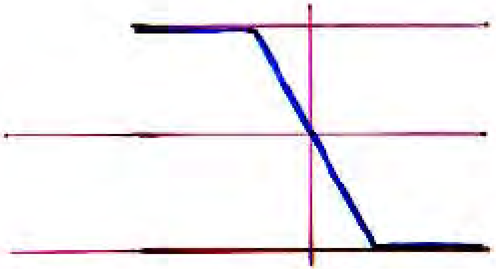

+90°
$\omega_o 5^{1/2Q}$
0°
$\angle Z_i$
-90°
$|Z_i|$
ω_0
R_o

$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1+\frac{\omega_o/Q}{s}\right)$$

ref. value

$\frac{1}{Z_i}$

inverted zero

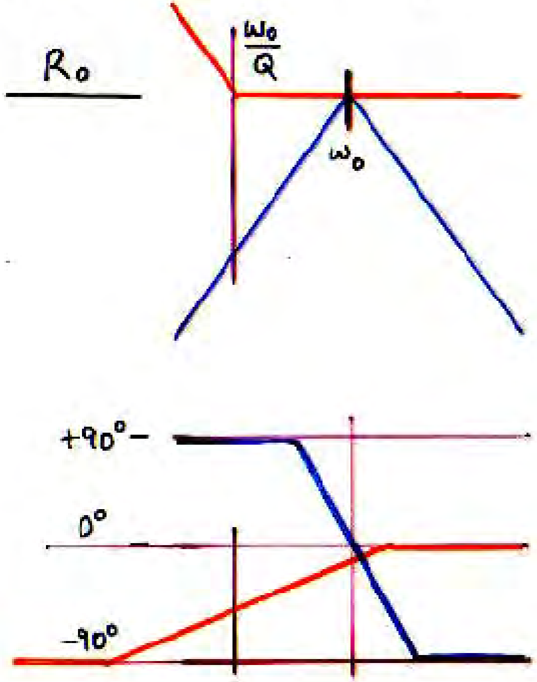


$$Z_o = R_o \times \left[\frac{\frac{s}{\omega_o}}{1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}\right] \times \left(1 + \frac{\omega_o/Q}{s}\right)$$

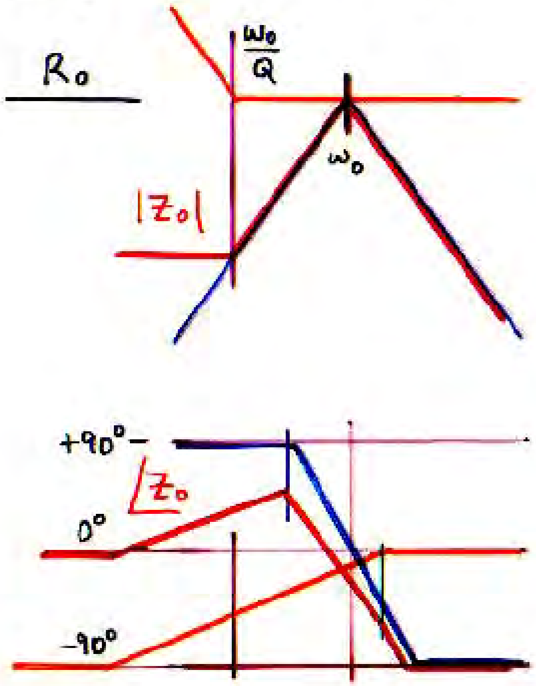
ref. value

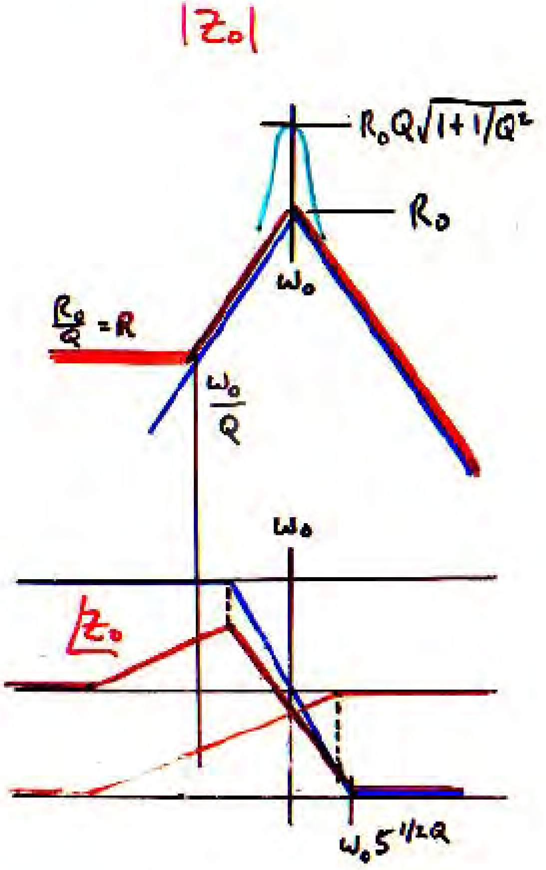
$\frac{1}{Z_i}$

inverted zero



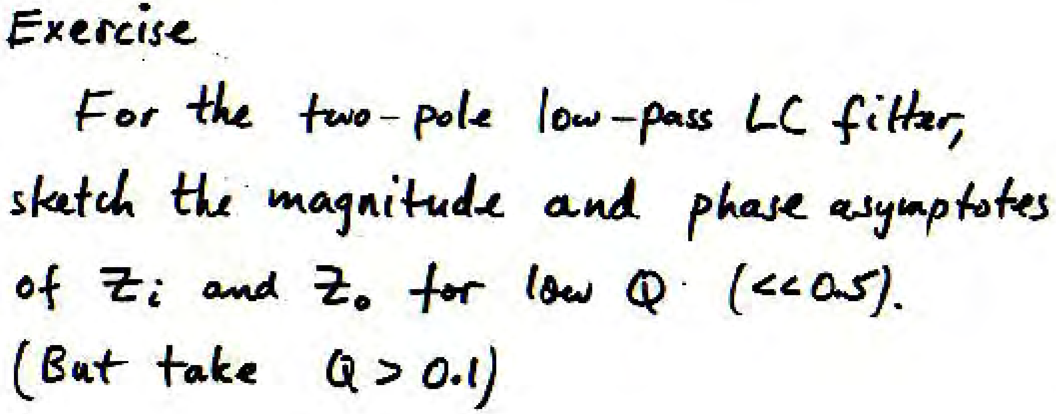
$$\underbrace{Z_o = R_o}_{\text{ref. value}} \times \underbrace{\left[\frac{\frac{s}{\omega_o}}{1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2}\right]}_{\frac{1}{Z_i}} \times \underbrace{\left(1+\frac{\omega_o/Q}{s}\right)}_{\text{inverted zero}}$$




|Z₀|
R₀Q√(1+1/Q²)
R₀
ω₀
R₀/Q = R
ω₀/Q
ω₀
∠Z₀
ω₀5^(1/2Q)

Exercise 5.1

Sketch asymptotes for Z_i and Z_o for low Q.



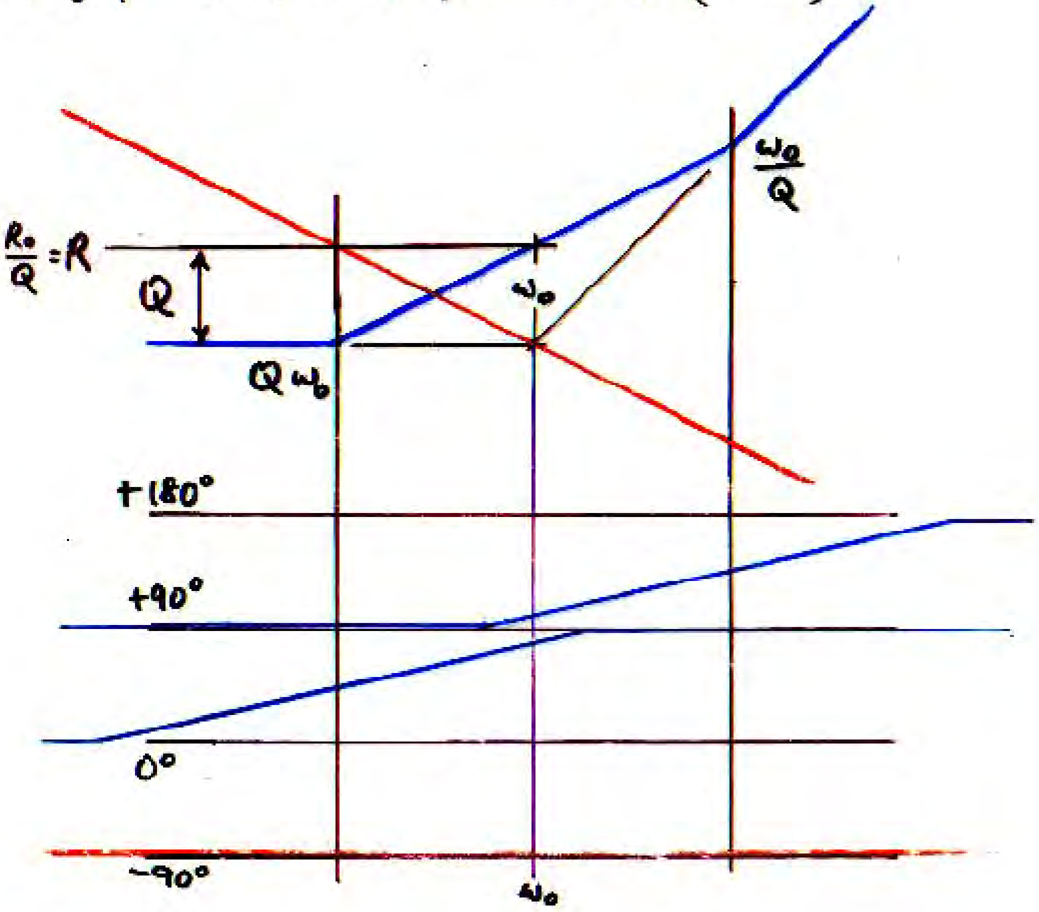
Exercise

For the two-pole low-pass LC filter, sketch the magnitude and phase asymptotes of Z_i and Z_o for low Q ($<<0.5$).

(But take $Q > 0.1$)

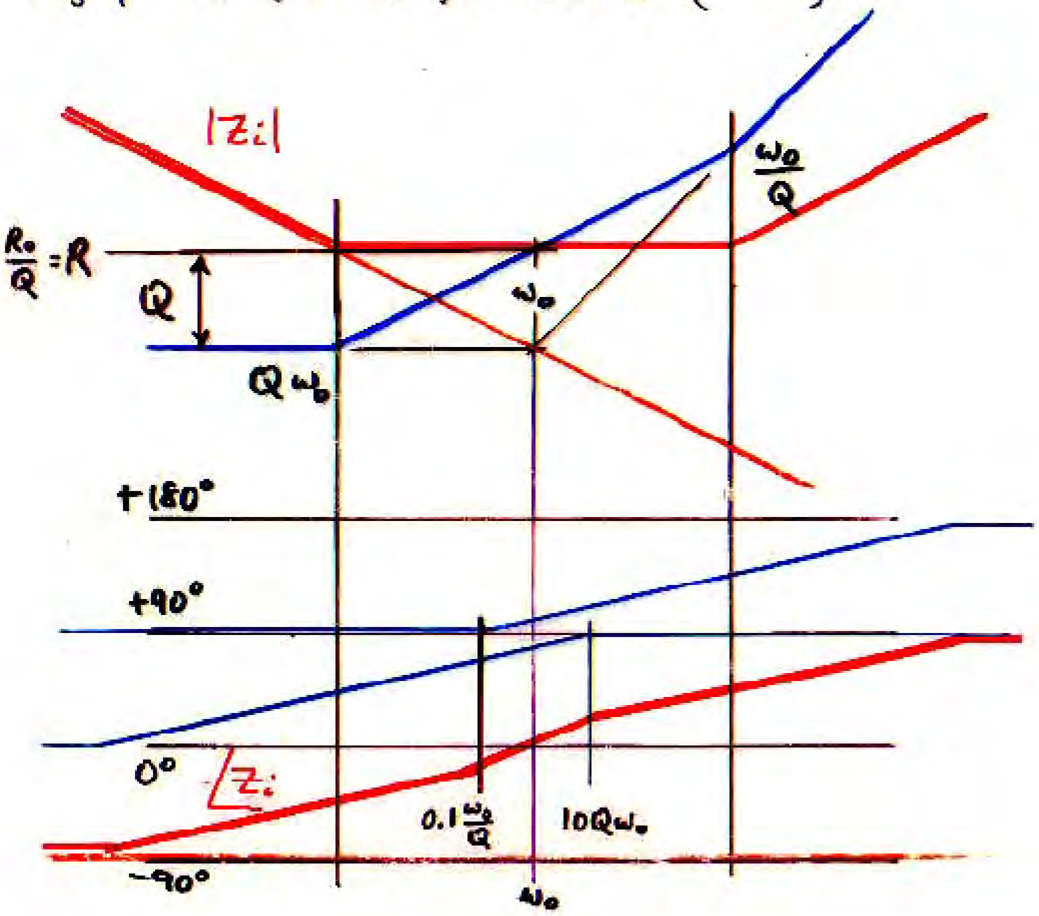
Exercise 5.1 - Solution

Asymptotes for Z_i for low Q ($<<0.5$):



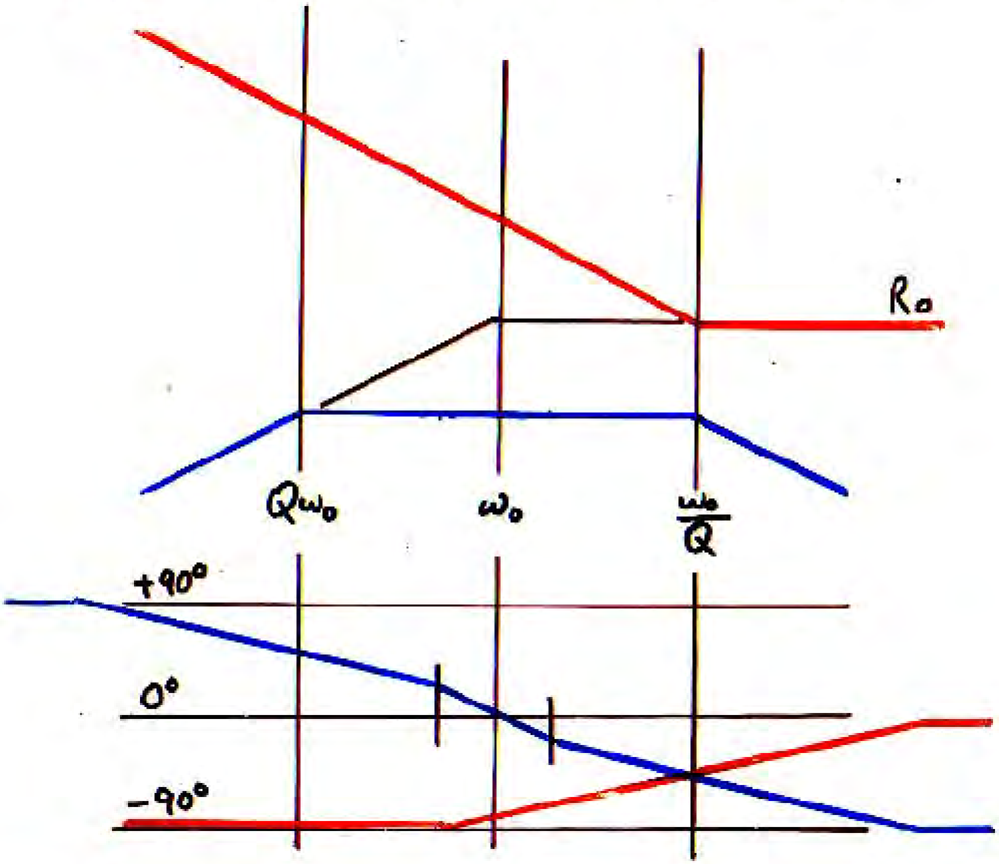
Exercise 5.1 - Solution

Asymptotes for Z_i for low Q (<<0.5):



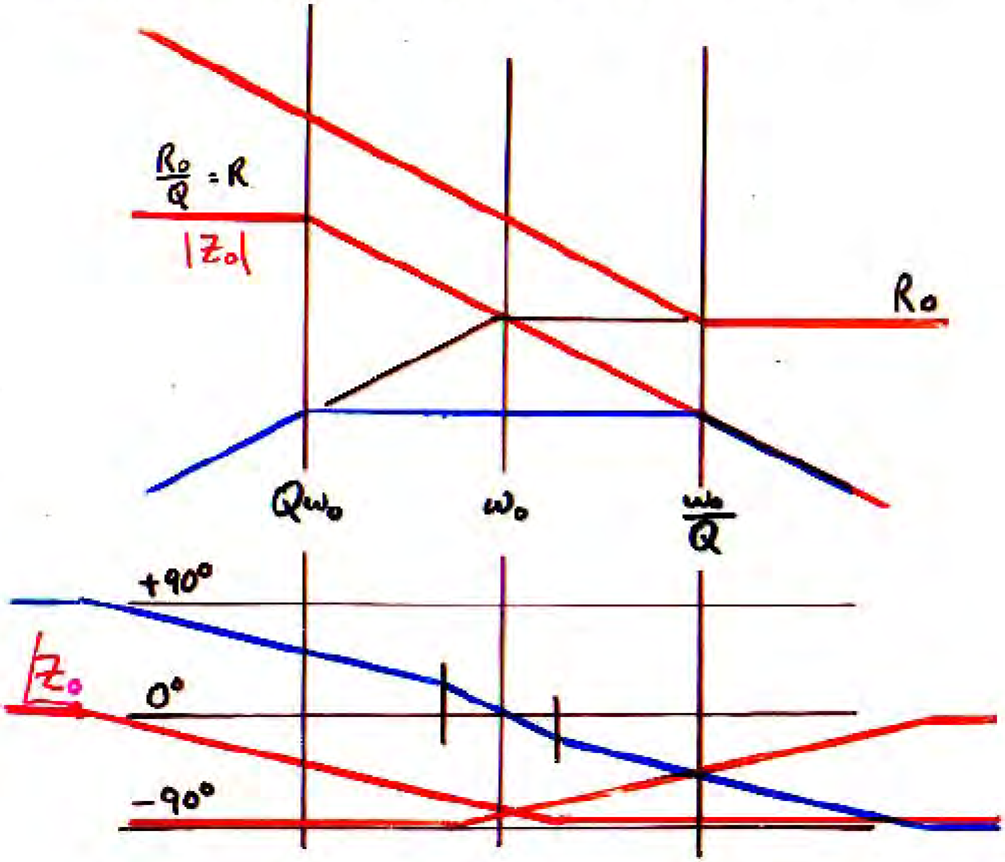
Exercise 5.1 - Solution

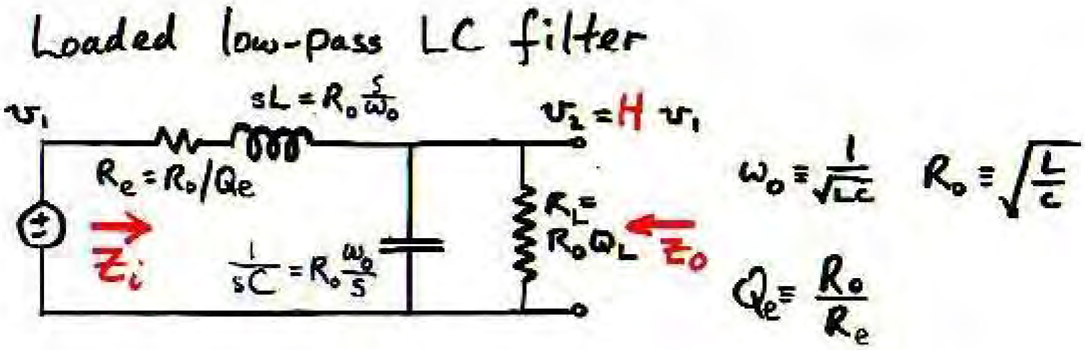
Asymptotes for Z_o for low Q (<< 0.5):



Exercise 5.1 - Solution

Asymptotes for Z_o for low Q ($<<0.5$):

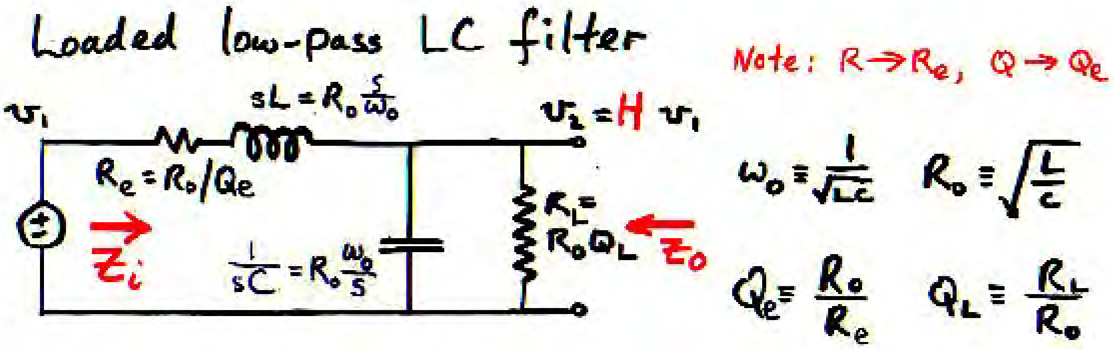




Since there are now two resistances, re-name

$$R \rightarrow R_e, \quad Q \rightarrow Q_e$$

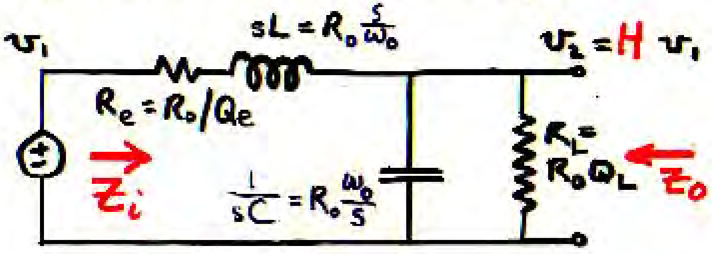
By analogy, define $Q_L \equiv \frac{R_L}{R_o}$



Why is Q_L defined "upside down" relative to Q_e ?

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

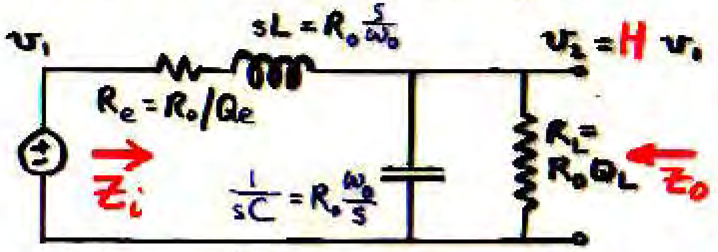
$$H = \frac{\dfrac{R_o Q_L}{1 + Q_L \frac{s}{\omega_o}}}{\dfrac{R_o Q_L}{1 + Q_L \frac{s}{\omega_o}} + \dfrac{R_o}{Q_e} + R_o \dfrac{s}{\omega_o}} = \frac{Q_L}{Q_L + \frac{1}{Q_e} + \left(\frac{Q_L}{Q_e} + 1\right)\left(\frac{s}{\omega_o}\right) + Q_L\left(\frac{s}{\omega_o}\right)^2}$$

$$= \frac{1}{1 + 1/Q_e Q_L} \; \frac{1}{1 + \dfrac{\left(\frac{1}{Q_e} + \frac{1}{Q_L}\right)}{1 + 1/Q_e Q_L}\left(\dfrac{s}{\omega_o}\right) + \dfrac{1}{1 + 1/Q_e Q_L}\left(\dfrac{s}{\omega_o}\right)^2}$$

Conventional result:

Loaded low-pass LC filter

Note: $R \to R_e$, $Q \to Q_e$



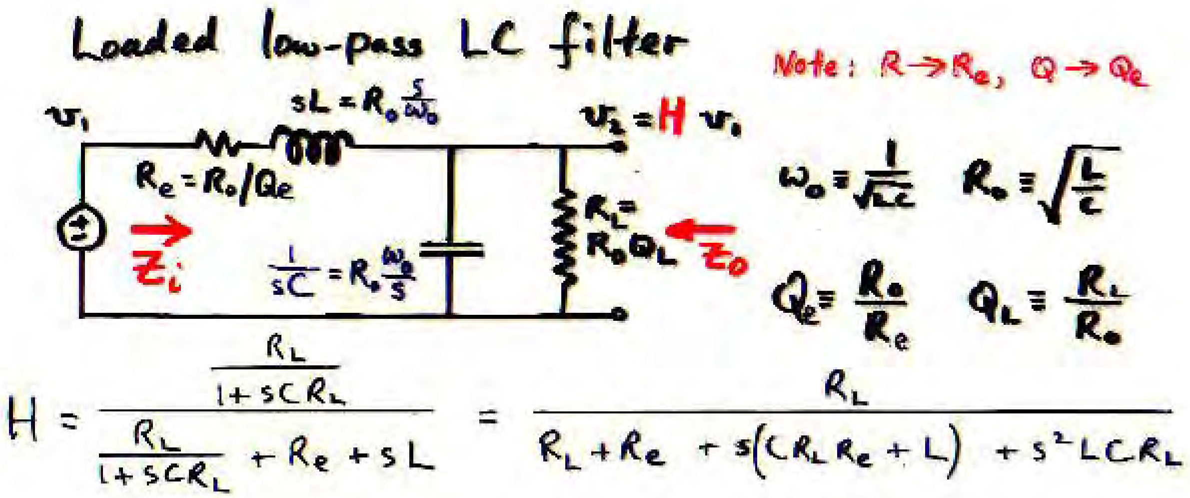
$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

$$H = \frac{\frac{R_L}{1+sCR_L}}{\frac{R_L}{1+sCR_L} + R_e + sL} = \frac{R_L}{R_L + R_e + s(CR_L R_e + L) + s^2 LCR_L}$$

Reveals no insight

Conventional result:



$$H = \frac{\frac{R_L}{1+sCR_L}}{\frac{R_L}{1+sCR_L} + R_e + sL} = \frac{R_L}{R_L + R_e + s(CR_LR_e + L) + s^2LCR_L}$$

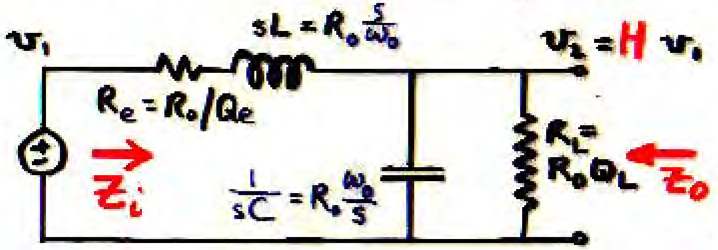
Reveals no insight

This high entropy result can be converted into the desired low entropy version by application of mental energy, but it takes quite an effort, and you have to know where you're going!

Conventional result:

Loaded low-pass LC filter

Note: $R \to R_e$, $Q \to Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

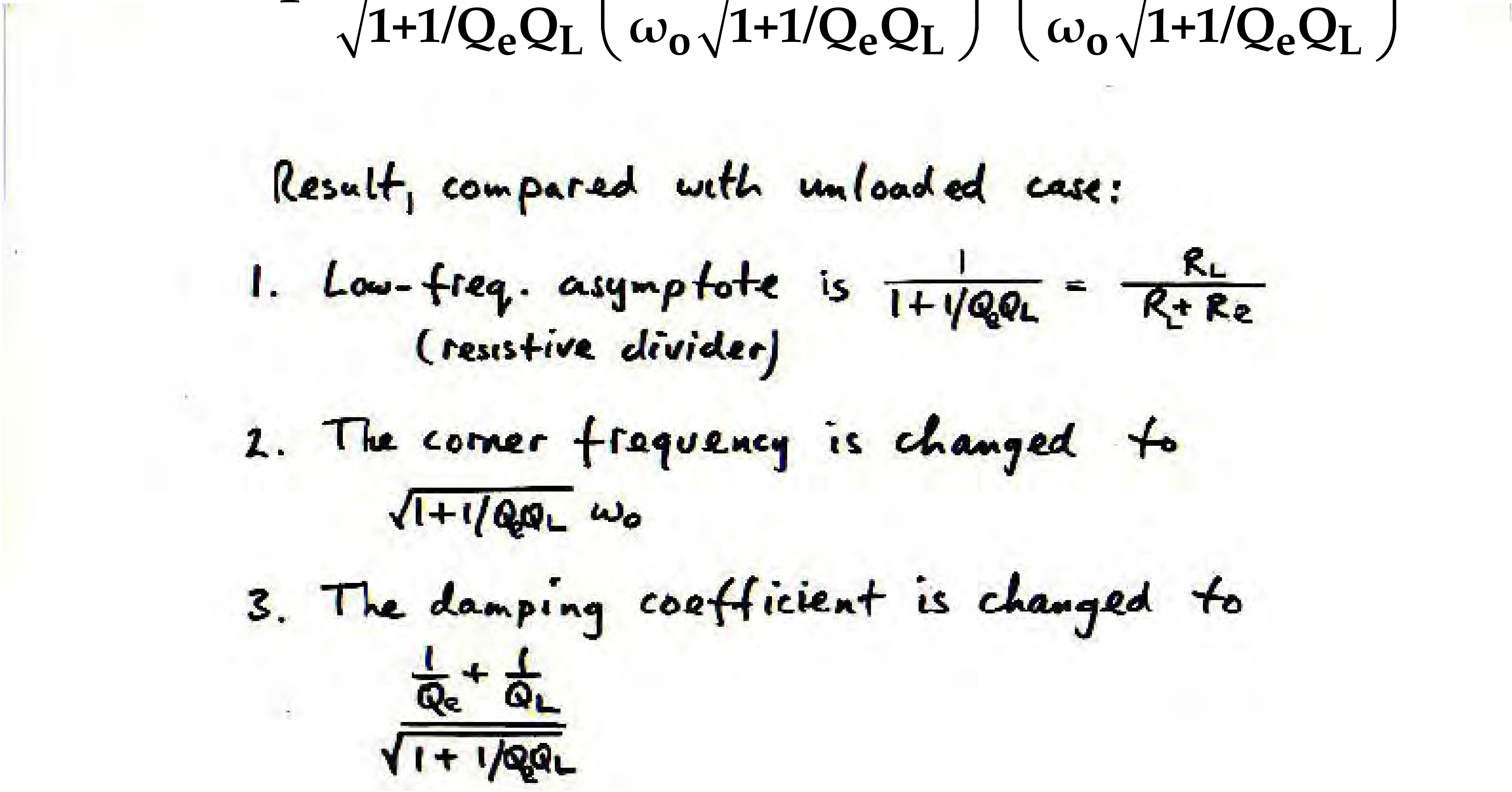
$$H = \frac{\frac{R_L}{1+sCR_L}}{\frac{R_L}{1+sCR_L} + R_e + sL} = \frac{R_L}{R_L + R_e + s(CR_LR_e + L) + s^2 LCR_L}$$

$$= \frac{R_L}{R_L + R_e} \frac{1}{1 + s\left(CR_e + \frac{L}{R_L}\right)\frac{R_L}{R_L+R_e} + s^2 LC \frac{R_L}{R_L+R_e}}$$

$$= \frac{R_L}{R_L + R_e} \frac{1}{1 + s\sqrt{LC}\left(R_e\sqrt{\frac{C}{L}} + \frac{1}{R_L}\sqrt{\frac{L}{C}}\right)\frac{R_L}{R_L+R_e} + s^2 LC \frac{R_L}{R_L+R_e}}$$

$$= \frac{1}{1 + 1/Q_eQ_L} \frac{1}{1 + \frac{\left(\frac{1}{Q_e} + \frac{1}{Q_L}\right)}{1 + 1/Q_eQ_L}\left(\frac{s}{\omega_0}\right) + \frac{1}{1 + 1/Q_eQ_L}\left(\frac{s}{\omega_0}\right)^2}$$

$$H=\frac{1}{1+1/Q_eQ_L}\frac{1}{1+\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)+\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)^2}$$



Result, compared with unloaded case:

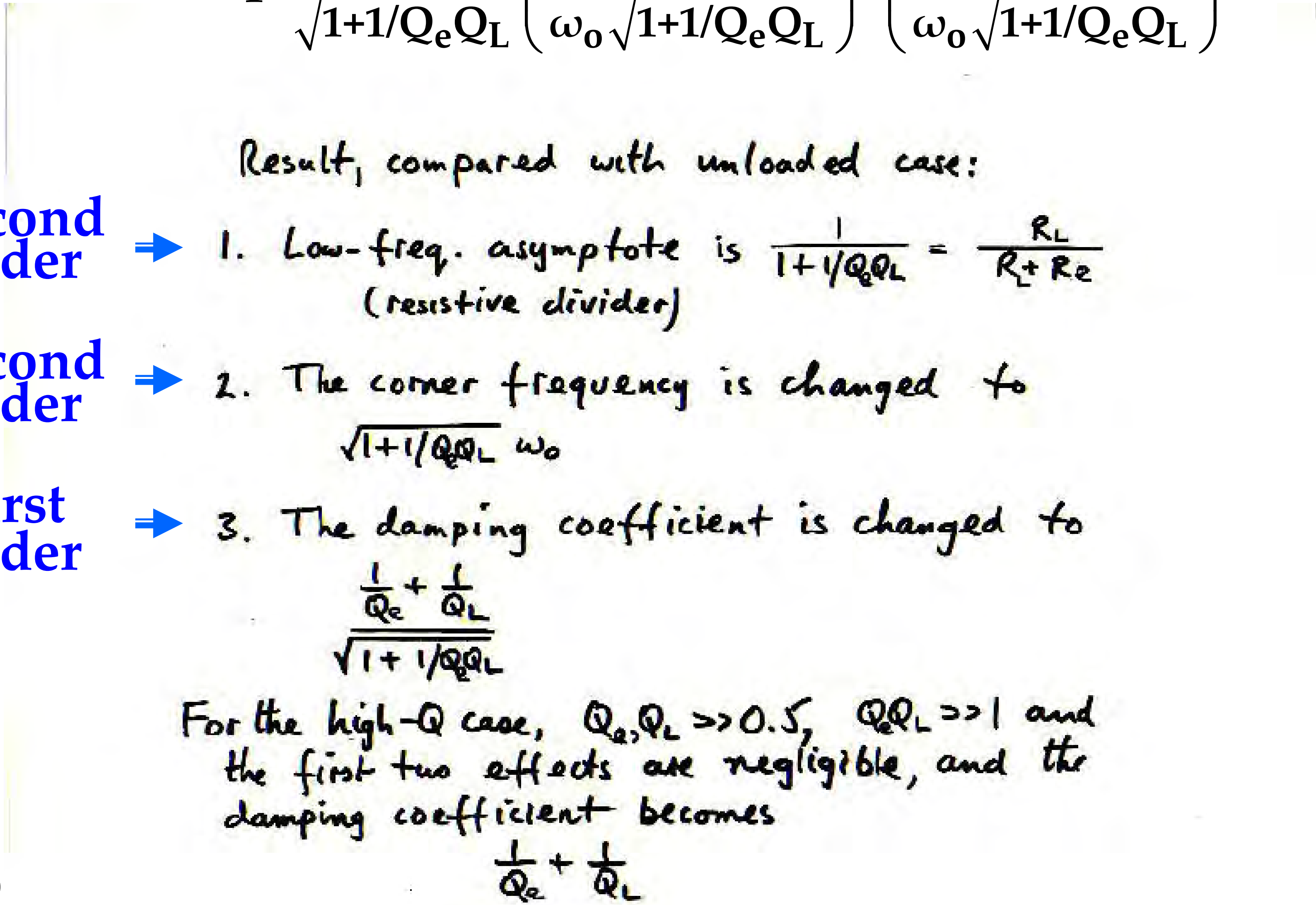
1. Low-freq. asymptote is $\frac{1}{1+1/Q_eQ_L}=\frac{R_L}{R_L+R_e}$
   (resistive divider)

2. The corner frequency is changed to
   $\sqrt{1+1/Q_eQ_L}\,\omega_o$

3. The damping coefficient is changed to
   $\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}$

This is a good example of how a low entropy format can allow one equation to disclose more than one useful piece of information.

$$H=\frac{1}{1+1/Q_eQ_L}\,\frac{1}{1+\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)+\left(\frac{s}{\omega_o\sqrt{1+1/Q_eQ_L}}\right)^2}$$



Result, compared with unloaded case:

second order ➡

1. Low-freq. asymptote is $\frac{1}{1+1/Q_eQ_L} = \frac{R_L}{R_L+R_e}$
(resistive divider)

second order ➡

2. The corner frequency is changed to
$\sqrt{1+1/Q_eQ_L}\;\omega_o$

first order ➡

3. The damping coefficient is changed to
$$\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}$$

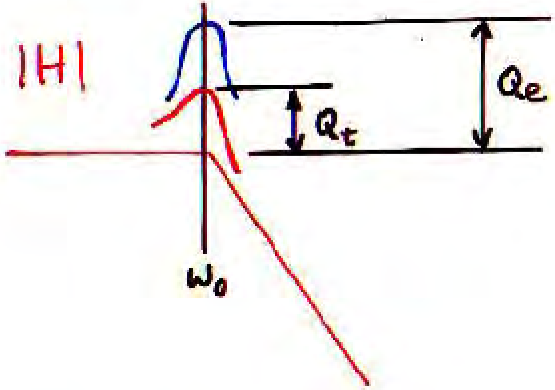
For the high-Q case, $Q_e, Q_L >> 0.5$, $Q_eQ_L >> 1$ and the first two effects are negligible, and the damping coefficient becomes
$$\frac{1}{Q_e}+\frac{1}{Q_L}$$

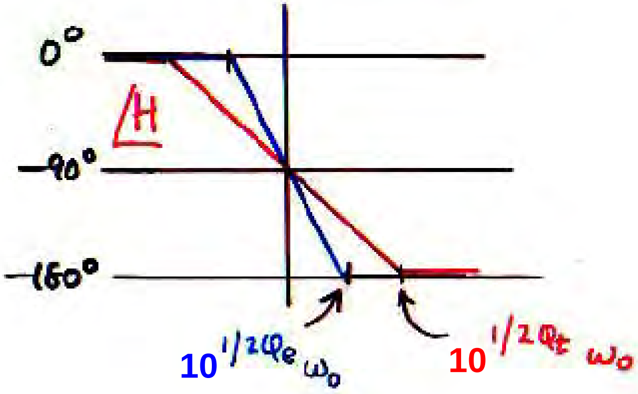
Hence, for the high-Q case,

$$H \approx \frac{1}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}$$

where Q_t is a "total" Q-factor given by the "parallel combination"

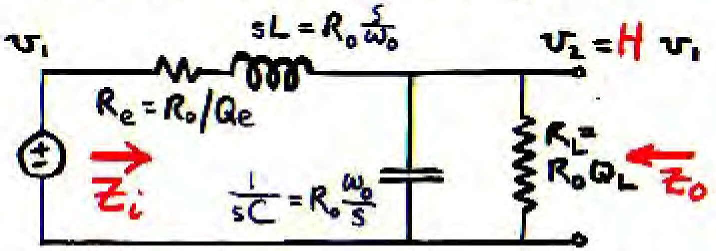
$$\frac{1}{Q_t} \equiv \frac{1}{Q_e} + \frac{1}{Q_L}$$


|H|
Q_t
Q_e
ω_0


0°
$\angle H$
−90°
−180°
$10^{1/2Q_e}\omega_0$
$10^{1/2Q_t}\omega_0$

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

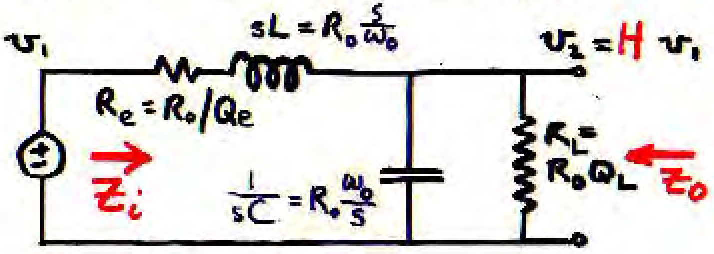
$$Z_i = \frac{R_0 Q_L}{1 + Q_L\left(\frac{s}{\omega_0}\right)} + \frac{R_0}{Q_e} + R_0 \frac{s}{\omega_0} = R_0 \frac{Q_L + \frac{1}{Q_e} + \left(\frac{Q_L}{Q_e} + 1\right)\left(\frac{s}{\omega_0}\right) + Q_L\left(\frac{s}{\omega_0}\right)^2}{1 + Q_L\left(\frac{s}{\omega_0}\right)}$$

$$= R_0\left(1 + 1/Q_e Q_L\right) \frac{1 + \frac{\frac{1}{Q_e} + \frac{1}{Q_L}}{1 + 1/Q_e Q_L}\left(\frac{s}{\omega_0}\right) + \frac{1}{1 + 1/Q_e Q_L}\left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

Same three effects as for H, but with addition of an inverted pole at ω_0/Q_L.

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



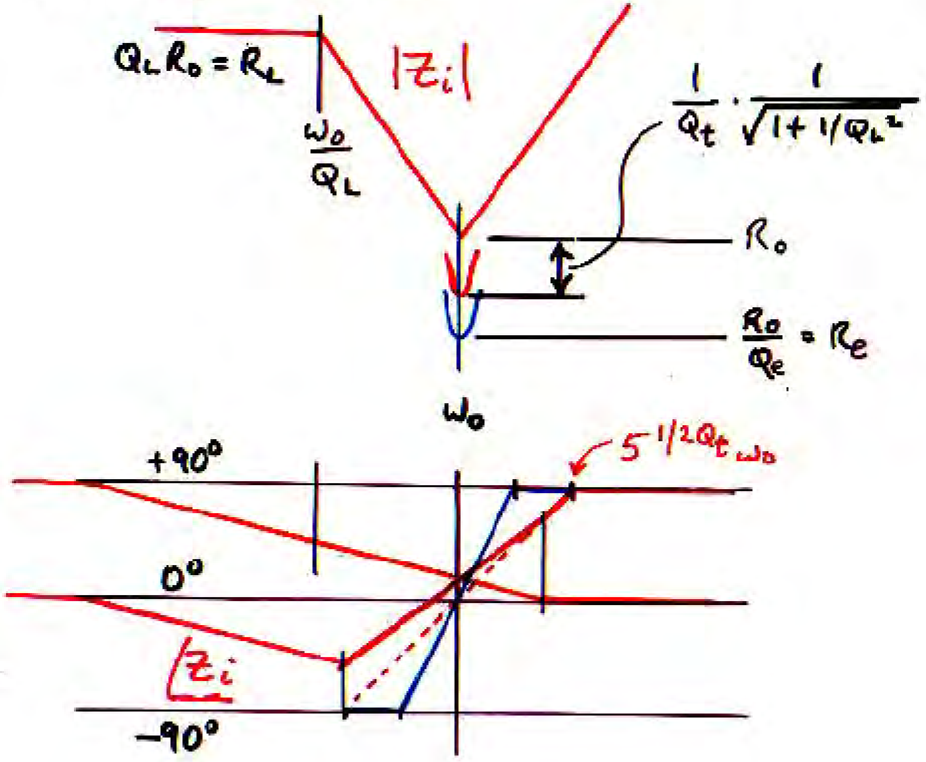
$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

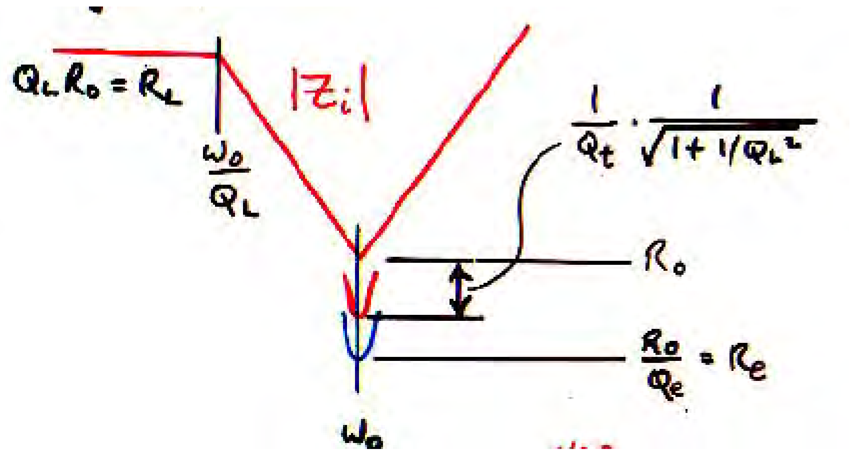
$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

Hence, for the high-Q case,

$$Z_i \approx R_0 \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

For high-Q case:

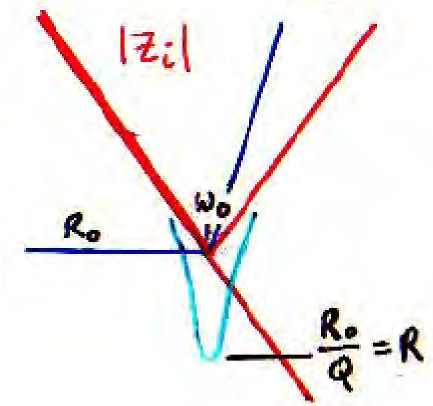



$Q_L R_o = R_L$
$|Z_i|$
$\frac{\omega_o}{Q_L}$
$\frac{1}{Q_t} \cdot \frac{1}{\sqrt{1+1/Q_L^2}}$
R_o
$\frac{R_o}{Q_e} = R_e$
ω_o

Compare the Z_i asymptotes with and without R_L:

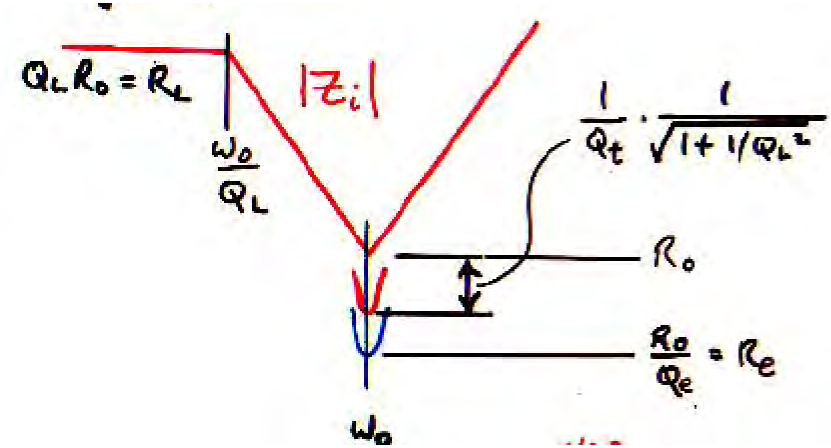
Without R_L:

($Q_L = \infty$)



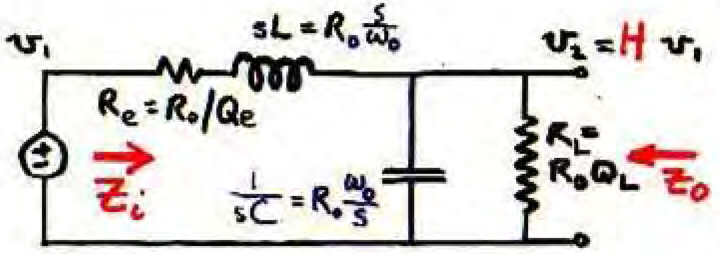
With R_L:

($Q_L \neq \infty$)



The appearance of the new corner frequency ω_o/Q_L can be confirmed by a mental frequency sweep:

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_0 \equiv \frac{1}{\sqrt{LC}} \qquad R_0 \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_0}{R_e} \qquad Q_L \equiv \frac{R_L}{R_0}$$

Hence, for the high-Q case,

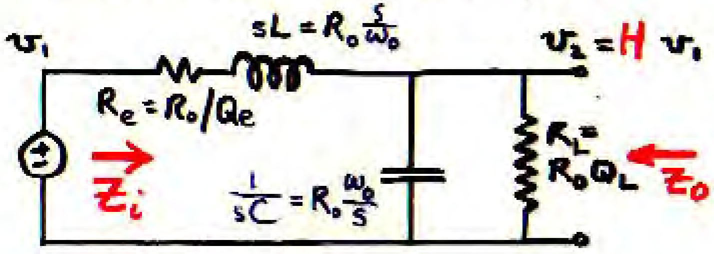
$$Z_i \approx R_0 \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

Without R_L, $Z_i \rightarrow \infty$ as $\omega \rightarrow 0$ because of the capacitive reactance.

With R_L, Z_i flattens, so a concave downwards corner is introduced, which is an inverted pole.

Loaded low-pass LC filter

Note: $R \rightarrow R_e$, $Q \rightarrow Q_e$



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_L \equiv \frac{R_L}{R_o}$$

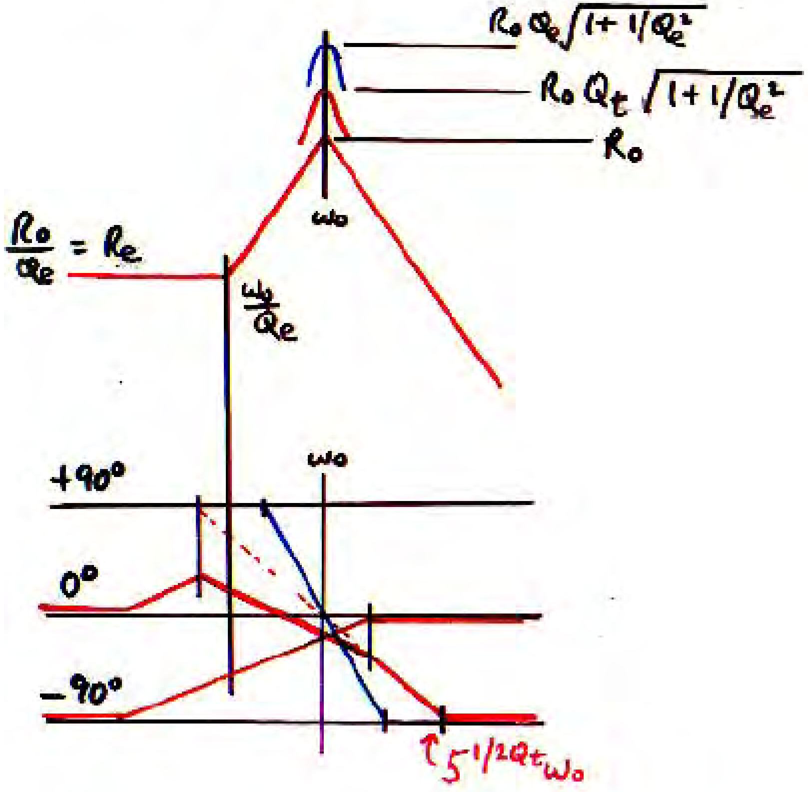
$$Z_o = \frac{\frac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)}\left(\frac{R_o}{Q_e}+R_o\frac{s}{\omega_o}\right)}{\frac{Q_L R_o}{1+Q_L\left(\frac{s}{\omega_o}\right)}+\frac{R_o}{Q_e}+R_o\frac{s}{\omega_o}} = R_o Q_L \frac{\frac{1}{Q_e}+\frac{s}{\omega_o}}{Q_L+\frac{1}{Q_e}+\left(\frac{Q_L}{Q_e}+1\right)\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2}$$

$$= R_o \cdot \frac{1}{1+1/Q_eQ_L} \cdot \frac{\left(\frac{s}{\omega_o}\right)\left(1+\frac{\omega_o/Q_e}{s}\right)}{1+\frac{\left(\frac{1}{Q_e}+\frac{1}{Q_L}\right)}{1+1/Q_eQ_L}\left(\frac{s}{\omega_o}\right)+\frac{1}{1+1/Q_eQ_L}\left(\frac{s}{\omega_o}\right)^2}$$

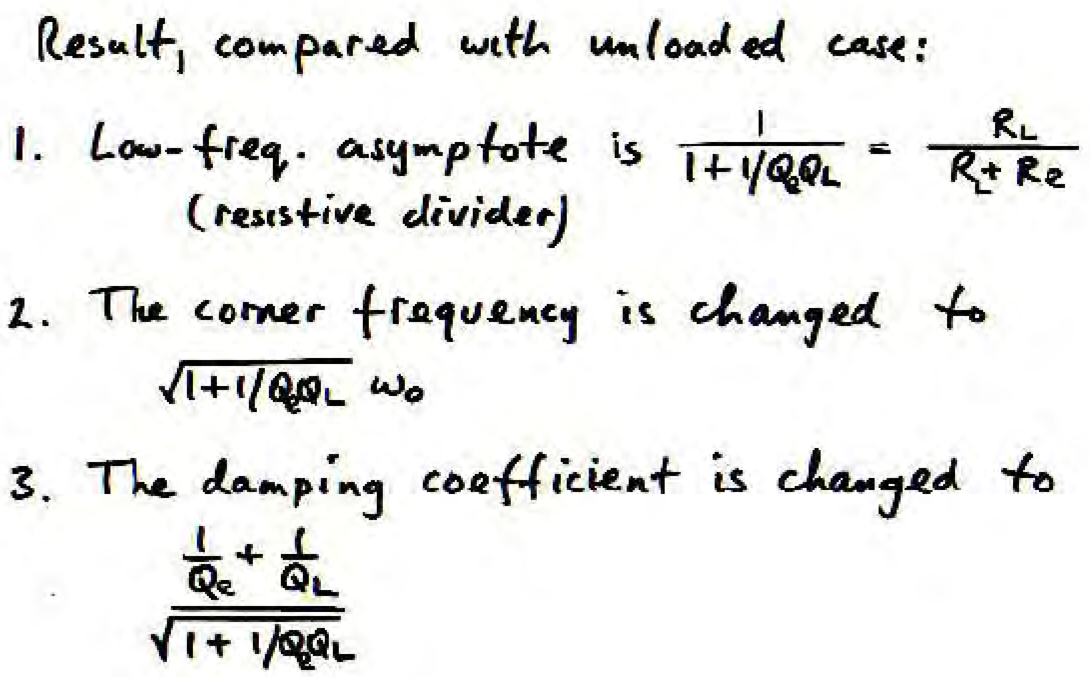
Same three effects as for H, so for high-Q case

$$Z_o \approx R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1+\frac{\omega_o/Q_e}{s}\right)}{1+\frac{1}{Q_t}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2}$$

For high-Q case:



When an LC filter is loaded, a 4th effect needs to be accounted for:



Result, compared with unloaded case:

second order →

1. Low-freq. asymptote is $\frac{1}{1+1/Q_eQ_L} = \frac{R_L}{R_L+R_e}$
(resistive divider)

second order →

2. The corner frequency is changed to
$\sqrt{1+1/Q_eQ_L}\;\omega_0$

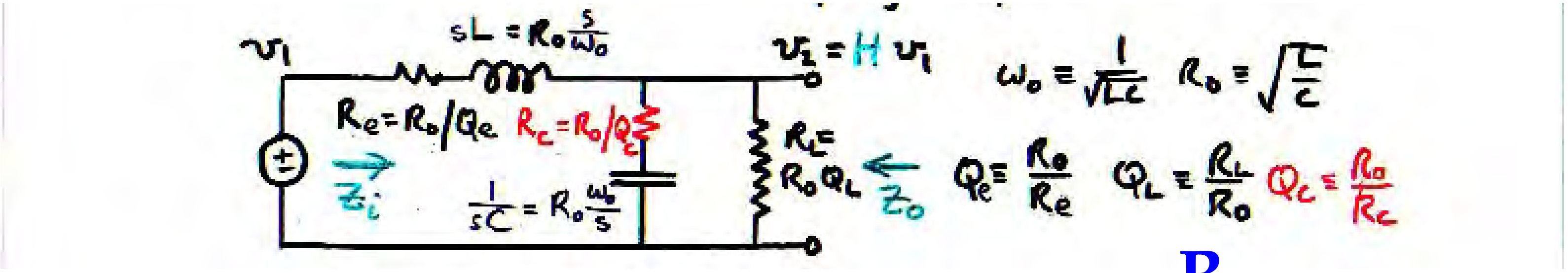
first order →

3. The damping coefficient is changed to
$$\frac{\frac{1}{Q_e}+\frac{1}{Q_L}}{\sqrt{1+1/Q_eQ_L}}$$

first order →

4. New corner frequencies may appear in some transfer functions

A third damping resistance R_c may be present, representing the capacitor esr:



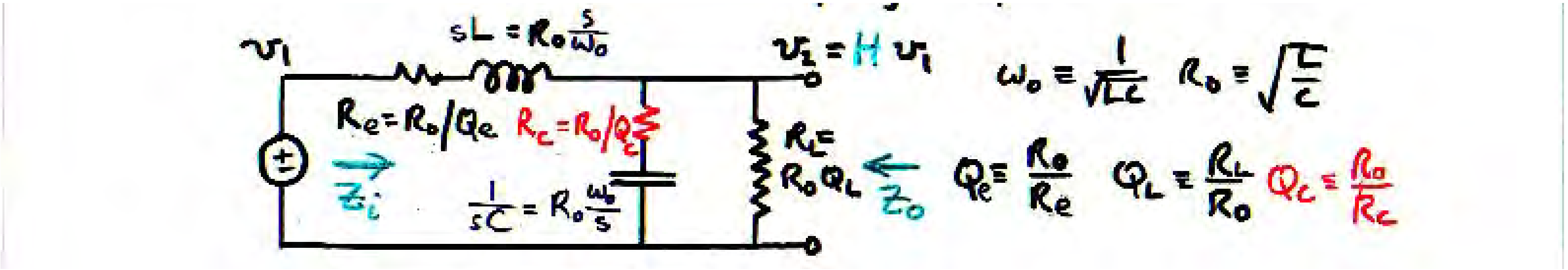
By analogy with Q_e, define $Q_c \equiv \dfrac{R_o}{R_c}$

The analysis for H, Z_i, and Z_o could be re-done in the same way.

Instead, let's *build* the result by applying what we already know about the two simpler cases.

The price we are willing to pay, in order to leap-frog directly to the result, is that the second-order effects will be omitted.

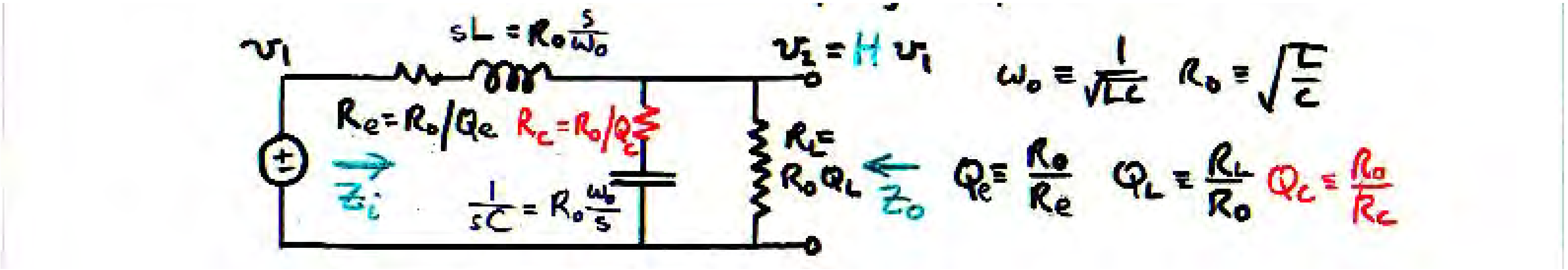
A third damping resistance R_c may be present, representing the capacitor esr:



One first-order effect of adding a second damping resistance was to lower the total Q_t to the parallel combination

$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L}$$

A third damping resistance R_c may be present, representing the capacitor esr:



A *good guess* would be that adding a third damping resistance would lower the total Q_t to the triple parallel combination

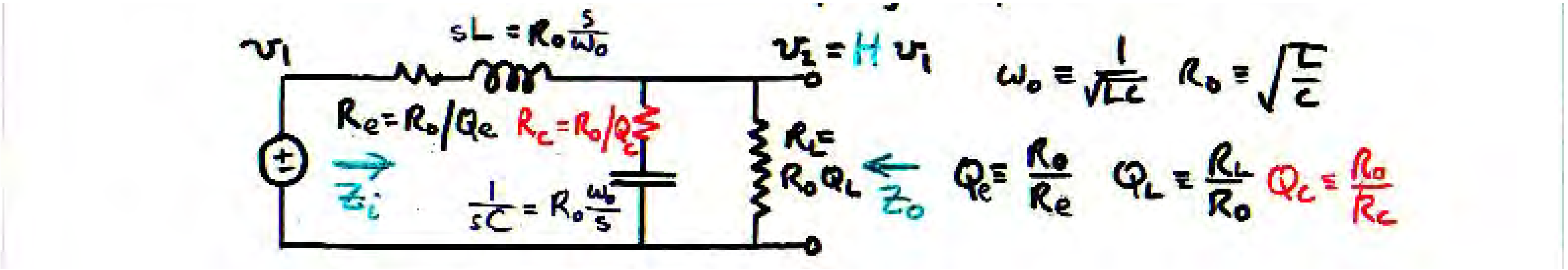
$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

Another possible first-order effect of adding a third damping resistance is the appearance of additional corner frequencies.

A mental frequency sweep can be used to verify an analytical result, but it can also be used "in reverse" to expose new corner frequencies.

The strategy is to determine whether or not the addition of the third damping resistance changes the asymptote slope as frequency approaches either zero or infinity.

For the voltage transfer function H:



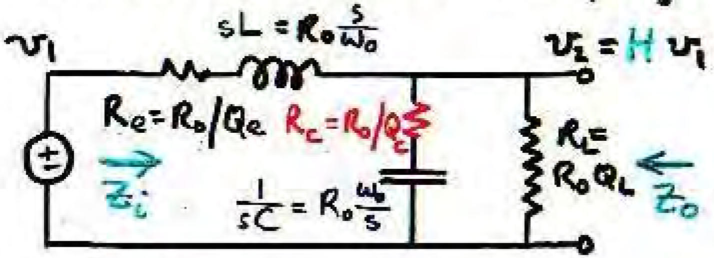
$\omega \rightarrow 0$: no change of slope, so no new inverted pole or zero;

$\omega \rightarrow \infty$: a concave upwards corner appears, so there is a new normal zero.

Further, the value of the corner is where $1/\omega C = R_c$, which is $1/RC = Q_c \omega_o$.

Assembled results:

Consider additional damping: Capacitor esr R_c



$\omega_0 \equiv \frac{1}{\sqrt{LC}} \quad R_0 \equiv \sqrt{\frac{L}{C}}$

$Q_e \equiv \frac{R_0}{R_e} \quad Q_L \equiv \frac{R_L}{R_0} \quad Q_c \equiv \frac{R_0}{R_c}$

For the high-Q case, the previous results can be extended by inspection:

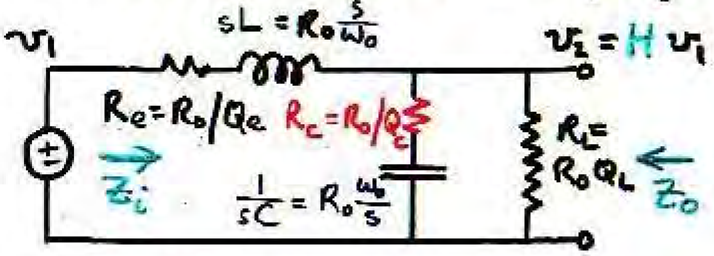
$$H = \frac{1 + \frac{1}{Q_c}\left(\frac{s}{\omega_0}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \Longleftarrow 1 + sR_cC$$

triple parallel combination:

$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

Assembled results:

Consider additional damping: Capacitor esr R_c



$\omega_0 \equiv \frac{1}{\sqrt{LC}}$ $R_0 \equiv \sqrt{\frac{L}{C}}$

$Q_e \equiv \frac{R_0}{R_e}$ $Q_L \equiv \frac{R_L}{R_0}$ $Q_c \equiv \frac{R_0}{R_c}$

For the high-Q case, the previous results can be extended by inspection:

$$H = \frac{1 + \frac{1}{Q_c}\left(\frac{s}{\omega_0}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2} \Longleftarrow 1 + sR_cC$$

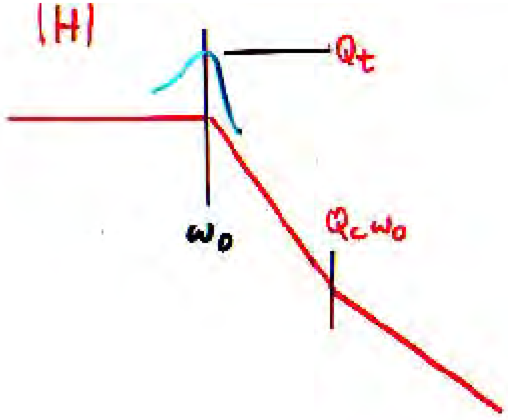
triple parallel combination:

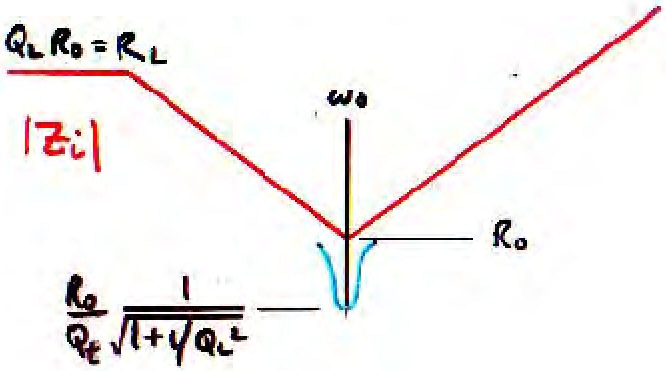
$$\frac{1}{Q_t} = \frac{1}{Q_e} + \frac{1}{Q_L} + \frac{1}{Q_c}$$

A similar process leads to the assembled result for Z_i:

$$Z_i = R_0 \frac{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_0}\right) + \left(\frac{s}{\omega_0}\right)^2}{\left(\frac{s}{\omega_0}\right)\left(1 + \frac{\omega_0/Q_L}{s}\right)}$$

(No new corners)


|H|
Q_t
ω_0
$Q_L\omega_0$


$Q_L R_0 = R_L$
$|Z_i|$
ω_0
R_0
$\frac{R_0}{Q_t}\frac{1}{\sqrt{1+1/Q_L^2}}$

Principle for extension of results to a more complicated case:

1. Determine the new total Q_t.
2. Add any additional pole or zero factors (Is there any change in the $\omega \to 0$ or $\omega \to \infty$ asymptotes?)

Exercise:

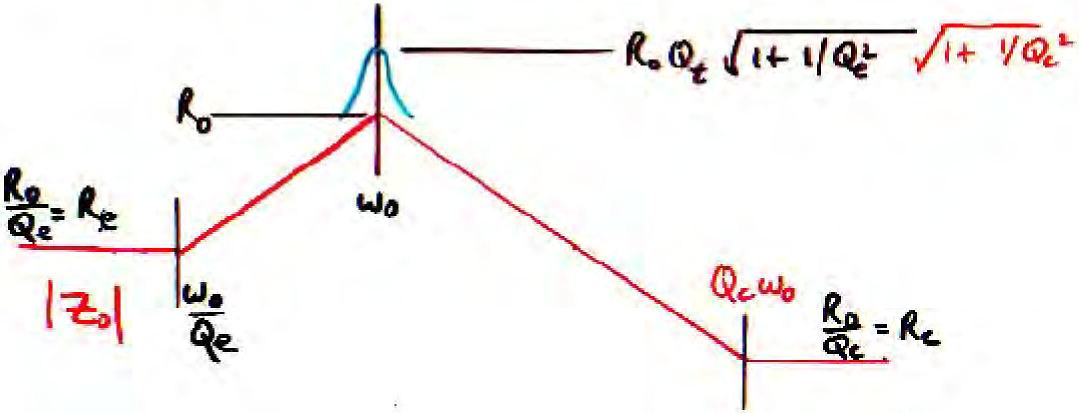
Obtain the corresponding results for Z_o.

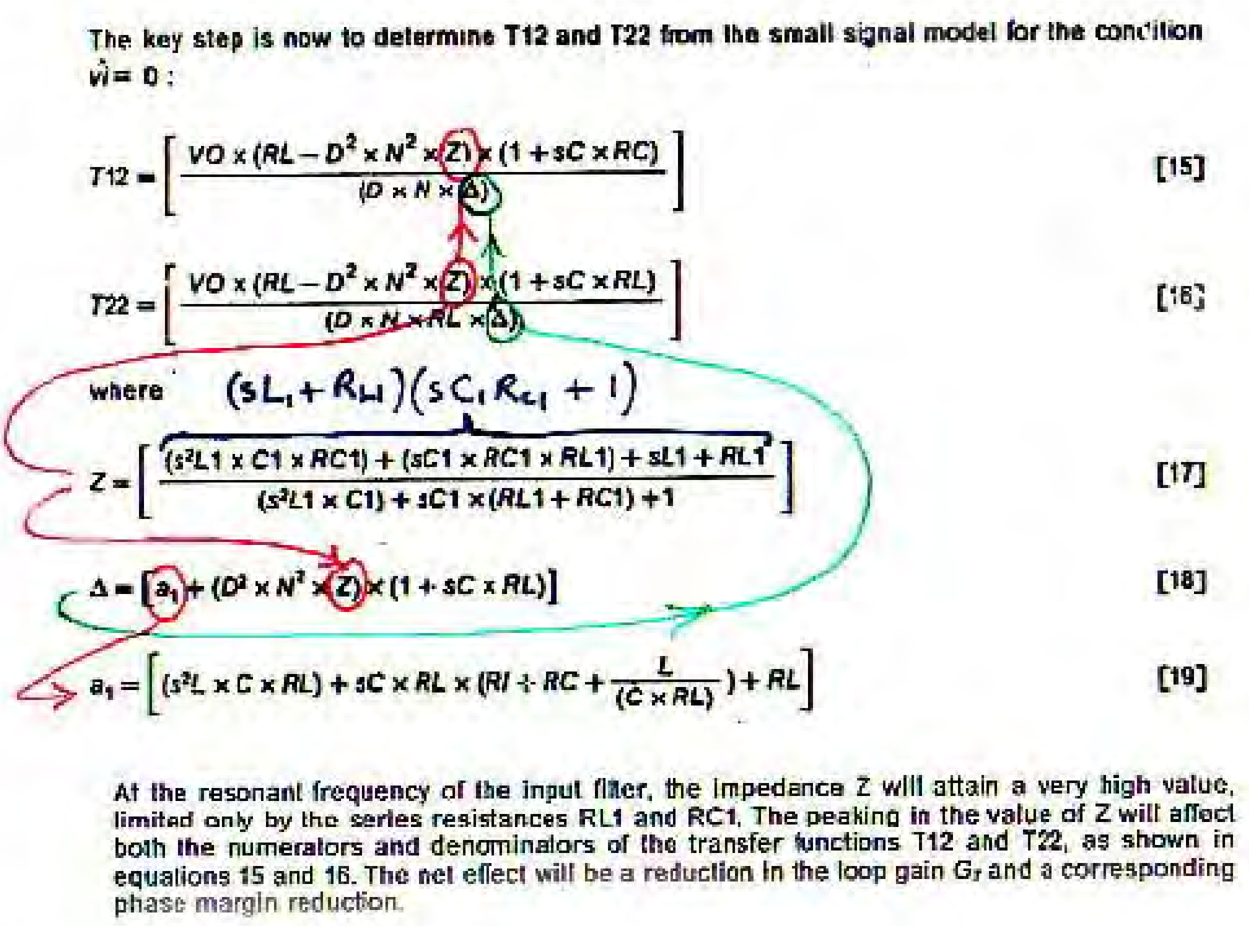
Exercise solution:

$$Z_o = R_o \frac{\left(\frac{s}{\omega_o}\right)\left(1 + \frac{\omega_o/Q_e}{s}\right)\left(1 + \frac{1}{Q_c}\frac{s}{\omega_o}\right)}{1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2}$$

Check high-frequency limit:

$$Z_o \xrightarrow{\omega \to \infty} \frac{R_o}{Q_c} = R_c$$





The key step is now to determine T12 and T22 from the small signal model for the condition $\hat{v}i = 0$:

$$T12 = \left[\frac{VO \times (RL - D^2 \times N^2 \times \boxed{Z}) \times (1 + sC \times RC)}{(D \times N \times \boxed{\Delta})} \right] \quad [15]$$

$$T22 = \left[\frac{VO \times (RL - D^2 \times N^2 \times \boxed{Z}) \times (1 + sC \times RL)}{(D \times N \times RL \times \boxed{\Delta})} \right] \quad [16]$$

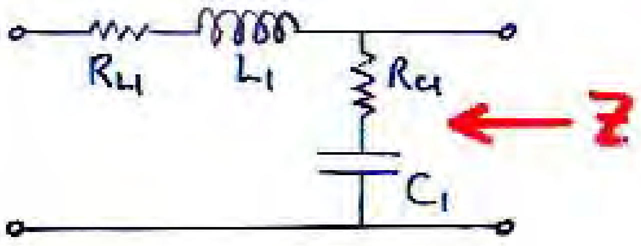
where

$$Z = \left[\frac{\overbrace{(s^2L1 \times C1 \times RC1) + (sC1 \times RC1 \times RL1) + sL1 + RL1}^{(sL_1 + R_{L1})(sC_1R_{C1} + 1)}}{(s^2L1 \times C1) + sC1 \times (RL1 + RC1) + 1} \right] \quad [17]$$

$$\Delta = \left[\boxed{a_1} + (D^2 \times N^2 \times \boxed{Z}) \times (1 + sC \times RL) \right] \quad [18]$$

$$a_1 = \left[(s^2L \times C \times RL) + sC \times RL \times (Rl \div RC + \frac{L}{(C \times RL)}) + RL \right] \quad [19]$$

At the resonant frequency of the input filter, the impedance Z will attain a very high value, limited only by the series resistances RL1 and RC1. The peaking in the value of Z will affect both the numerators and denominators of the transfer functions T12 and T22, as shown in equations 15 and 16. The net effect will be a reduction in the loop gain G_f and a corresponding phase margin reduction.



6. PRODUCTS AND SUMS OF FACTORED POLE-ZERO EXPRESSIONS

Doing the Algebra on the Graph

Functions expressed in factored pole-zero form often need to be combined, either by multiplication or addition.

Multiplication is straightforward:

$$A = A_1 A_2$$

$$A_1 = A_{1ref}\frac{\left(1+\frac{s}{\omega_{z11}}\right)\left(1+\frac{s}{\omega_{z12}}\right)\ldots}{\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p12}}\right)\ldots} \qquad A_2 = A_{2ref}\frac{\left(1+\frac{s}{\omega_{z21}}\right)\left(1+\frac{s}{\omega_{z22}}\right)\ldots}{\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)\ldots}$$

$$A = A_{1ref}A_{2ref}\frac{\left(1+\frac{s}{\omega_{z11}}\right)\left(1+\frac{s}{\omega_{z12}}\right)\ldots\left(1+\frac{s}{\omega_{z21}}\right)\left(1+\frac{s}{\omega_{z22}}\right)\ldots}{\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p12}}\right)\ldots\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)\ldots}$$

The product contains the poles and zeros of both functions.

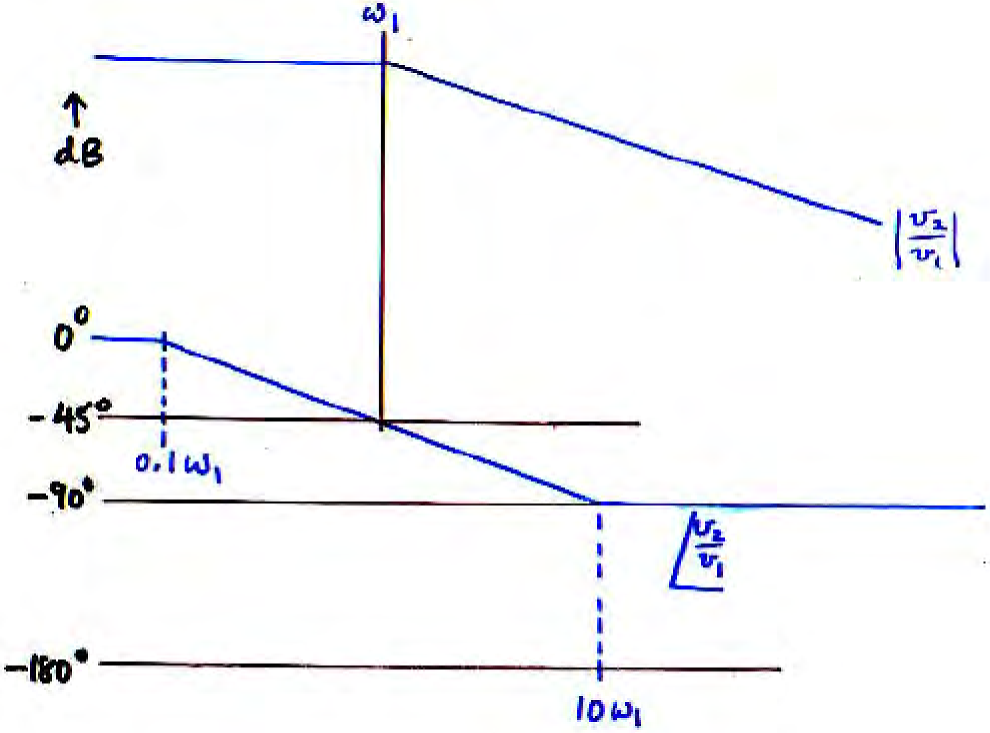
Double-pole low-pass RC filters

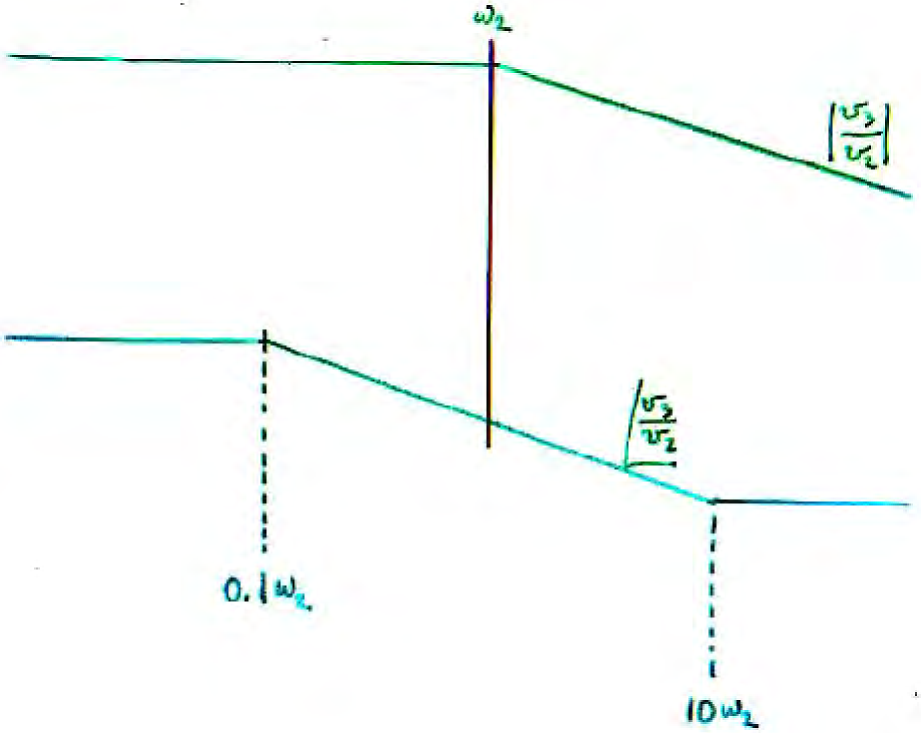
$$\frac{v_3}{v_1} = \frac{1}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$ where $\omega_1 \equiv \frac{1}{C_1R_1}$ $\omega_2 \equiv \frac{1}{C_2R_2}$

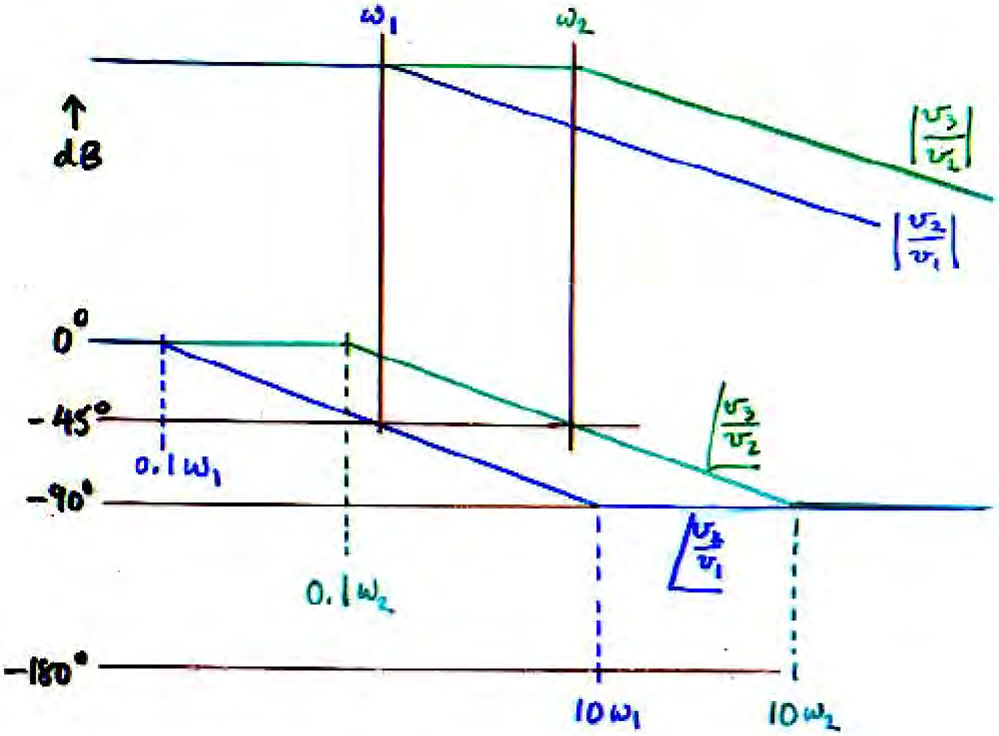
$$\left|\frac{v_3}{v_1}\right|_{dB} = -20\log\sqrt{1+\left(\frac{\omega}{\omega_1}\right)^2} - 20\log\sqrt{1+\left(\frac{\omega}{\omega_2}\right)^2}$$

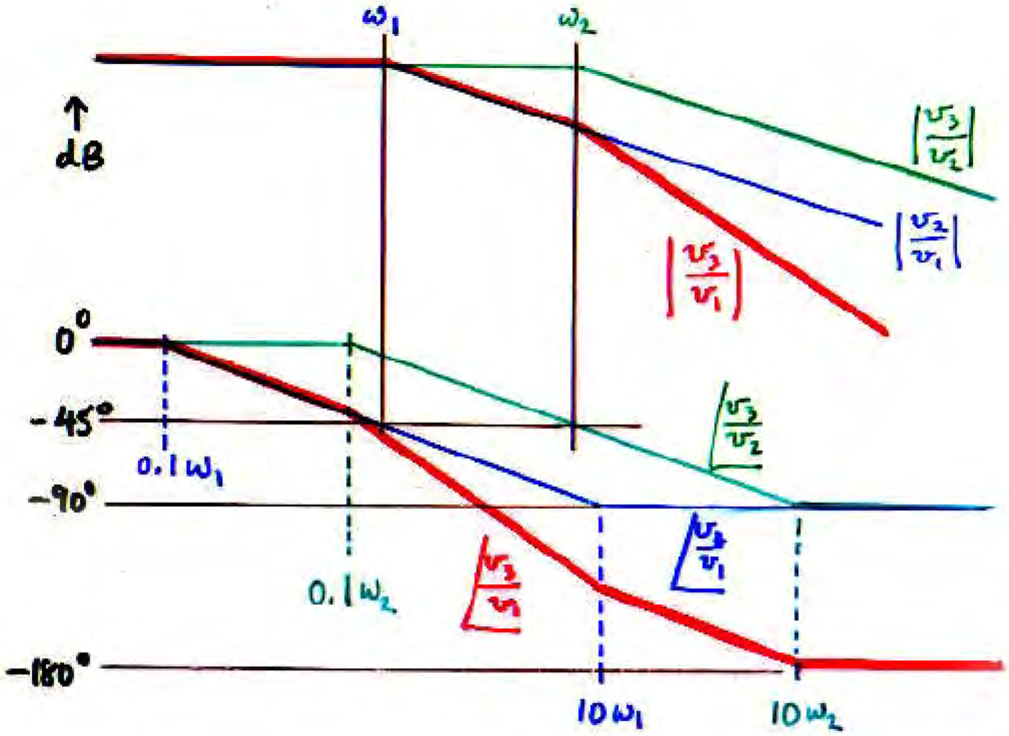
superposition

$$\angle\frac{v_3}{v_1} = -\tan^{-1}\left(\frac{\omega}{\omega_1}\right) - \tan^{-1}\left(\frac{\omega}{\omega_2}\right)$$

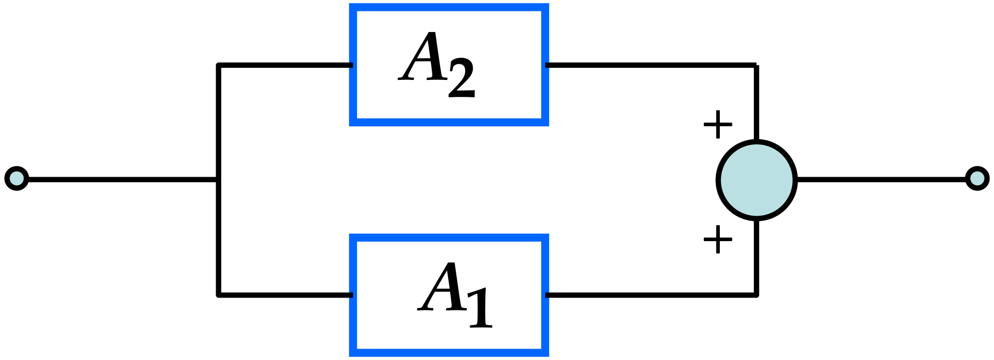

ω_1
↑ dB
$\left|\frac{v_2}{v_1}\right|$
0°
-45°
-90°
-180°
$0.1\omega_1$
$10\omega_1$
$\angle\frac{v_2}{v_1}$


ω_2
$\left|\frac{v_3}{v_2}\right|$
$\angle\frac{v_3}{v_2}$
$0.1\omega_2$
$10\omega_2$


ω_1
ω_2
↑ dB
$\left|\frac{v_3}{v_2}\right|$
$\left|\frac{v_2}{v_1}\right|$
0°
-45°
-90°
-180°
$0.1\omega_1$
$0.1\omega_2$
$\angle\frac{v_3}{v_2}$
$\angle\frac{v_2}{v_1}$
$10\omega_1$
$10\omega_2$


ω_1
ω_2
dB
$\left|\frac{v_3}{v_2}\right|$
$\left|\frac{v_2}{v_1}\right|$
$\left|\frac{v_3}{v_1}\right|$
0°
-45°
-90°
-180°
$0.1\omega_1$
$0.1\omega_2$
$10\omega_1$
$10\omega_2$
$\angle\frac{v_3}{v_2}$
$\angle\frac{v_2}{v_1}$
$\angle\frac{v_3}{v_1}$

Addition is more complicated:

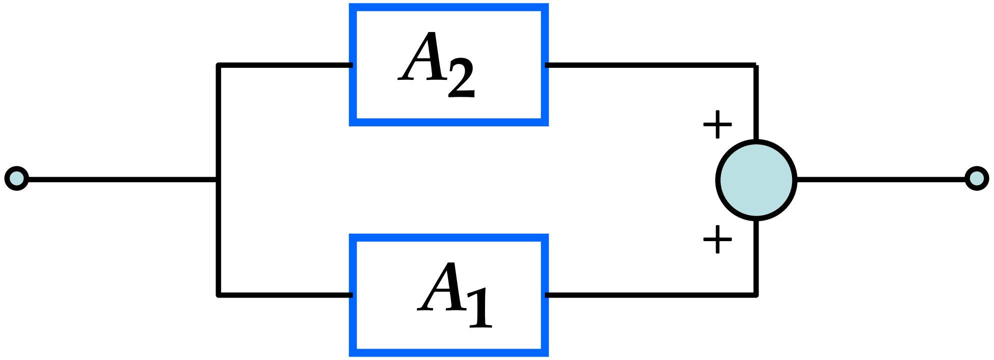


$$A = A_1 + A_2$$

$$A = A_{1ref}\frac{\left(1+\frac{s}{\omega_{z11}}\right)\left(1+\frac{s}{\omega_{z12}}\right)...}{\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p12}}\right)...} + A_{2ref}\frac{\left(1+\frac{s}{\omega_{z21}}\right)\left(1+\frac{s}{\omega_{z22}}\right)...}{\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)...}$$

$$A = \frac{A_{1ref}\left(1+\frac{s}{\omega_{z11}}\right)\left(1+\frac{s}{\omega_{z12}}\right)...\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)...+A_{2ref}\left(1+\frac{s}{\omega_{z21}}\right)\left(1+\frac{s}{\omega_{z22}}\right)...\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p22}}\right)...}{\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p22}}\right)...\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)...}$$

Addition is more complicated:



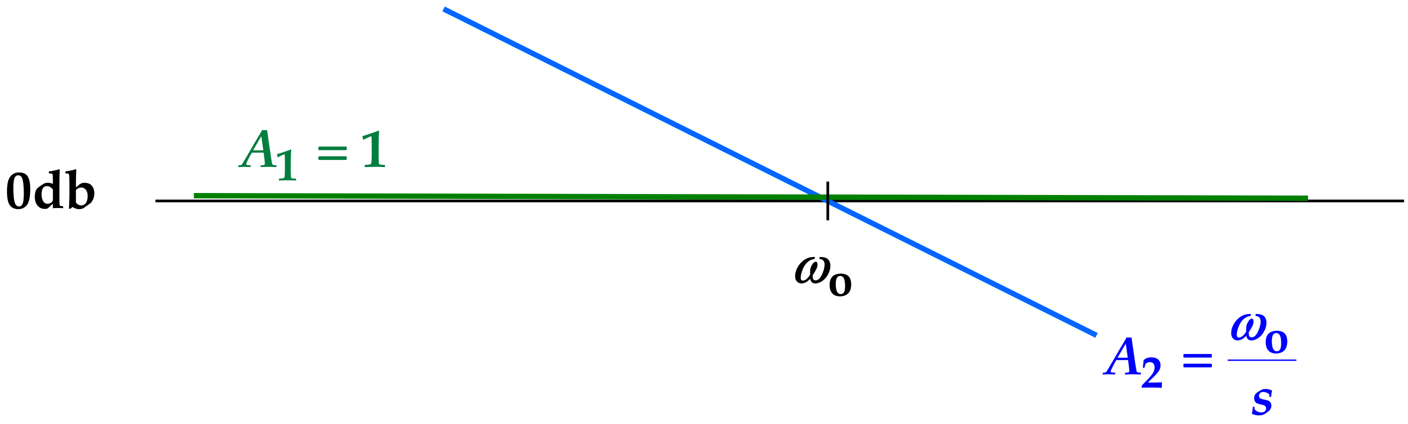
$$A = A_1 + A_2$$

$$A = \frac{A_{1ref}\left(1+\frac{s}{\omega_{z11}}\right)\left(1+\frac{s}{\omega_{z12}}\right)\ldots\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)\ldots+A_{2ref}\left(1+\frac{s}{\omega_{z21}}\right)\left(1+\frac{s}{\omega_{z22}}\right)\ldots\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p22}}\right)\ldots}{\left(1+\frac{s}{\omega_{p11}}\right)\left(1+\frac{s}{\omega_{p22}}\right)\ldots\left(1+\frac{s}{\omega_{p21}}\right)\left(1+\frac{s}{\omega_{p22}}\right)\ldots}$$

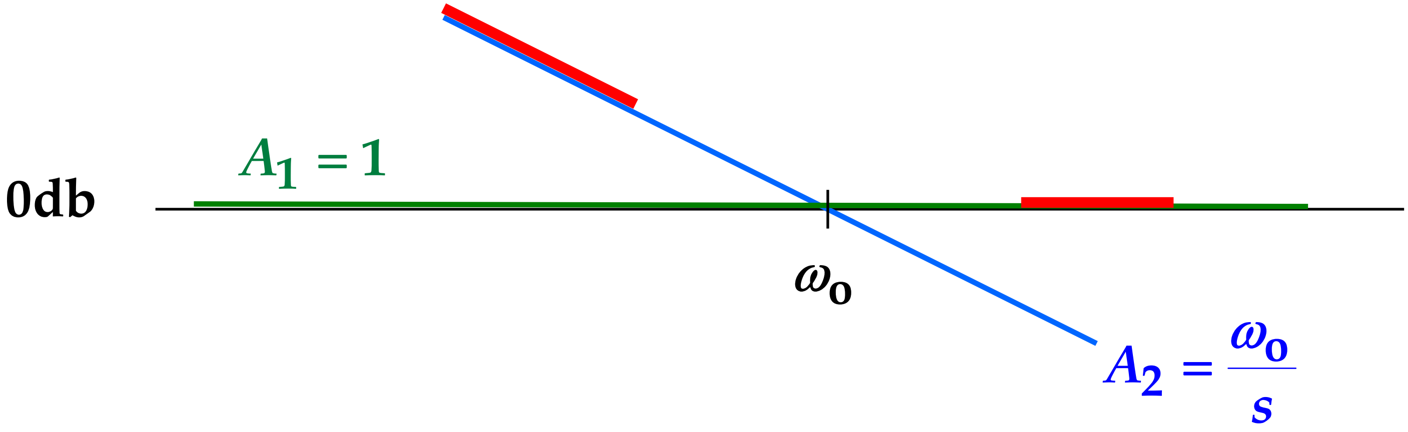
The sum contains the poles of both functions, but the numerator consists of the sums of cross-products of poles and zeros, and is a new polynomial that has to be renormalized and refactored.

This can be very tedious, and requires approximations if the numerator is higher than a quadratic in *s*.

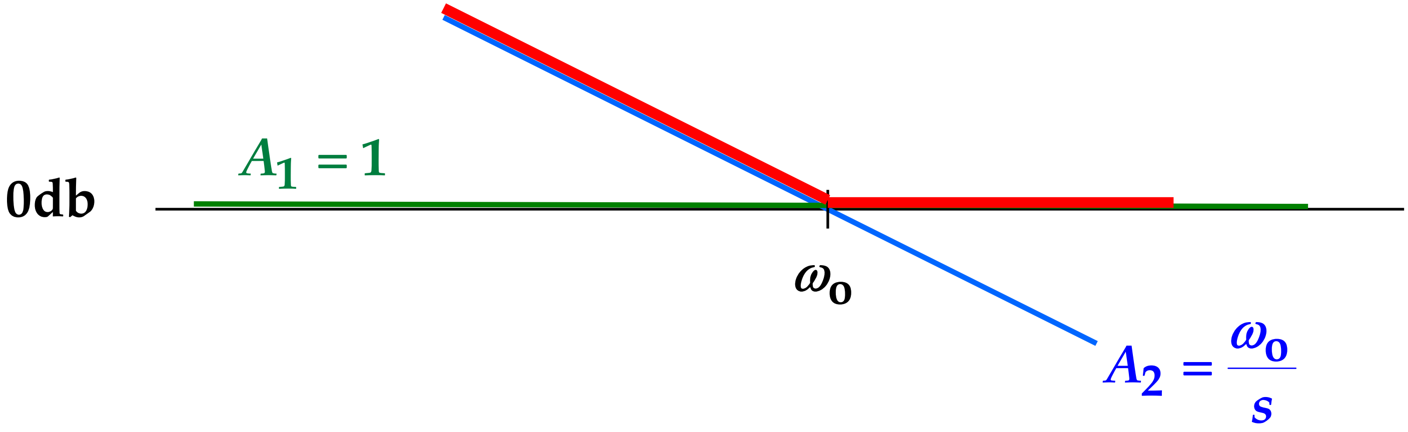
"Doing the algebra on the graph" makes suitable approximations obvious.


$A_1 = 1$
0db
ω_o
$A_2 = \frac{\omega_o}{s}$

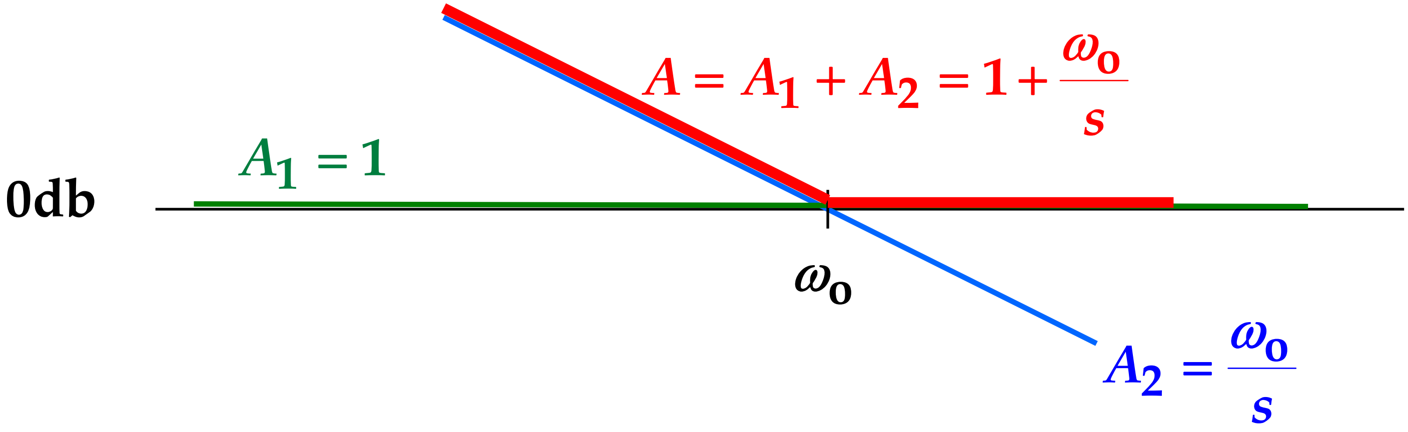
A guess is that the sum follows the larger:



A guess is that the sum follows the larger:

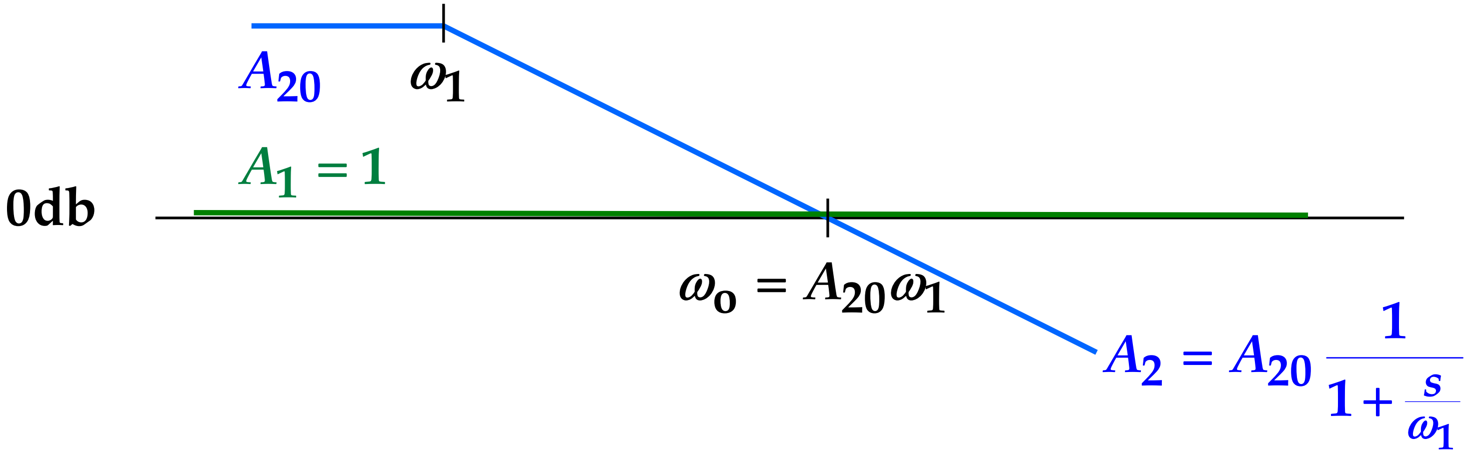


A guess is that the sum follows the larger:

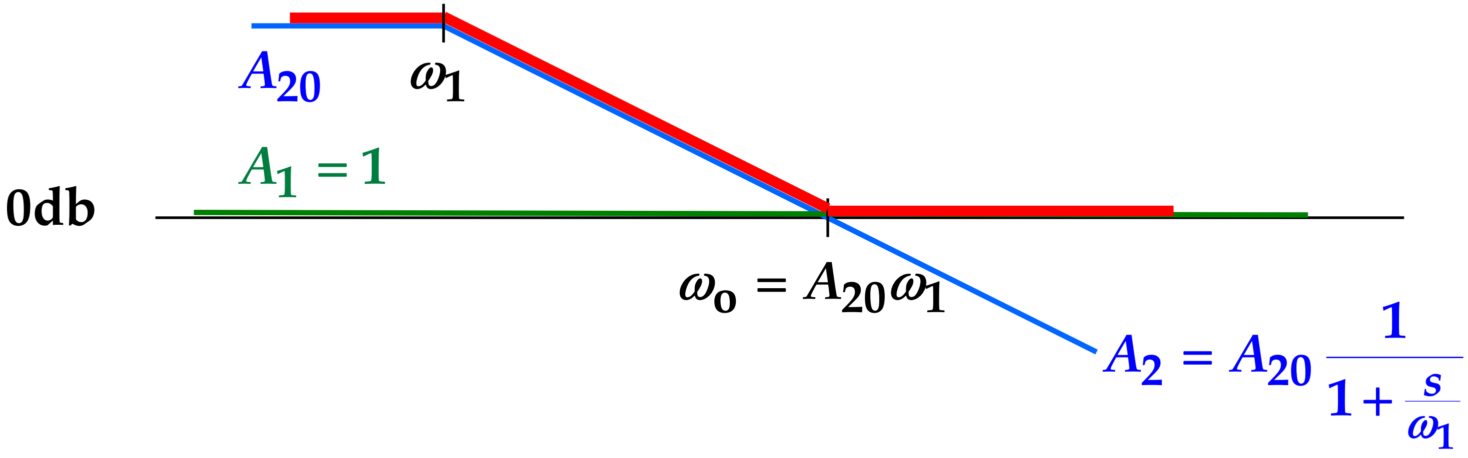


This is confirmed algebraically:

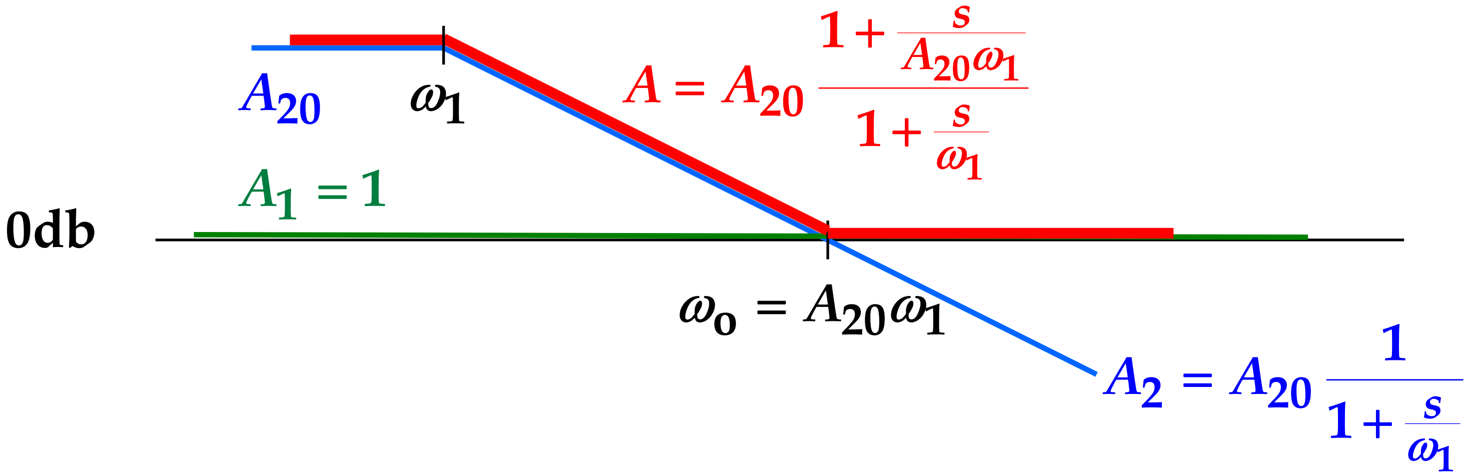
$$A = A_1 + A_2 = 1 + \frac{\omega_o}{s}$$


A_{20}
ω_1
$A_1 = 1$
0db
$\omega_o = A_{20}\omega_1$
$A_2 = A_{20} \dfrac{1}{1 + \frac{s}{\omega_1}}$

If the sum follows the larger, the result is:



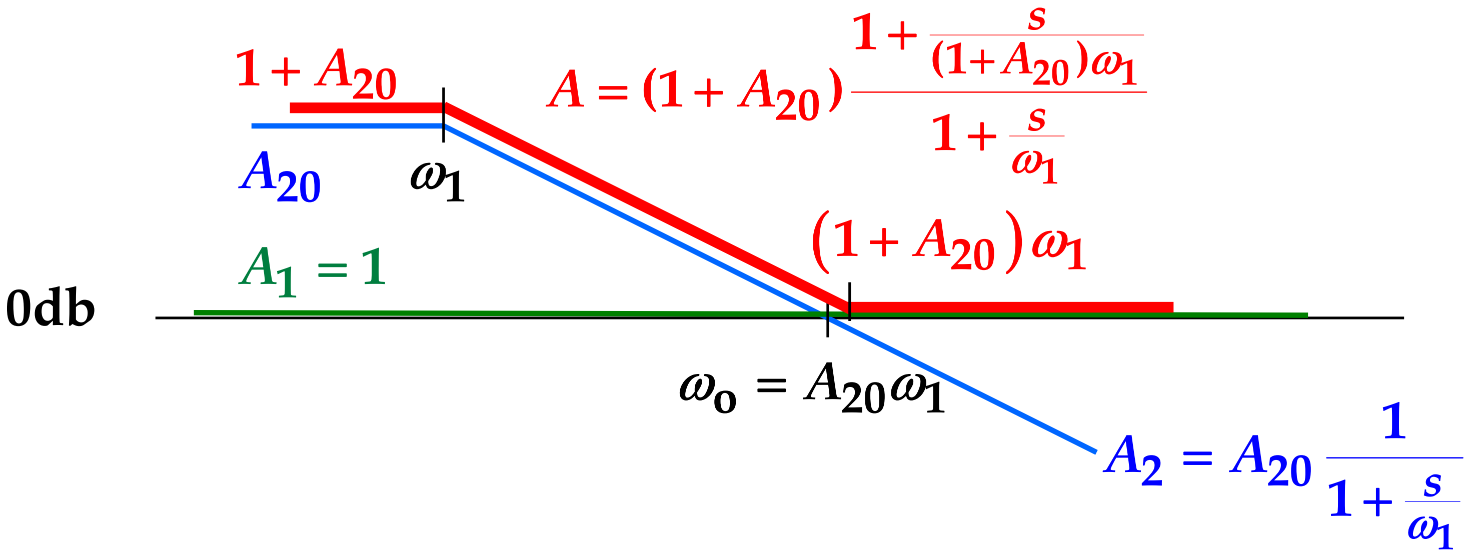
If the sum follows the larger, the result is:



However, the algebra shows that this is an approximation:

$$A = 1 + A_{20}\frac{1}{1+\frac{s}{\omega_1}} = \frac{1+A_{20}+\frac{s}{\omega_1}}{1+\frac{s}{\omega_1}}$$

$$= (1+A_{20})\frac{1+\frac{s}{(1+A_{20})\omega_1}}{1+\frac{s}{\omega_1}}$$



$1+A_{20}$

$$A=(1+A_{20})\frac{1+\frac{s}{(1+A_{20})\omega_1}}{1+\frac{s}{\omega_1}}$$

A_{20}

ω_1

$(1+A_{20})\omega_1$

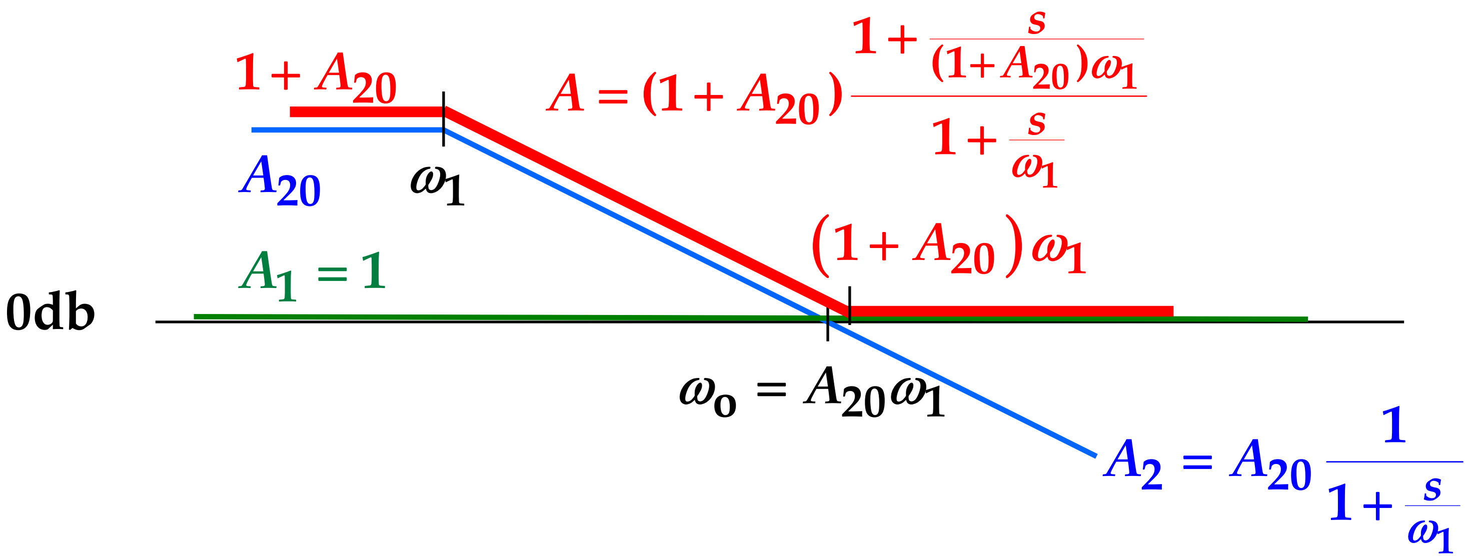
$A_1=1$

0db

$\omega_o=A_{20}\omega_1$

$$A_2=A_{20}\frac{1}{1+\frac{s}{\omega_1}}$$

This is the exact answer.



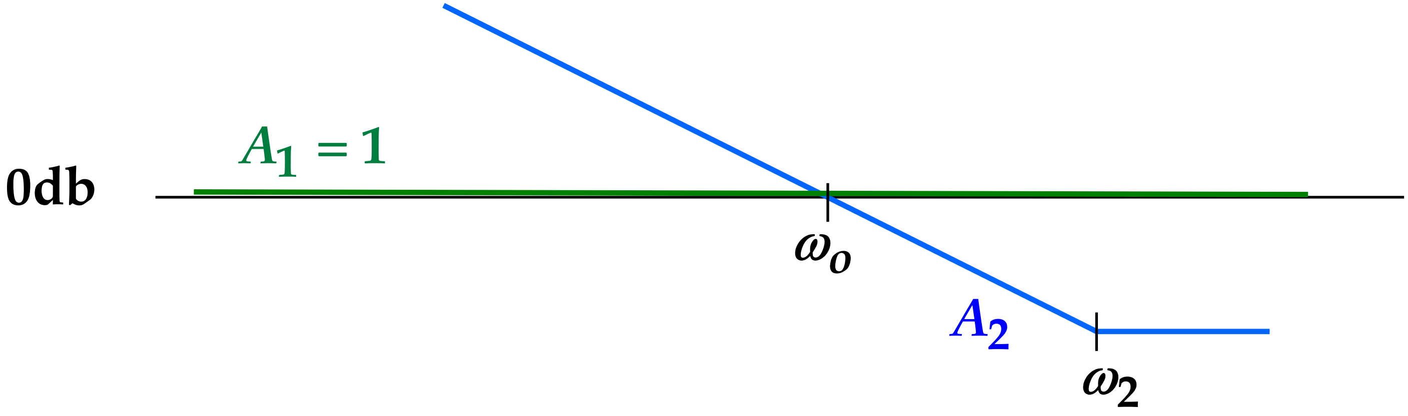
This is the exact answer.

It's your decision as to whether the approximate answer is good enough.

Exercise 6.1

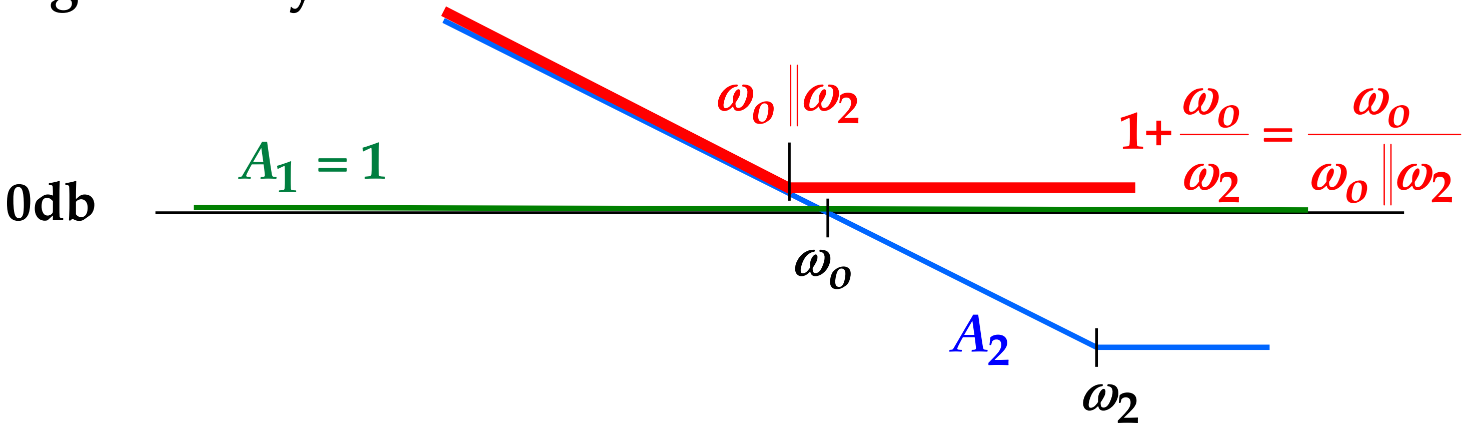
Guess an exact sum

Guess the exact sum $A = A_1 + A_2$ on the graph, then find it algebraically.

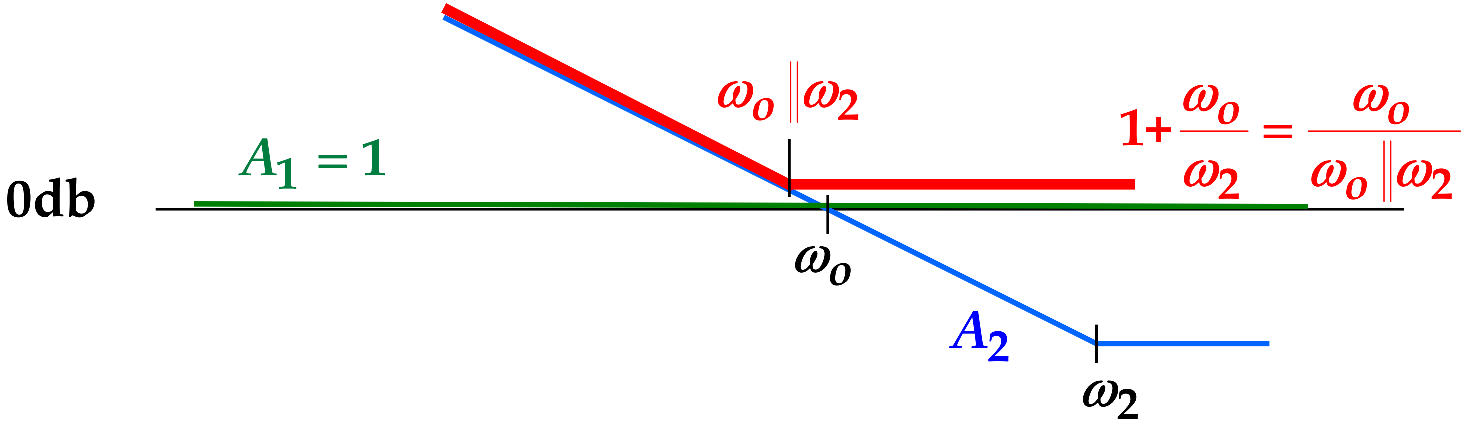


Exercise 6.1 - Solution

Guess the exact sum $A = A_1 + A_2$ on the graph, then find it algebraically.



Exercise 6.1 - Solution



$$A = 1 + \frac{\omega_o}{s}\left(1 + \frac{s}{\omega_2}\right) = 1 + \frac{\omega_o}{\omega_2} + \frac{\omega_o}{s}$$

$$= \left(1 + \frac{\omega_o}{\omega_2}\right)\left(1 + \frac{\omega_o \| \omega_2}{s}\right)$$

Guidelines:

In ranges where both functions have the same slope, the combination has the same slope and is the sum of the separate values.

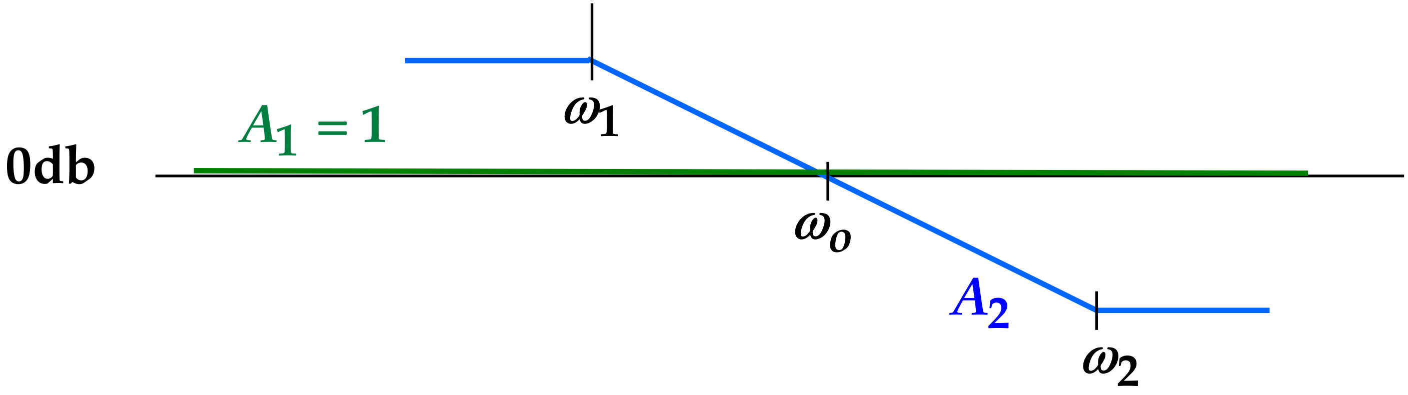
The poles of the sum are the poles of the two functions.

The "gain-bandwidth tradeoff" relates the corner frequencies to the flat values.

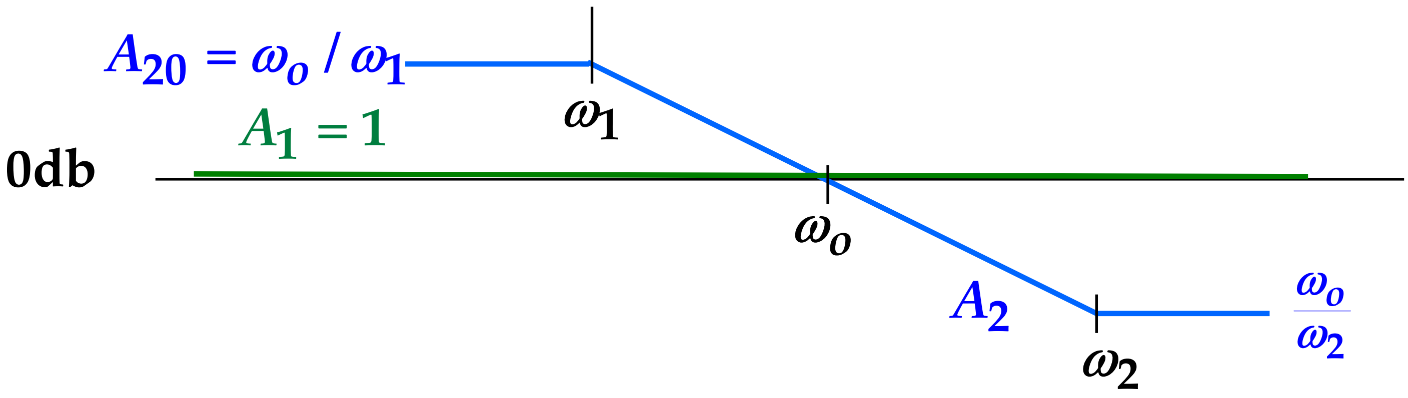
Exercise 6.2

Find an exact sum graphically

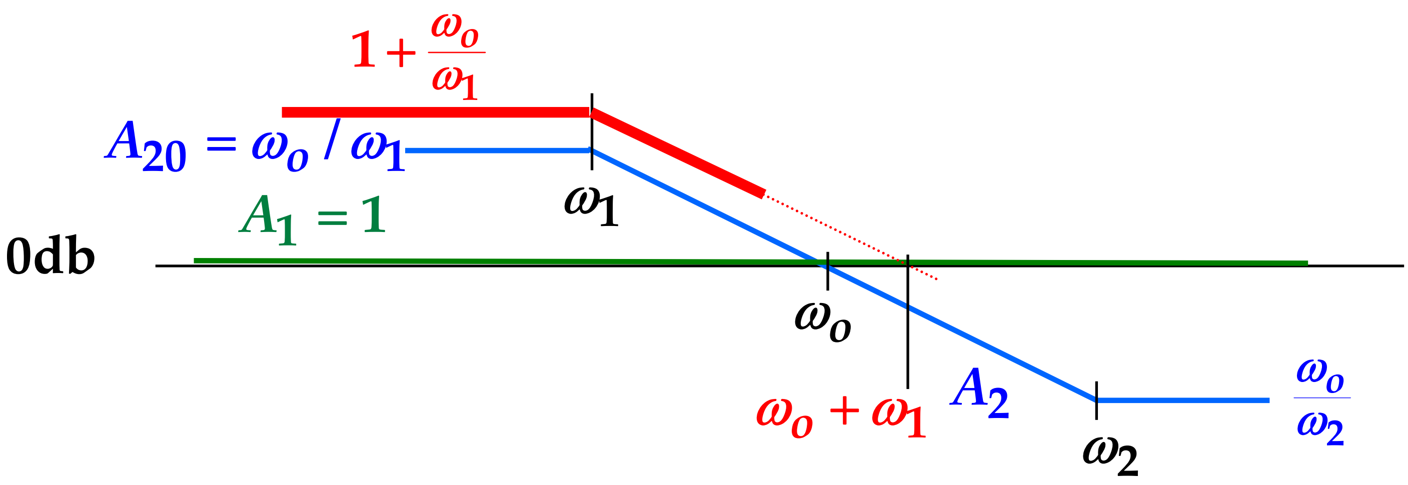
Use the Guidelines to construct the exact sum of the two functions, without doing any algebra:



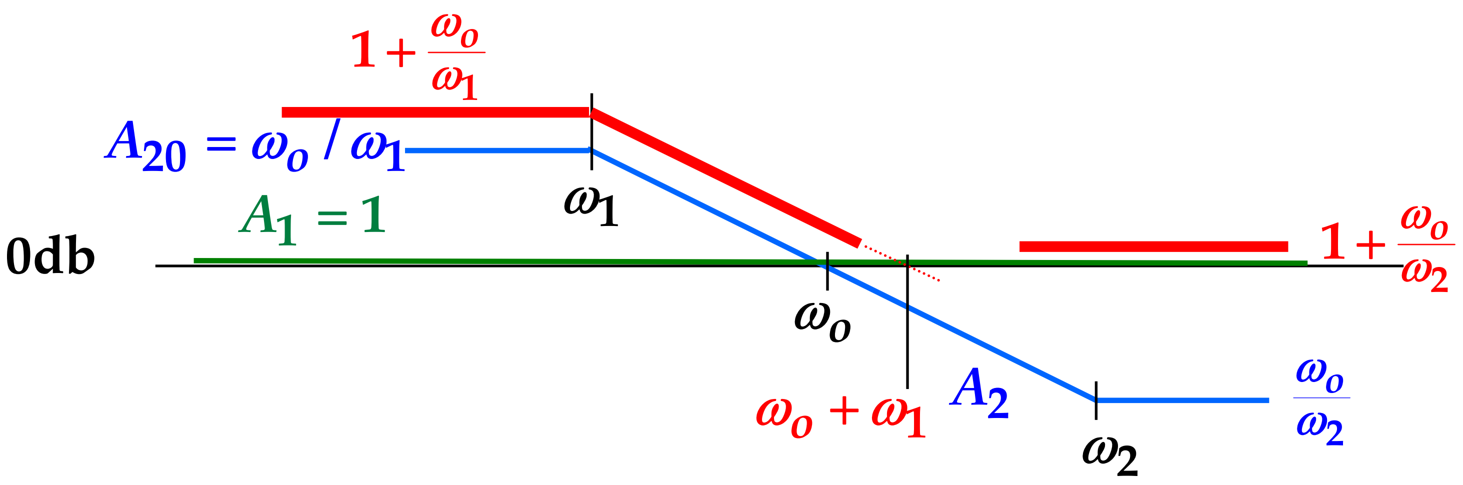
Exercise 6.2 - Solution



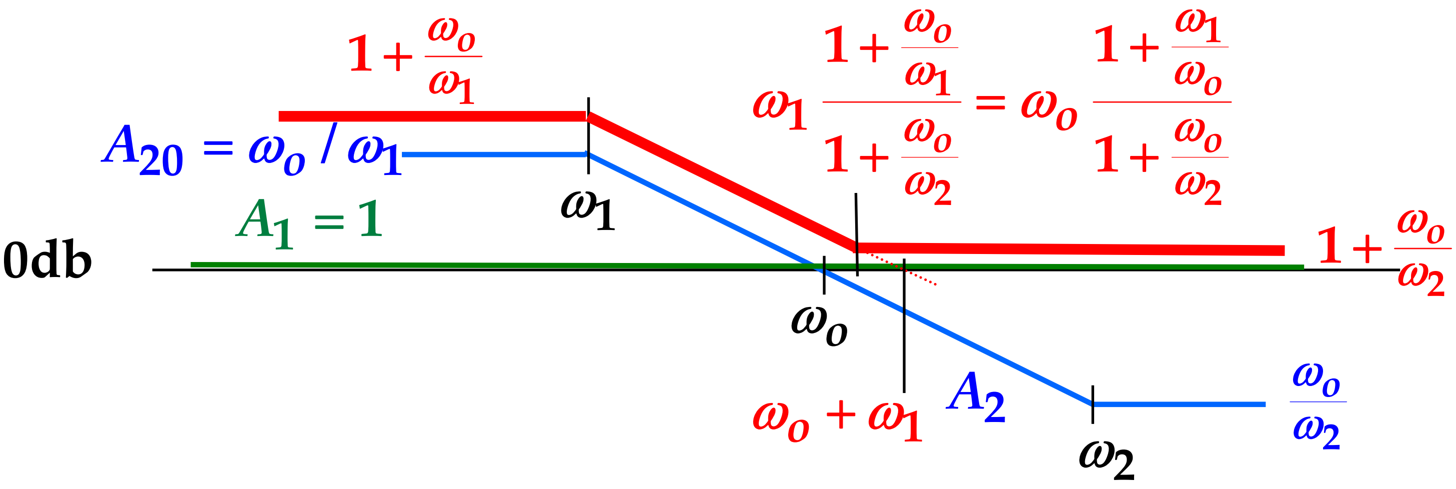
Exercise 6.2 - Solution



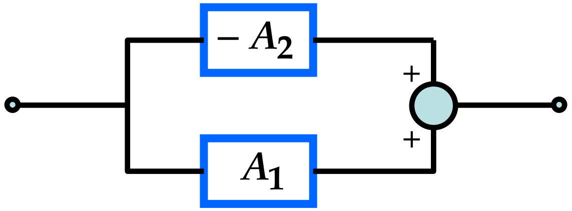
Exercise 6.2 - Solution



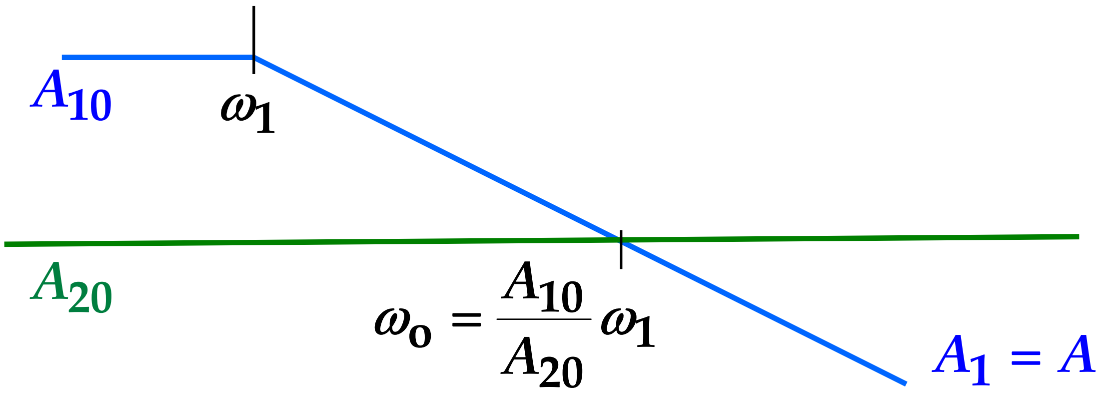
Exercise 6.2 - Solution



Difference of two functions:



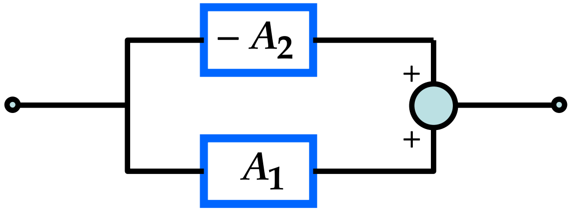
$$A = A_1 - A_2$$



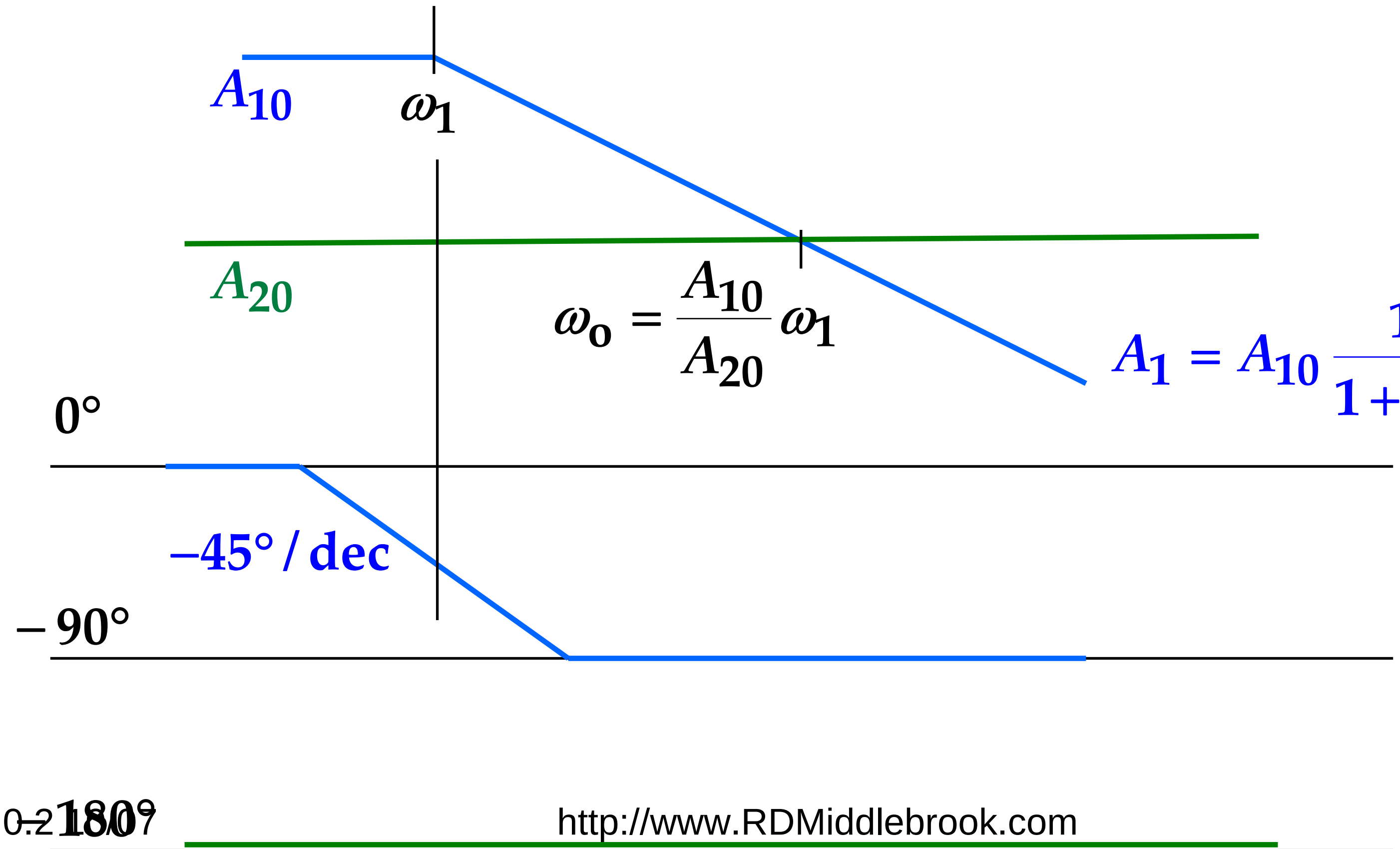
$$\omega_o = \frac{A_{10}}{A_{20}}\omega_1$$

$$A_1 = A_{10}\frac{1}{1+\frac{s}{\omega_1}}$$

Difference of two functions:



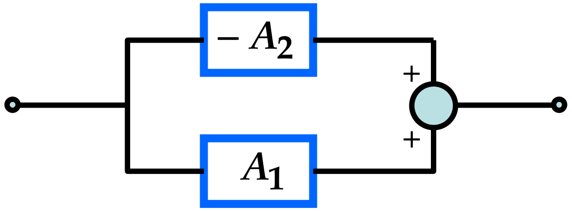
$$A = A_1 - A_2$$



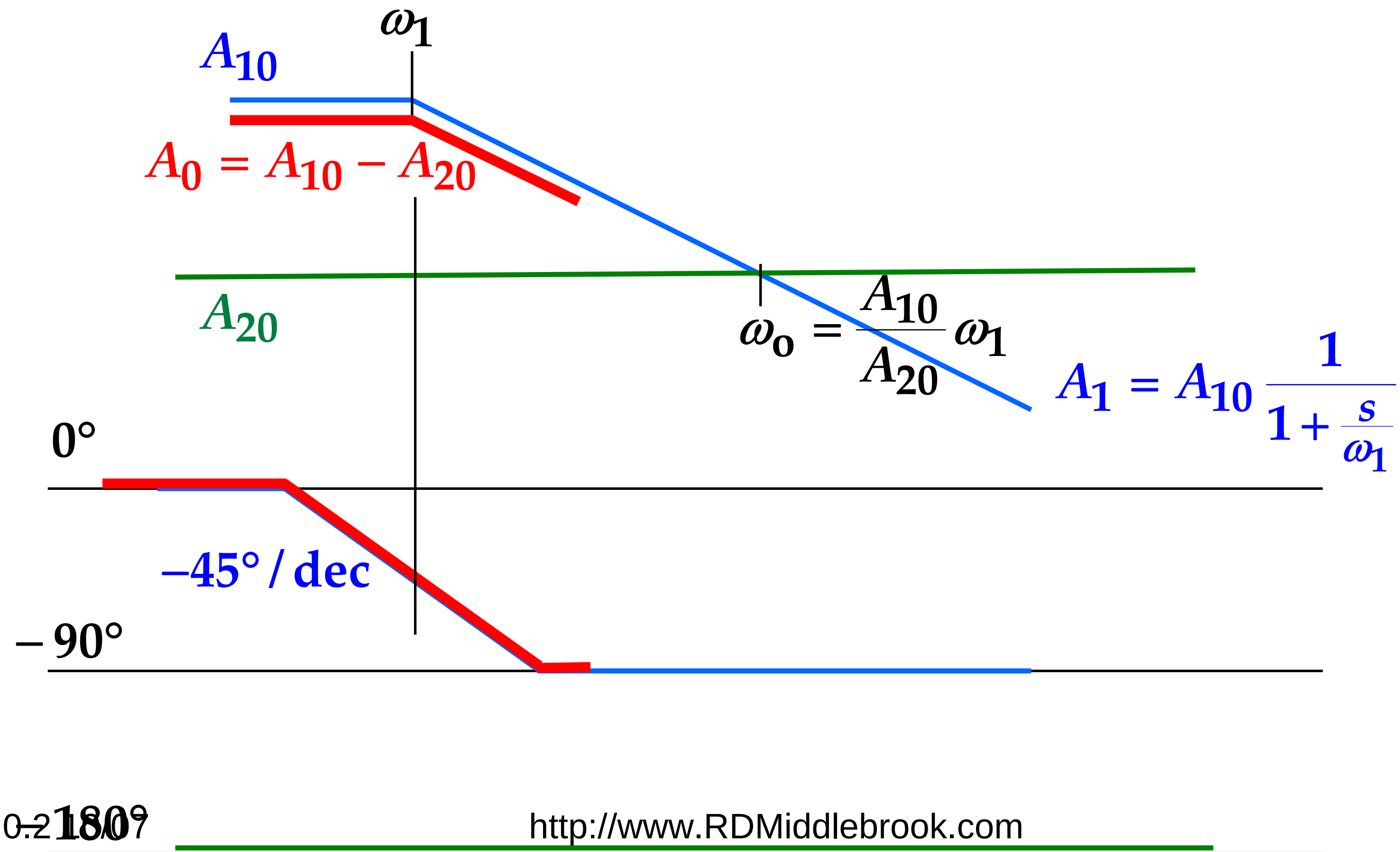
$$\omega_o = \frac{A_{10}}{A_{20}} \omega_1$$

$$A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}}$$

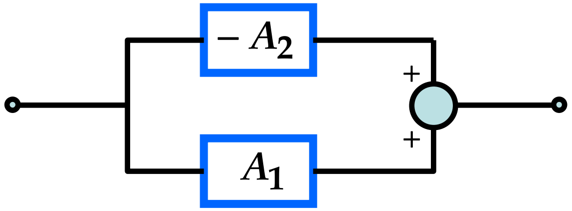
Difference of two functions:



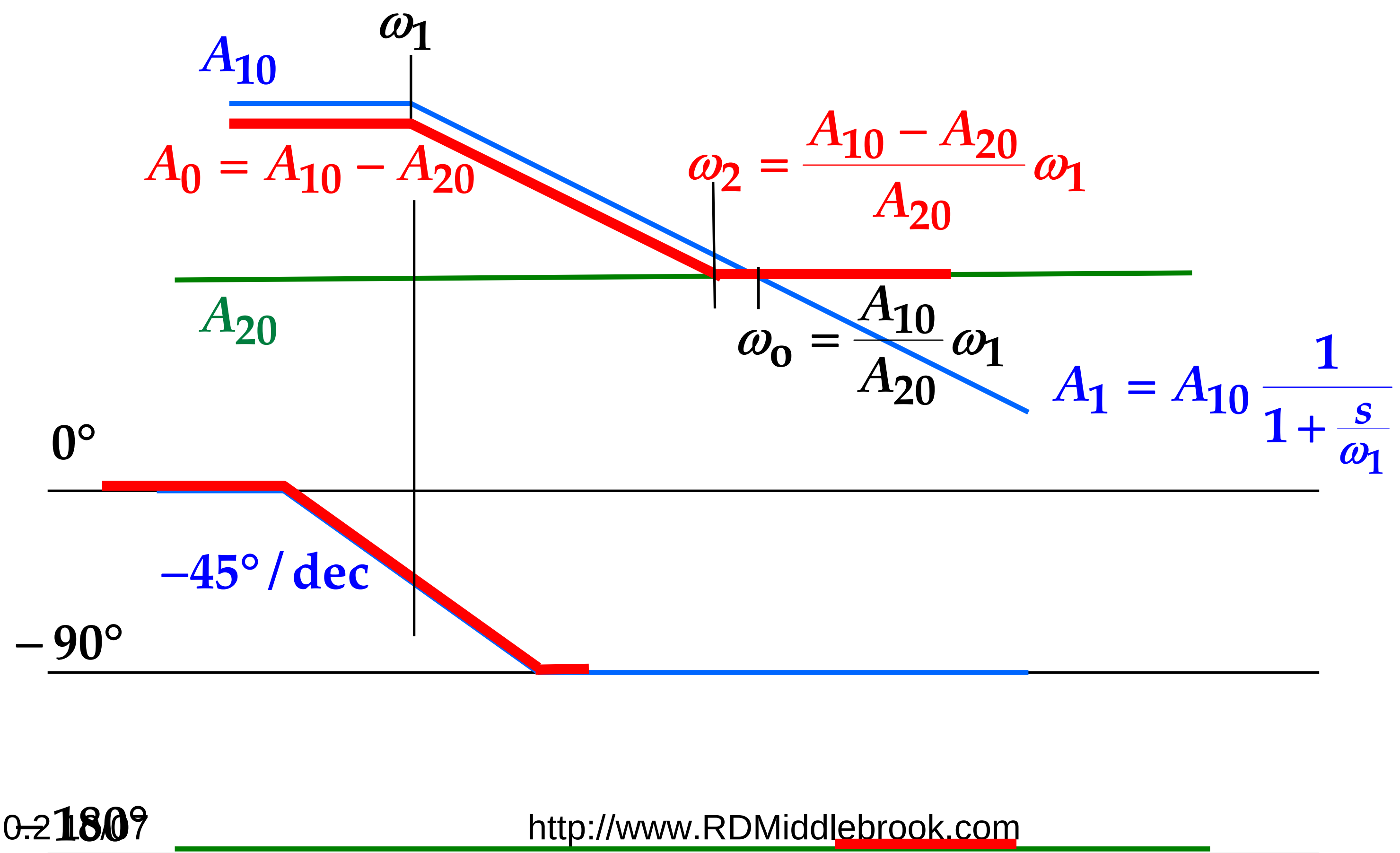
$$A = A_1 - A_2$$



Difference of two functions:



$$A = A_1 - A_2$$



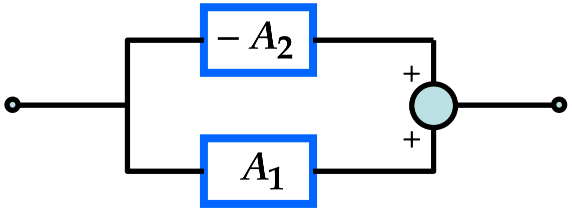
$$A_0 = A_{10} - A_{20}$$

$$\omega_2 = \frac{A_{10} - A_{20}}{A_{20}} \omega_1$$

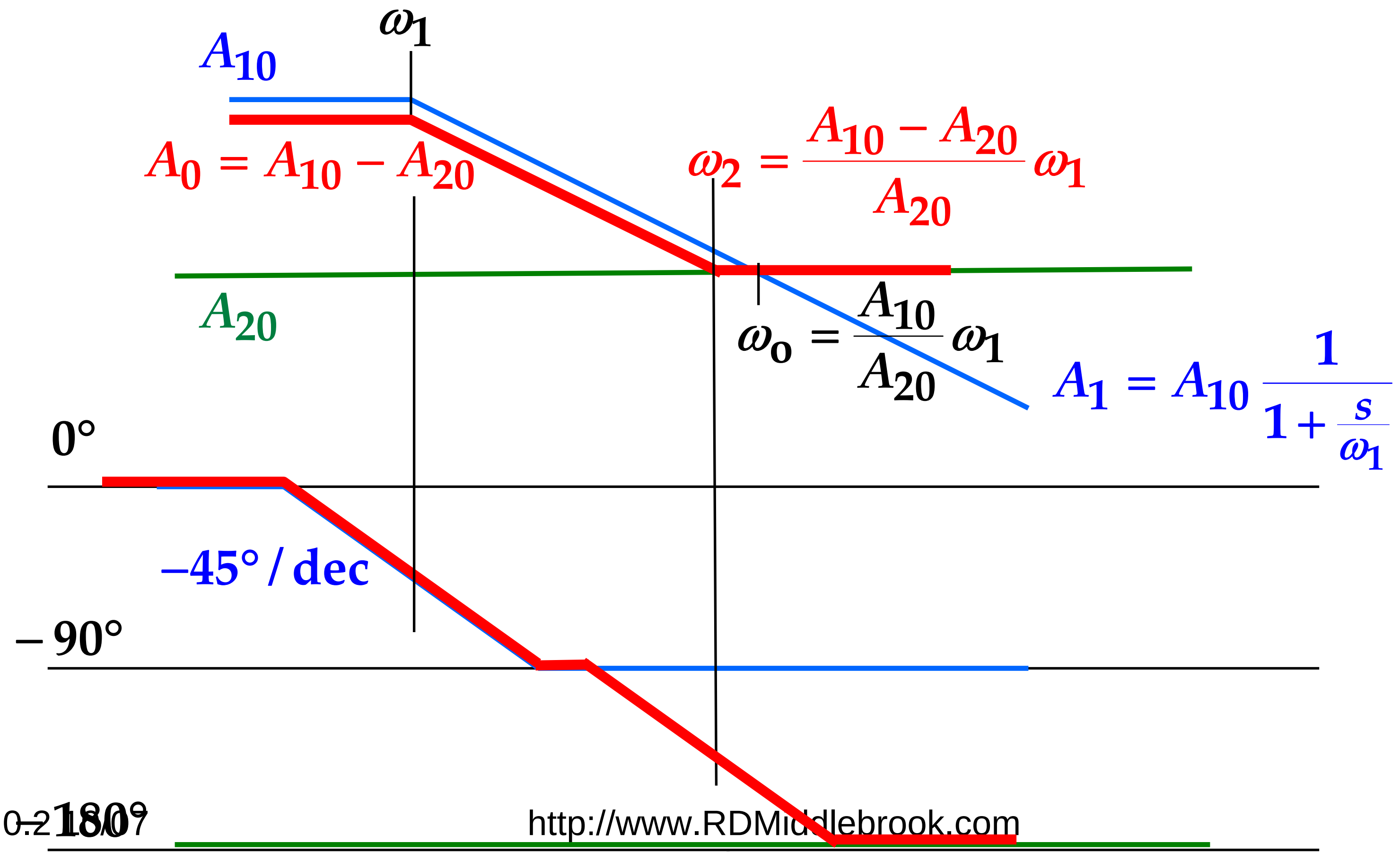
$$\omega_o = \frac{A_{10}}{A_{20}} \omega_1$$

$$A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}}$$

Difference of two functions:



$$A = A_1 - A_2$$



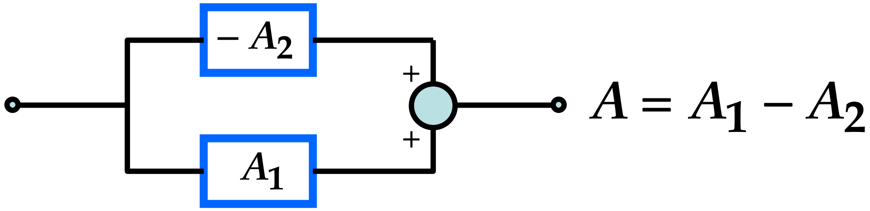
$$A_0 = A_{10} - A_{20}$$

$$\omega_2 = \frac{A_{10} - A_{20}}{A_{20}}\omega_1$$

$$\omega_o = \frac{A_{10}}{A_{20}}\omega_1$$

$$A_1 = A_{10}\frac{1}{1+\frac{s}{\omega_1}}$$

Difference of two functions:

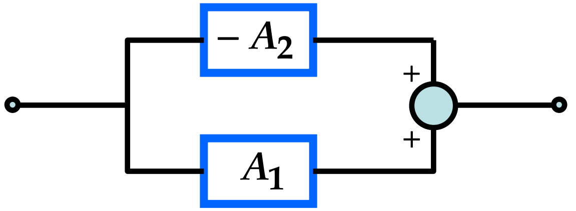


The result is

$$A = A_0 \frac{1 - \dfrac{s}{\omega_2}}{1 + \dfrac{s}{\omega_1}}$$

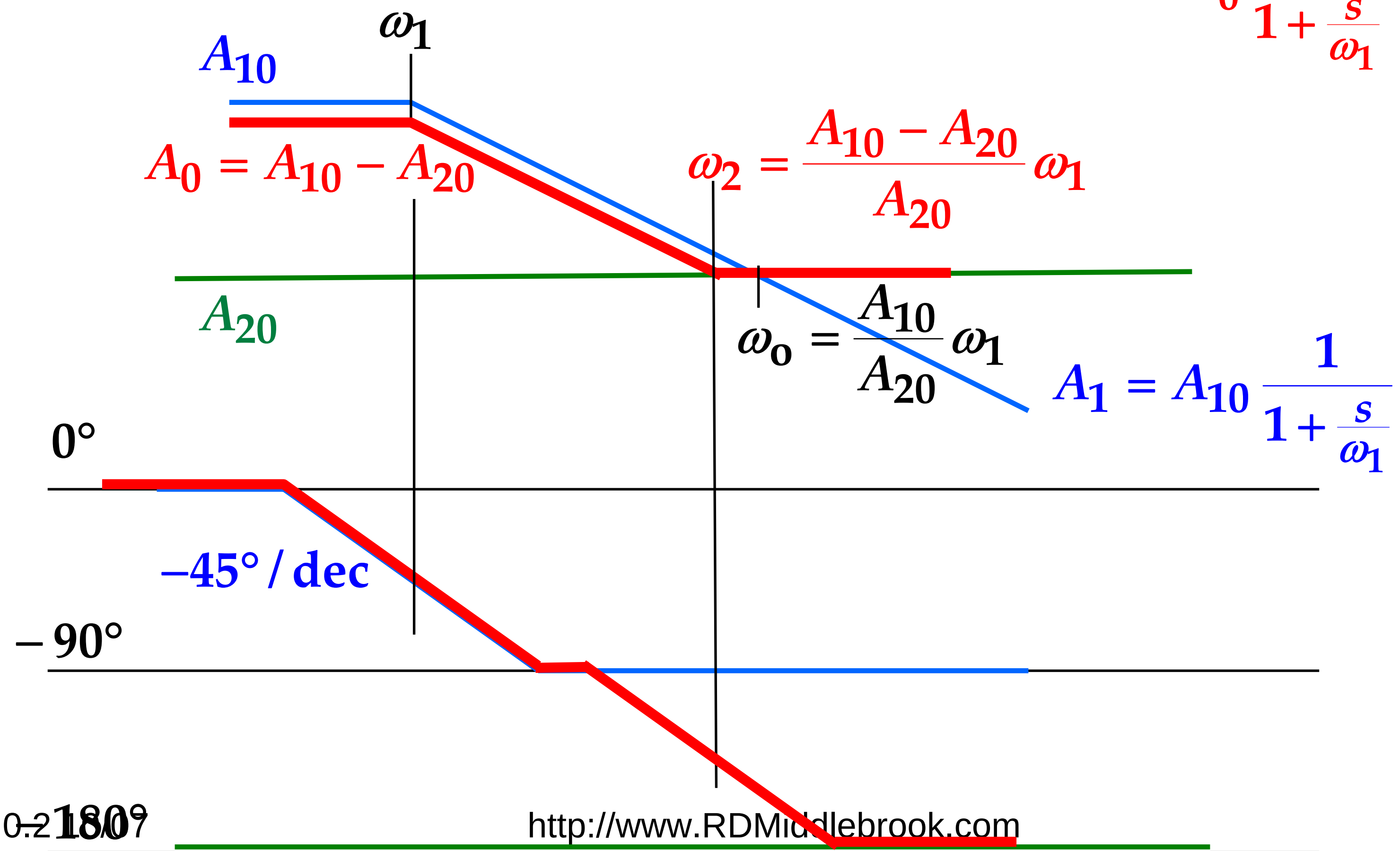
The corner ω_2 is a right half plane (rhp) zero: it has a concave upward magnitude response, but a phase lag, not a phase lead.

Difference of two functions:



$$A = A_1 - A_2$$

$$A = A_0 \frac{1 - \frac{s}{\omega_2}}{1 + \frac{s}{\omega_1}}$$



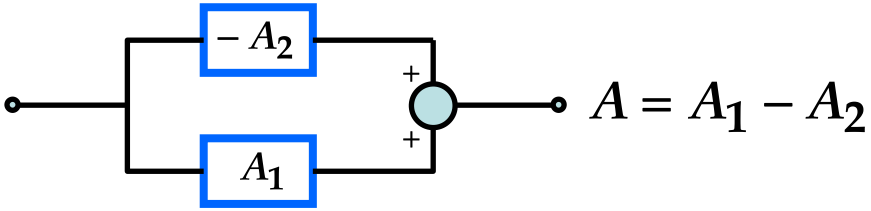
$$A_0 = A_{10} - A_{20}$$

$$\omega_2 = \frac{A_{10} - A_{20}}{A_{20}} \omega_1$$

$$\omega_o = \frac{A_{10}}{A_{20}} \omega_1$$

$$A_1 = A_{10} \frac{1}{1 + \frac{s}{\omega_1}}$$

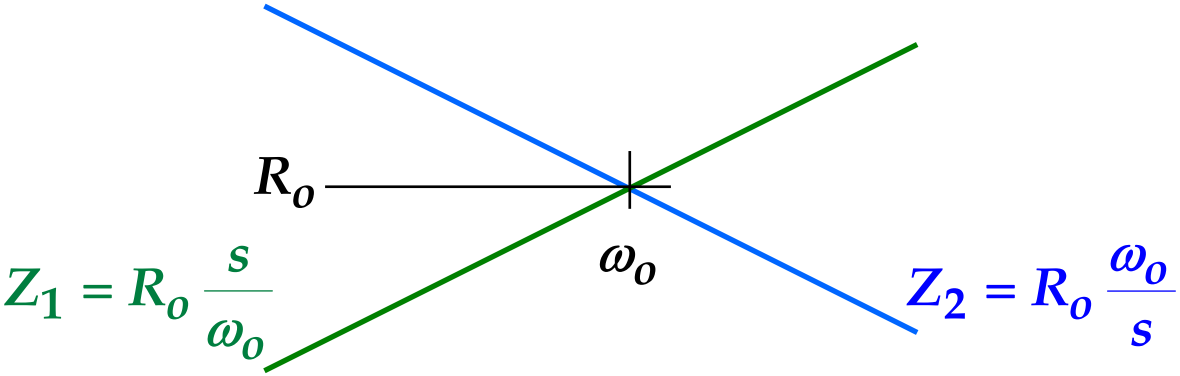
Difference of two functions:



A rhp zero occurs when a signal can go from input to output by two paths, one inverting and one not, with one path dominating at low frequencies, and the other dominating at high frequencies.

Every common-emitter or common-source amplifier stage potentially exhibits a rhp zero.

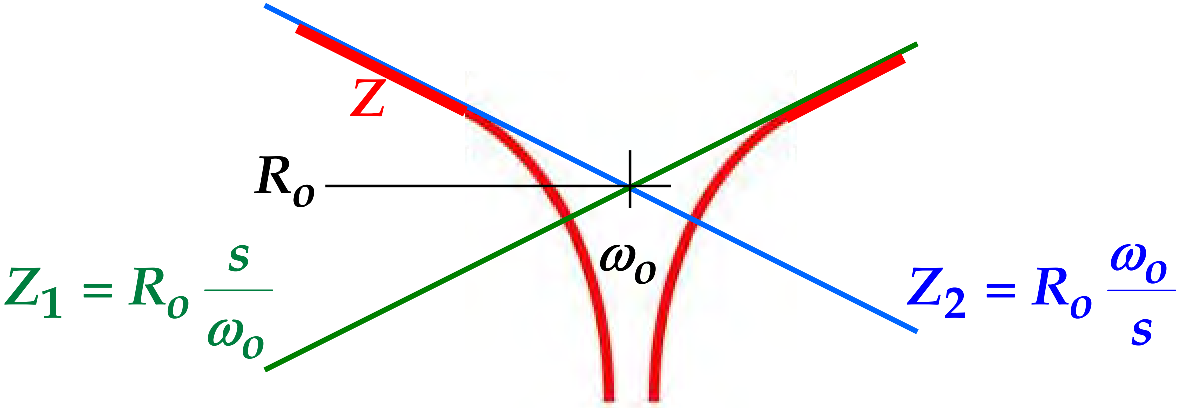
Consider sums of functions that result in quadratics in *s*



$$Z_1 = R_o \frac{s}{\omega_o}$$

$$Z_2 = R_o \frac{\omega_o}{s}$$

Consider sums of functions that result in quadratics in *s*

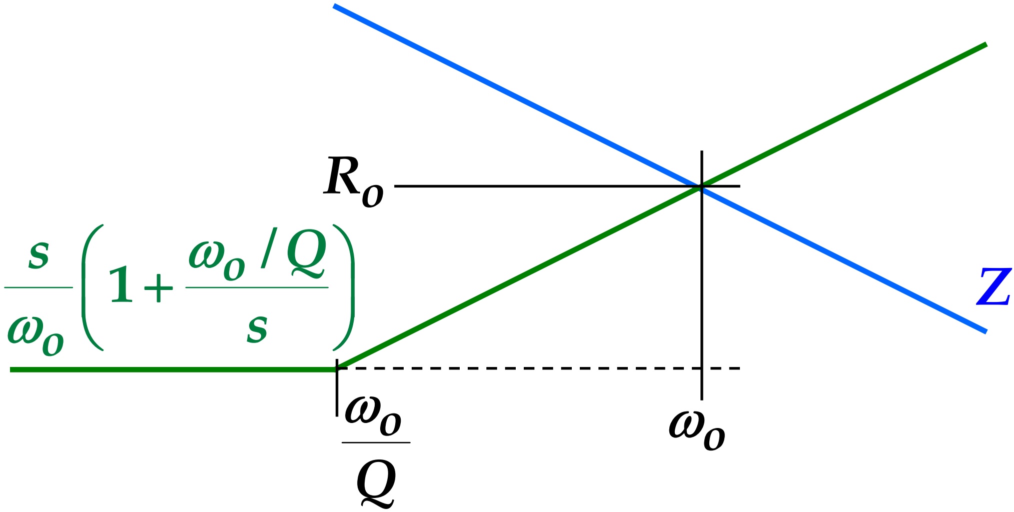


$$Z = Z_1 + Z_2 = R_o\left(\frac{s}{\omega_o} + \frac{\omega_o}{s}\right) = R_o \frac{\left[1+\left(\frac{s}{\omega_o}\right)^2\right]}{\frac{s}{\omega_o}}$$

The numerator is a quadratic pair of zeros with infinite *Q*

Consider sums of functions that result in quadratics in *s*

In any realistic case, there will be at least one additional corner:

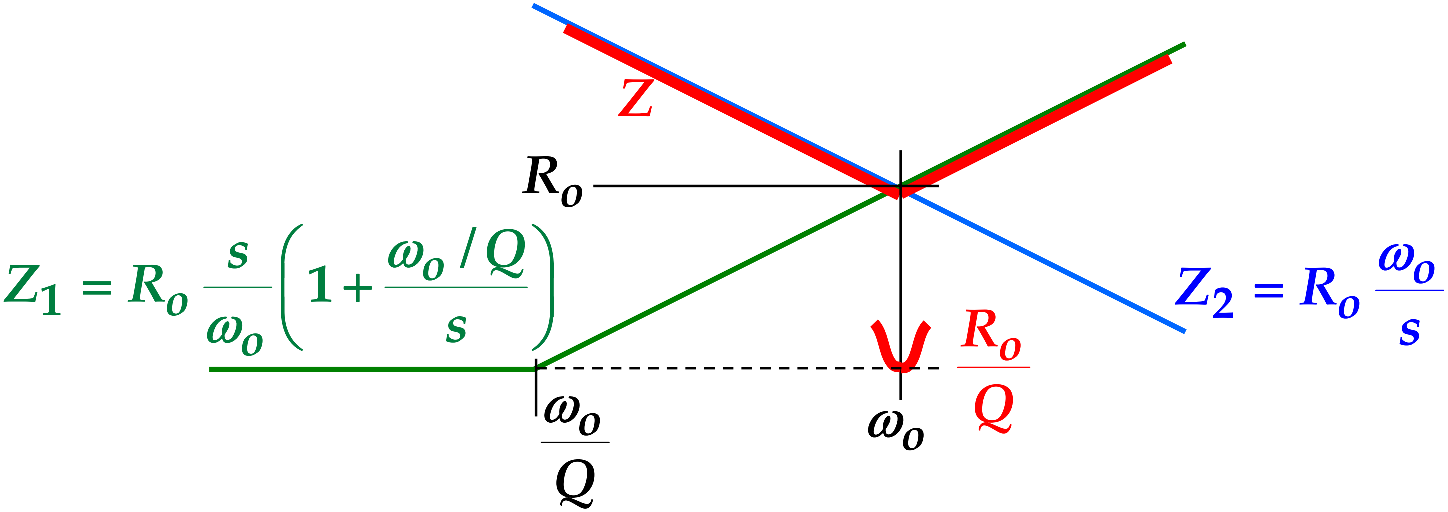


$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}$$

Consider sums of functions that result in quadratics in *s*

In any realistic case, there will be at least one additional corner:

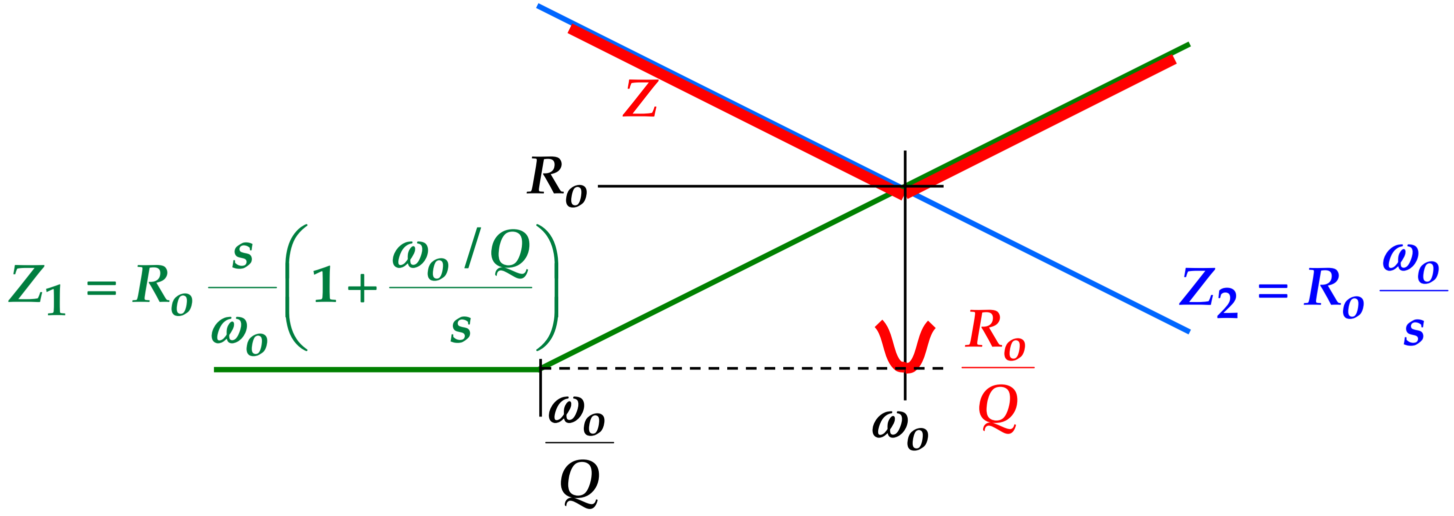


$$Z = R_o \frac{\left[1+\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right]}{\frac{s}{\omega_o}} = R_o\left[\frac{\omega_o}{s}+\frac{1}{Q}+\frac{s}{\omega_o}\right]$$

The second version exposes the symmetry.

Consider sums of functions that result in quadratics in *s*

In any realistic case, there will be at least one additional corner:



$$Z = R_o \frac{\left[1 + \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right]}{\frac{s}{\omega_o}}$$

Conclusion: The *Q* of a quadratic corner is affected by a nearby corner.

Short cut to find the quadratic Q-factor:

Evaluate Z_1 and Z_2 separately at $s = j\omega_o$:

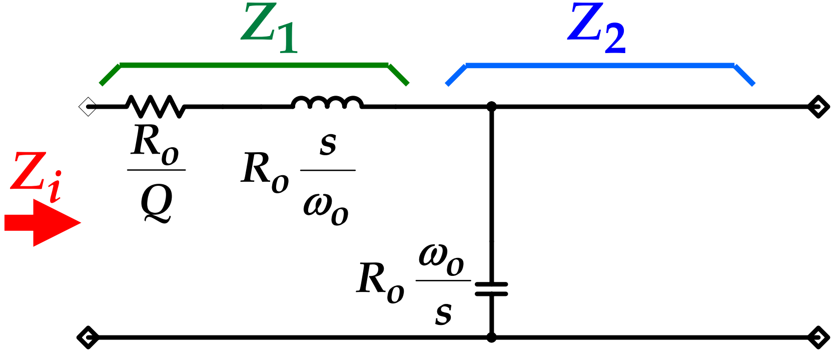
$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q}{s}\right) \qquad Z_1(j\omega_o) = R_o j\left(1 + \frac{1}{jQ}\right) = R_o\left(\frac{1}{Q} + j\right)$$

$$Z_2 = R_o \frac{\omega_o}{s} \qquad Z_2(j\omega_o) = R_o \frac{1}{j} \qquad = R_o\left(\quad - j\right)$$

When the two are added, the imaginary parts cancel, and the real part is the sum of the separate real parts :

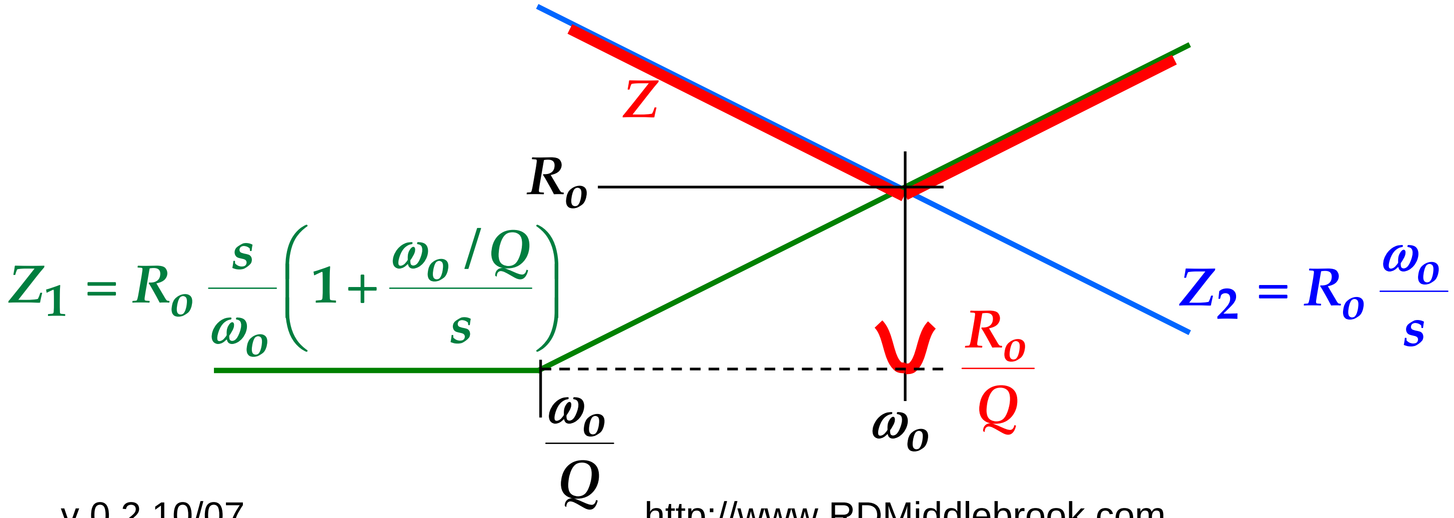
$$Z(j\omega_o) = \frac{R_o}{Q}$$

In the above example, Z_1 and Z_2 are the series and parallel branches of the single-damped LC low-pass filter, and Z is the input impedance Z_i :



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

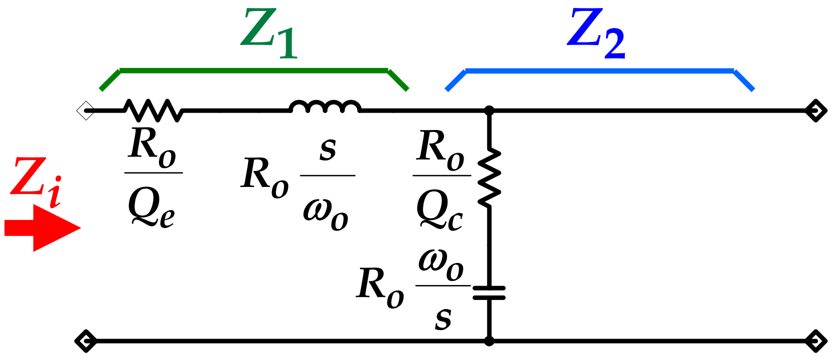
$$Q \equiv \frac{R_o}{R}$$



Doing the algebra on the graph can be extended to the double- and triple-damped filters.

Exercise 6.3

Find Z_i for the double-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.

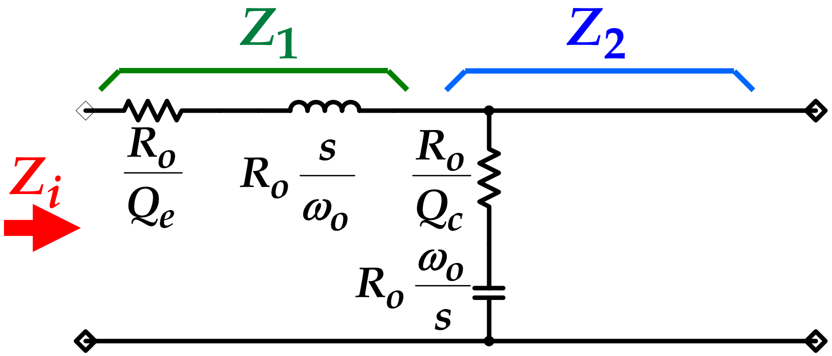


$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c}$$

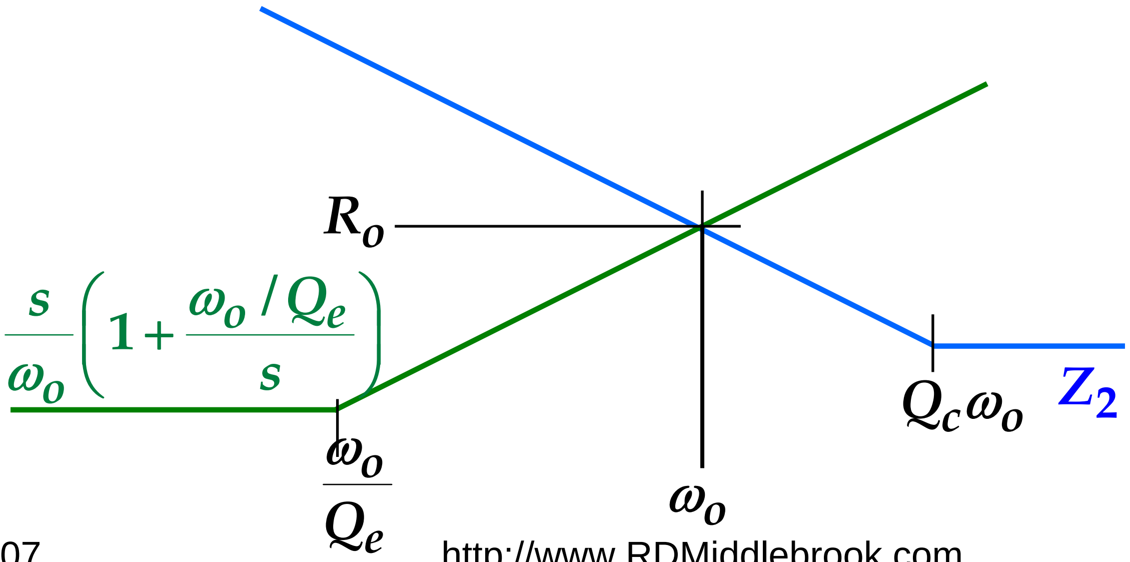
Exercise 6.3 - Solution

Find Z_i for the double-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance Z_i=Z_1+Z_2, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c}$$

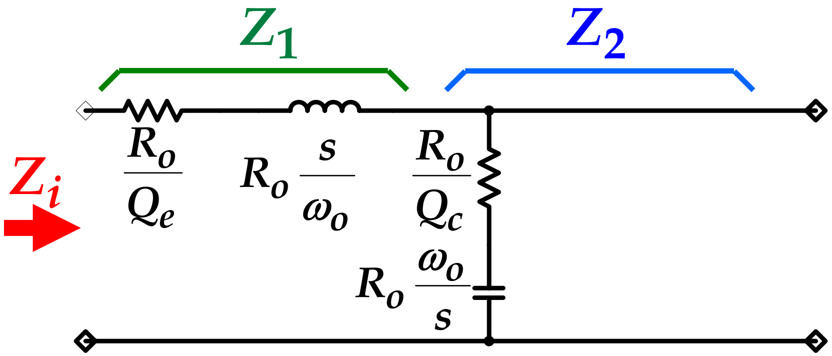


$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}\left(1 + \frac{s}{Q_c\omega_o}\right)$$

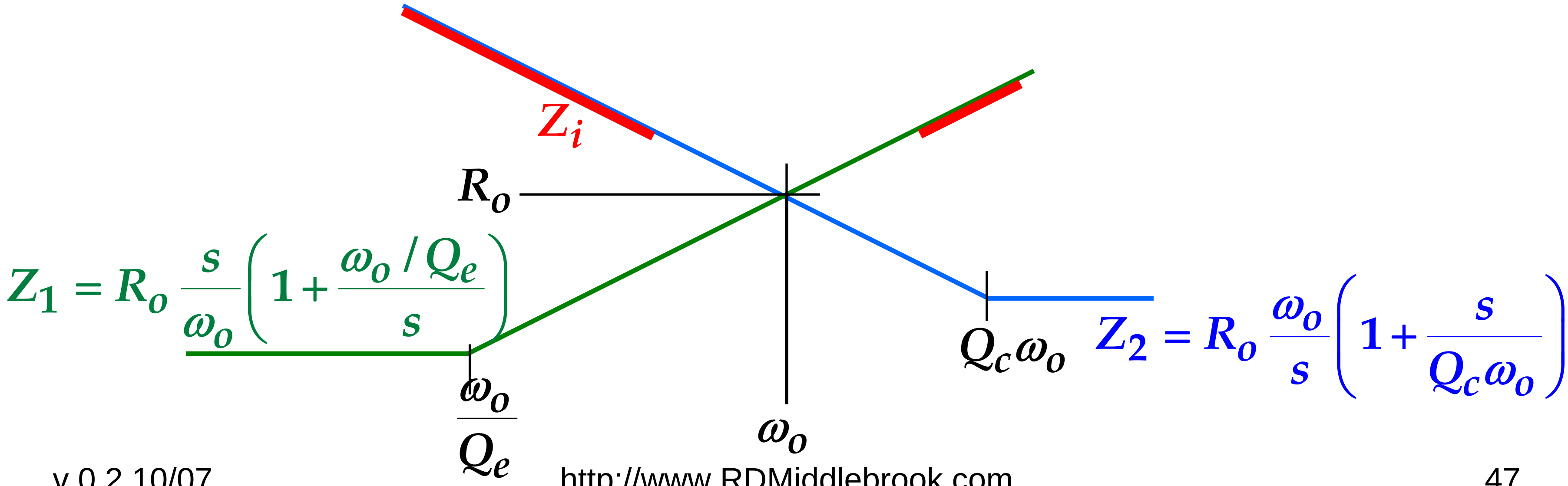
Exercise 6.3 - Solution

Find Z_i for the double-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c}$$



$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}\left(1 + \frac{s}{Q_c\omega_o}\right)$$

Exercise 6.3 - Solution

To find the quadratic Q-factor, evaluate Z_1 and Z_2 separately at $s = j\omega_o$:

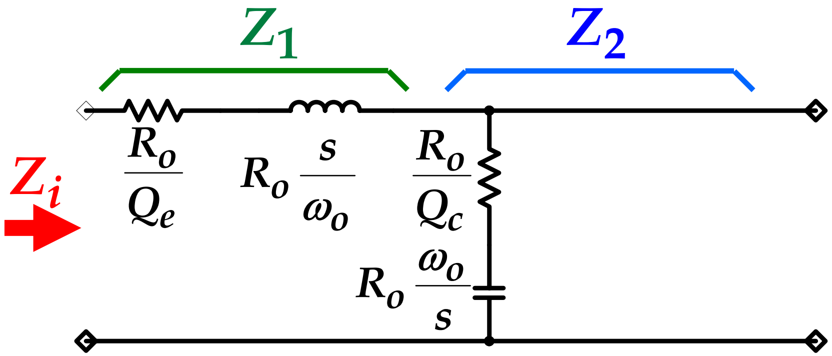
$$Z_1(j\omega_o) = R_o j\left(1 + \frac{1}{jQ_e}\right) \qquad = R_o\left(\frac{1}{Q_e} + j\right)$$

$$Z_2(j\omega_o) = R_o(-j)\left(1 + j\frac{1}{Q_c}\right) = R_o\left(\frac{1}{Q_c} - j\right)$$

Hence $\mathrm{Z_i}(j\omega_o) = Z_1(j\omega_o) + Z_2(j\omega_o) = R_o\left(\frac{1}{Q_e} + \frac{1}{Q_c}\right)$

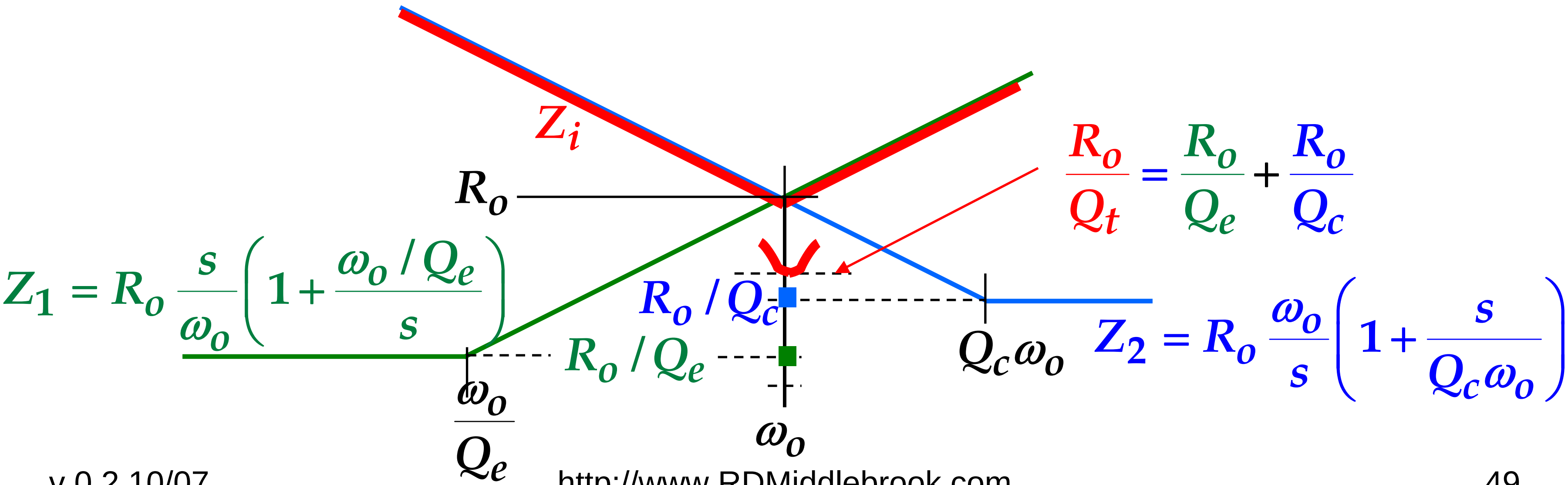
Exercise 6.3 - Solution

Find Z_i for the double-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance Z_i=Z$_1$+Z$_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c}$$



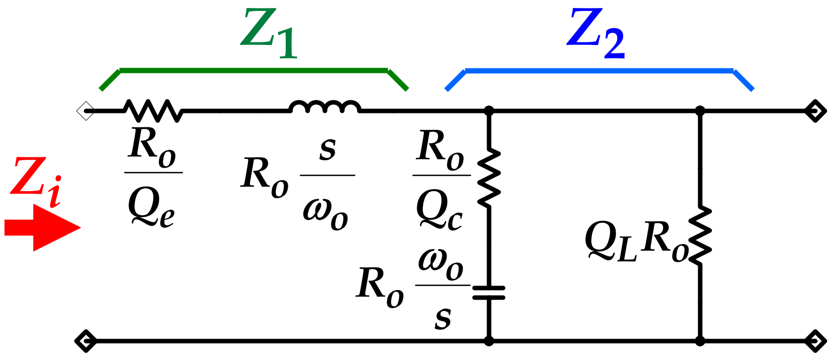
$$\frac{R_o}{Q_t} = \frac{R_o}{Q_e} + \frac{R_o}{Q_c}$$

$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s}\left(1 + \frac{s}{Q_c \omega_o}\right)$$

Exercise 6.4

Find Z_i for the triple-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.

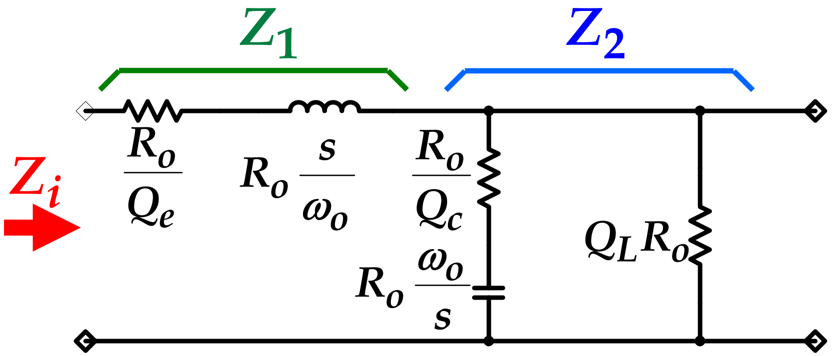


$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$

Exercise 6.4 - Solution

Find Z_i for the triple-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$

$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

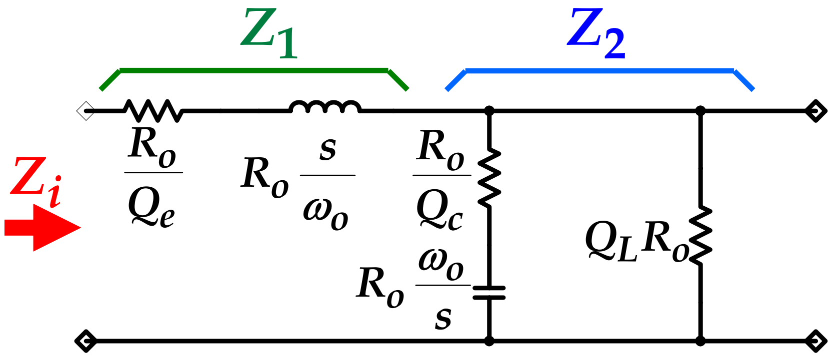
$$Z_2 = R_o \frac{Q_L\left(\frac{1}{Q_c} + \frac{\omega_o}{s}\right)}{Q_L + \left(\cancel{\frac{1}{Q_c}} + \frac{\omega_o}{s}\right)} \approx R_o \frac{\omega_o}{s} \frac{1 + \frac{s}{Q_c \omega_o}}{1 + \frac{\omega_o / Q_L}{s}}$$

$$Z_i(0) = Z_1(0) + Z_2(0) \;\; = \cancel{\frac{R_o}{Q_e}} + R_o Q_L \approx R_o Q_L$$

With neglect of the second-order effects, Z_i follows the higher asymptote:

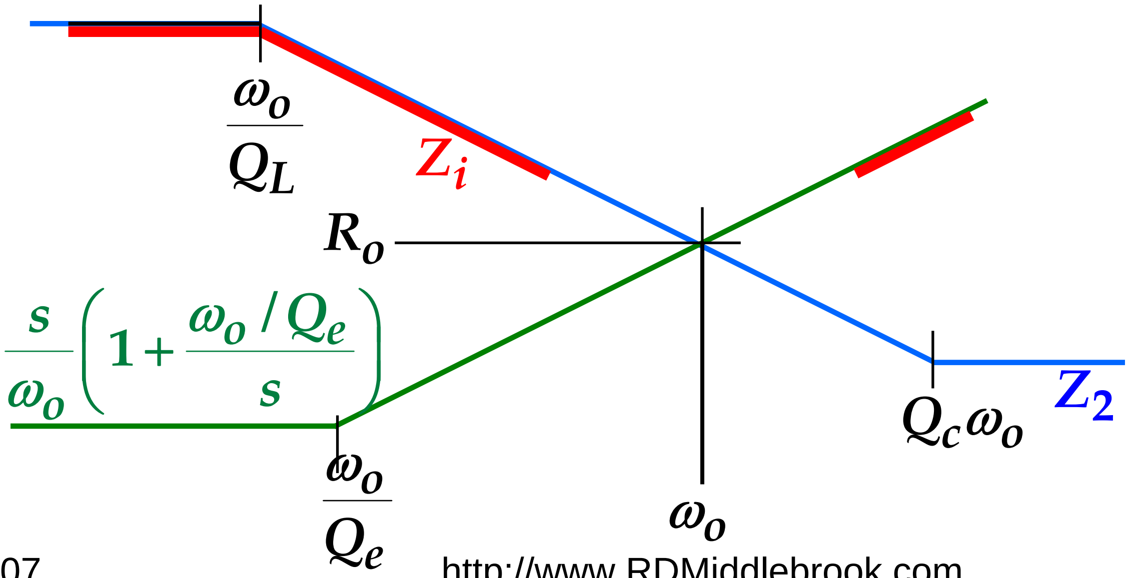
Exercise 6.4 - Solution

Find Z_i for the triple-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$



$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s} \frac{1 + \frac{s}{Q_c \omega_o}}{1 + \frac{\omega_o / Q_L}{s}}$$

Exercise 6.4 - Solution

To find the quadratic Q-factor, evaluate Z_1 and Z_2 separately at $s = j\omega_o$:

$$Z_1(j\omega_o) = R_o j\left(1+\frac{1}{jQ_e}\right) \qquad = R_o\left(\frac{1}{Q_e} + j\right)$$

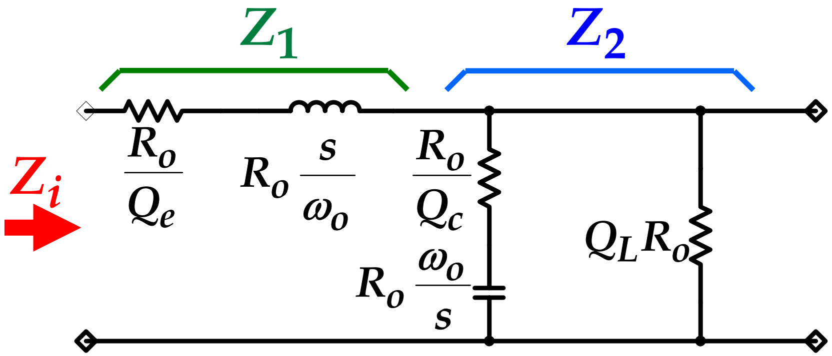
$$Z_2(j\omega_o) = R_o(-j)\frac{1+j\frac{1}{Q_c}}{1-j\frac{1}{Q_L}} = R_o\frac{\left(-j+\frac{1}{Q_c}\right)\left(1+j\frac{1}{Q_L}\right)}{1+\cancel{\frac{1}{{Q_L}^2}}}$$

$$\approx R_o\left[\frac{1}{Q_c}+\frac{1}{Q_L}-j\left(1-\cancel{\frac{1}{Q_cQ_L}}\right)\right] \qquad \approx R_o\left(\frac{1}{Q_c}+\frac{1}{Q_L}-j\right)$$

Hence $Z_i(j\omega_o) = Z_1(j\omega_o) + Z_2(j\omega_o)$ $\qquad = R_o\left(\frac{1}{Q_e}+\frac{1}{Q_c}+\frac{1}{Q_L}\right)$

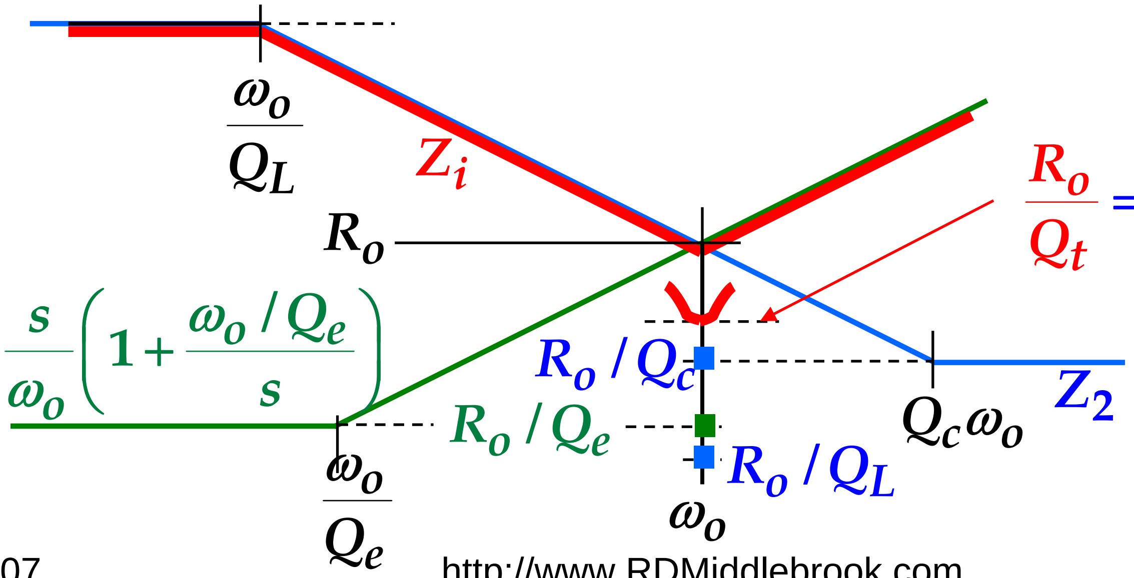
Exercise 6.4 - Solution

Find Z_i for the triple-damped LC filter. Draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the input impedance $Z_i = Z_1 + Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$



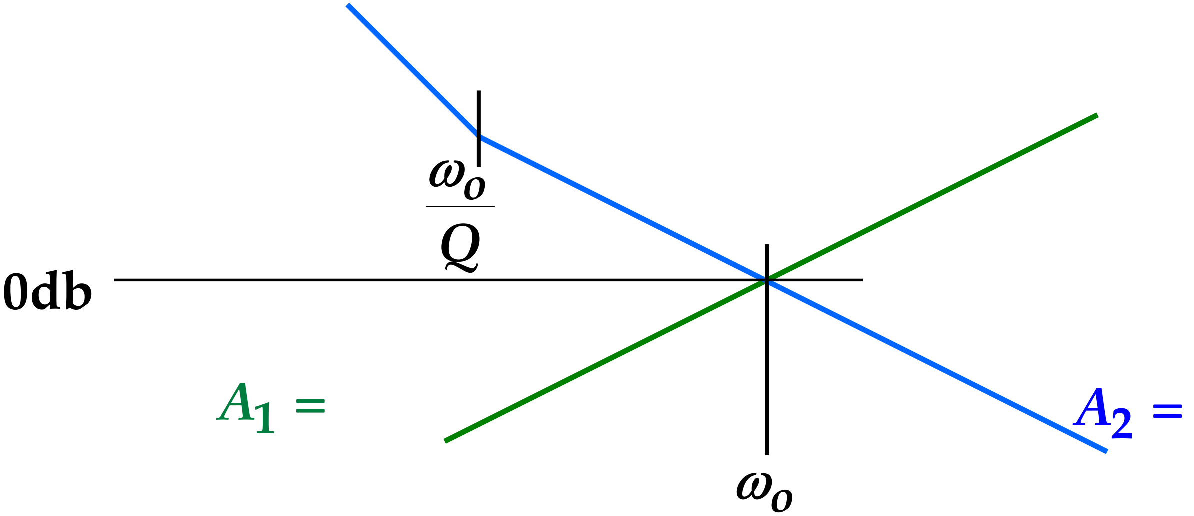
$$\frac{R_o}{Q_t} = \frac{R_o}{Q_e} + \frac{R_o}{Q_c} + \frac{R_o}{Q_L}$$

$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s} \frac{1 + \frac{s}{Q_c \omega_o}}{1 + \frac{\omega_o / Q_L}{s}}$$

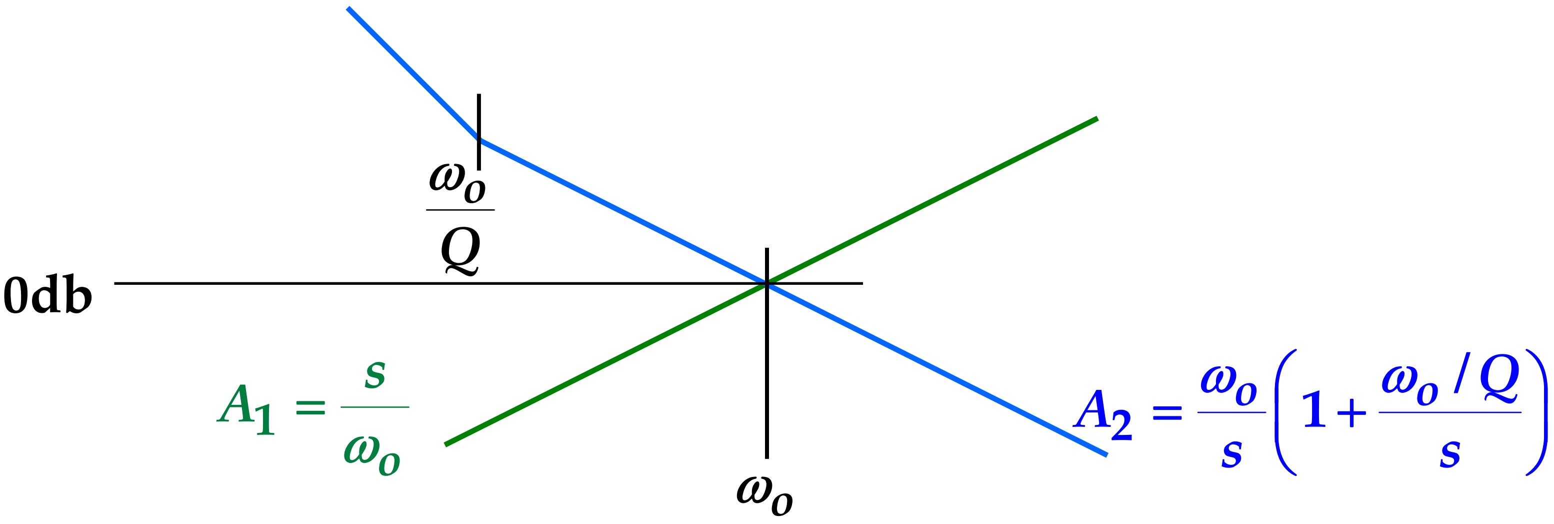
Exercise 6.5

Construct $A = A_1 + A_2$ in both magnitude and phase asymptotes, starting from A_1 and A_2 in suitable factored pole-zero forms.



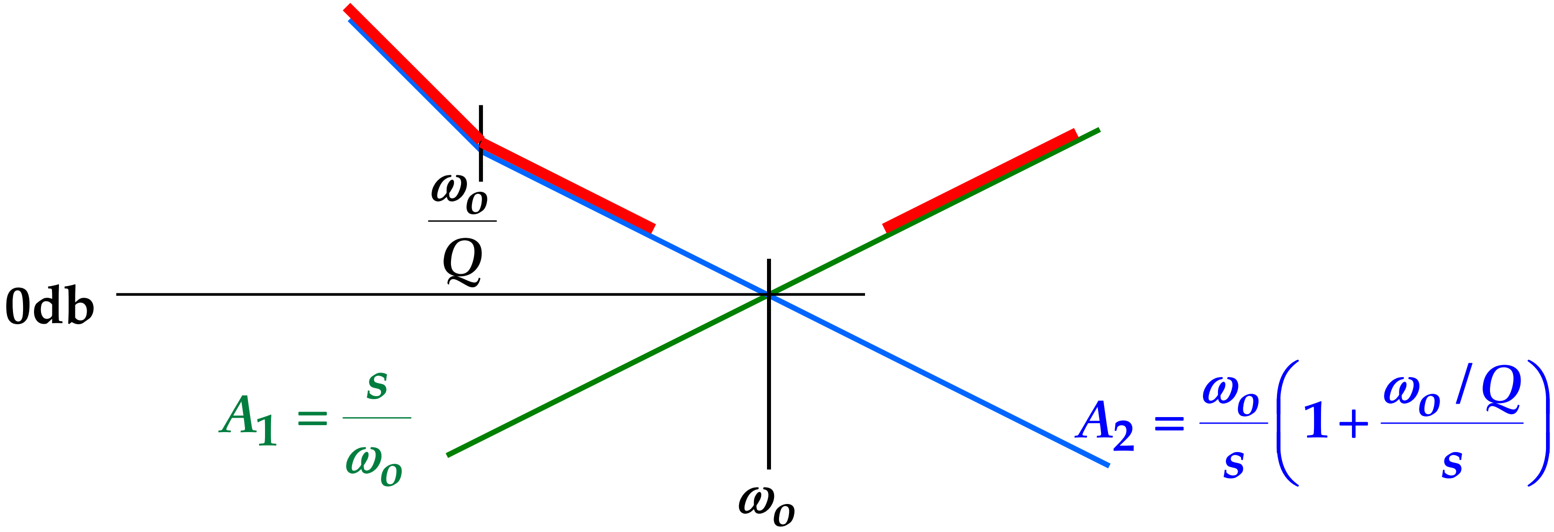
Exercise 6.5 - Solution

Construct $A = A_1 + A_2$ in both magnitude and phase asymptotes, starting from A_1 and A_2 in suitable factored pole-zero forms.



Exercise 6.5 - Solution

With neglect of second-order effects, the sum will follow the higher function:



The form is:

$$A = \frac{\omega_o}{s}\left(1 + \frac{\omega_o / Q}{s}\right)\left[1 + \frac{1}{Q_t}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right]$$

Exercise 6.5 - Solution

Find the quadratic Q-factor Q_t :

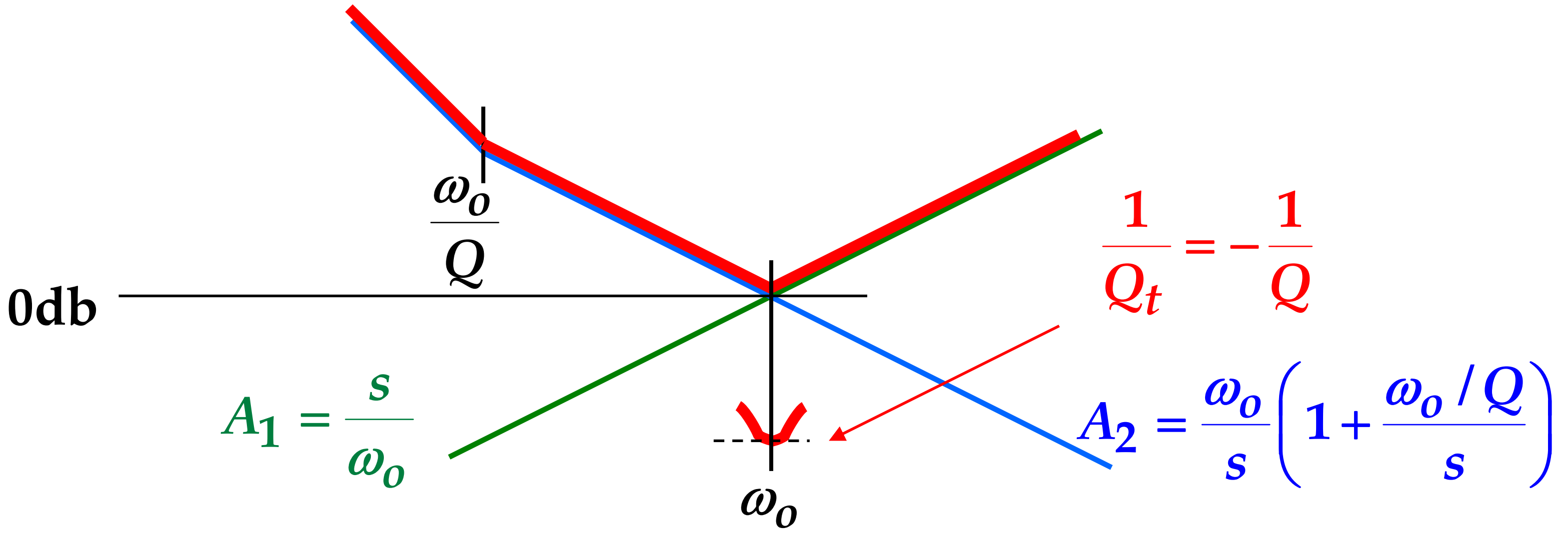
$$A = \frac{s}{\omega_o} + \frac{\omega_o}{s}\left(1 + \frac{\omega_o / Q}{s}\right)$$

$$A(j\omega_o) = j + \frac{1}{j}\left(1 + \frac{1}{jQ}\right) = j - j\left(1 - j\frac{1}{Q}\right) = -\frac{1}{Q}$$

So $Q_t = -Q$ and the final form is

$$A = \frac{\omega_o}{s}\left(1 + \frac{\omega_o / Q}{s}\right)\left[1 - \frac{1}{Q}\left(\frac{s}{\omega_o}\right) + \left(\frac{s}{\omega_o}\right)^2\right]$$

Exercise 6.5 - Solution



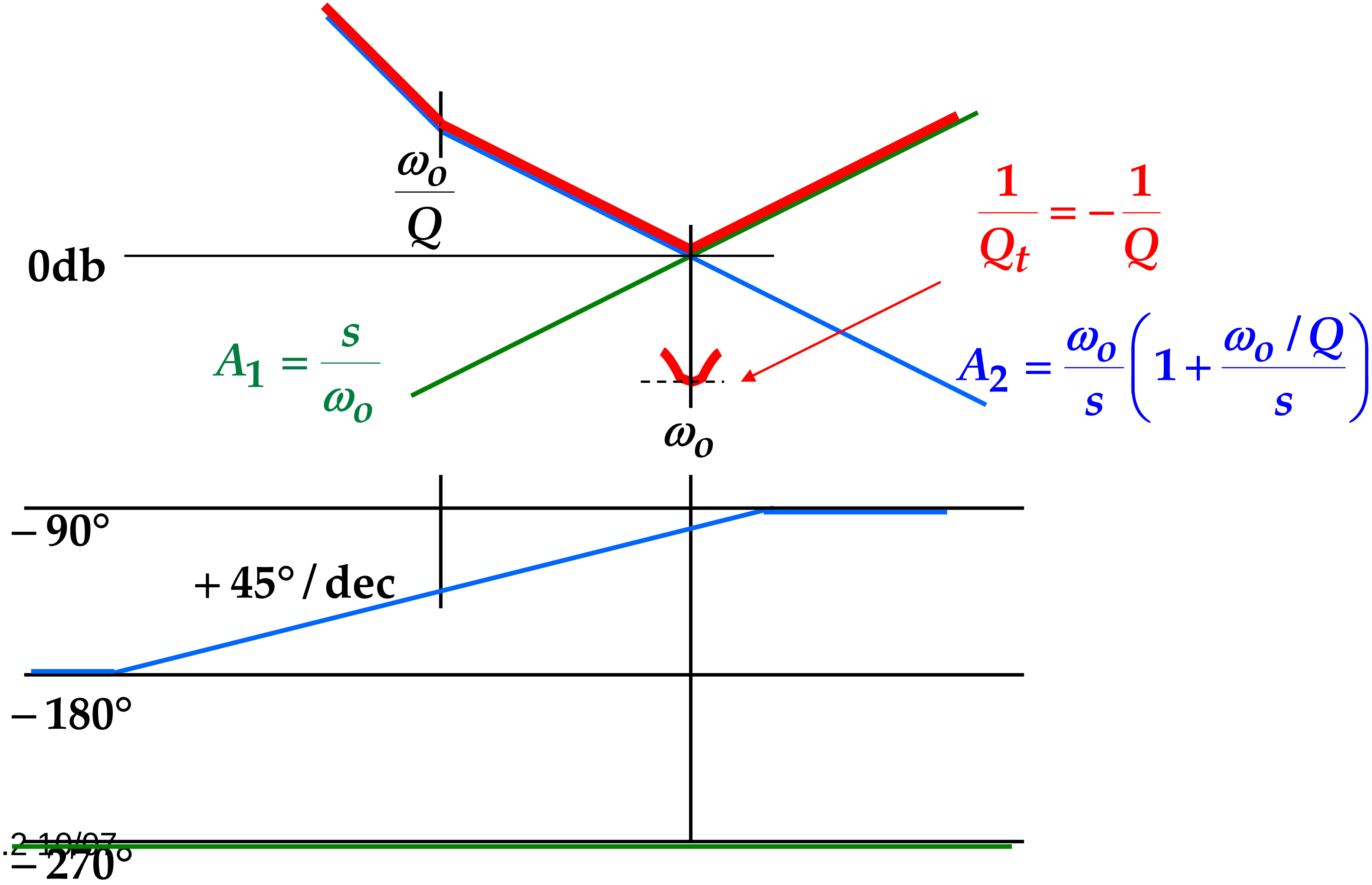
$$A_1 = \frac{s}{\omega_o}$$

$$\frac{1}{Q_t} = -\frac{1}{Q}$$

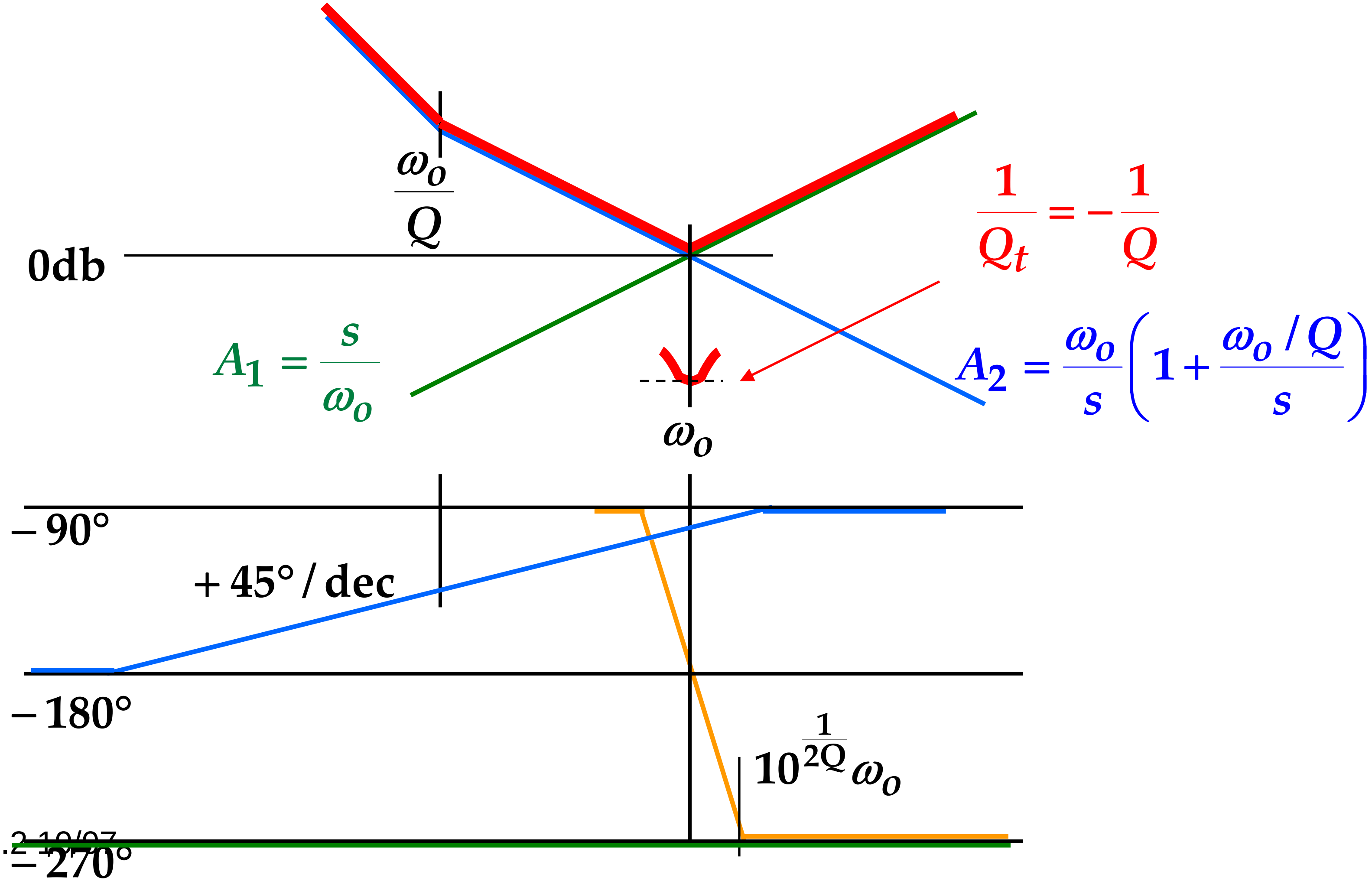
$$A_2 = \frac{\omega_o}{s}\left(1 + \frac{\omega_o / Q}{s}\right)$$

The negative Q_t means that the quadratic is two rhp zeros, so the magnitude asymptotes have a concave upwards corner at ω_o, and the phase is a 180° lag, not a lead.

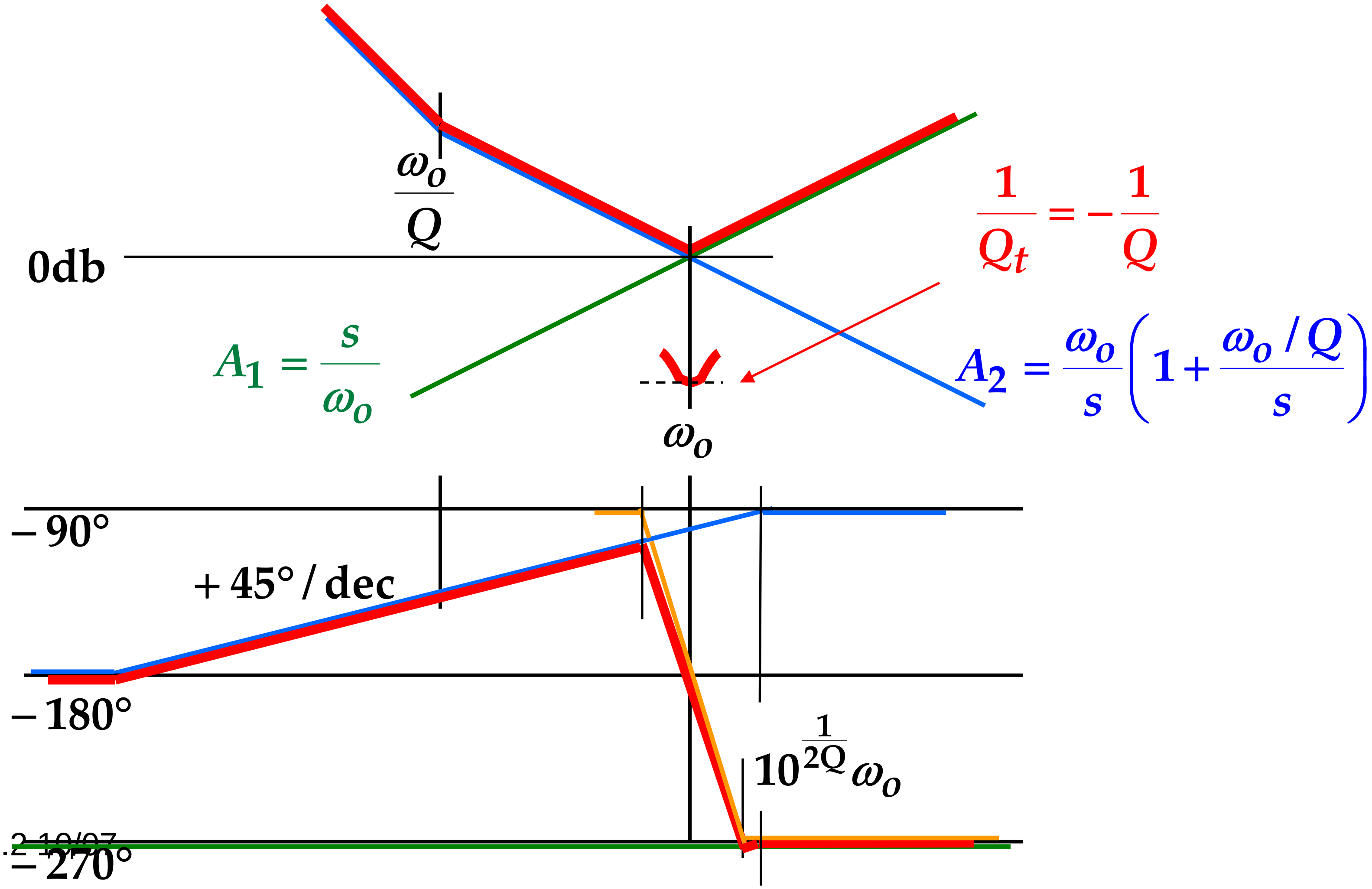
Exercise 6.5 - Solution



Exercise 6.5 - Solution



Exercise 6.5 - Solution



Exercise 6.5 - Solution

By doing the algebra on the graph to set up

$$A=\frac{\omega_o}{s}\left(1+\frac{\omega_o/Q}{s}\right)\left[1-\frac{1}{Q}\left(\frac{s}{\omega_o}\right)+\left(\frac{s}{\omega_o}\right)^2\right]$$

you have effectively found the symbolic roots of a cubic equation!

Algebraically,

$$A=\frac{s}{\omega_o}+\frac{\omega_o}{s}\left(1+\frac{\omega_o/Q}{s}\right)$$

$$=\frac{1}{Q}\left(\frac{\omega_o}{s}\right)^2+\frac{\omega_o}{s}+\frac{s}{\omega_o}$$

which is a cubic in (s/ω_o).

Extensions of the graphical method

1. In the sum of two functions, any one can be extracted to reduce the sum to the form $1+T$:

$$Z = Z_1 + Z_2 = Z_1\left(1+\frac{Z_2}{Z_1}\right) \text{ etc.}$$

Extensions of the graphical method

1. In the sum of two functions, any one can be extracted to reduce the sum to the form $1+T$:

$$Z = Z_1 + Z_2 = Z_1\left(1+\frac{Z_2}{Z_1}\right) \text{ etc.}$$

2. The sum of any number of impedances in series can be found graphically:

$$Z = Z_1 + Z_2 + Z_3 + \ldots$$

The result follows whichever contribution is the largest.

Extensions of the graphical method

1. In the sum of two functions, any one can be extracted to reduce the sum to the form $1+T$:

$$Z = Z_1 + Z_2 = Z_1\left(1+\frac{Z_2}{Z_1}\right) \text{ etc.}$$

2. The sum of any number of impedances in series can be found graphically:

$$Z = Z_1 + Z_2 + Z_3 + ...$$

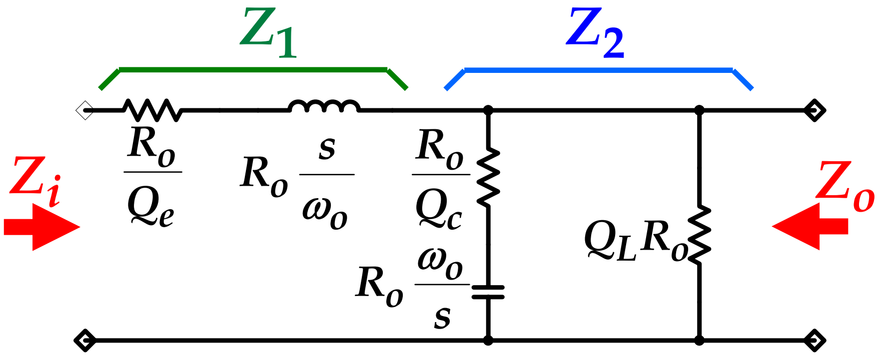
The result follows whichever contribution is the largest.

3. Similarly, the sum of any number of impedances in parallel can be found graphically:

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + ...$$

The result follows whichever contribution is the smallest.

For the triple-damped LC filter, draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the output impedance $Z_o = Z_1 \| Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



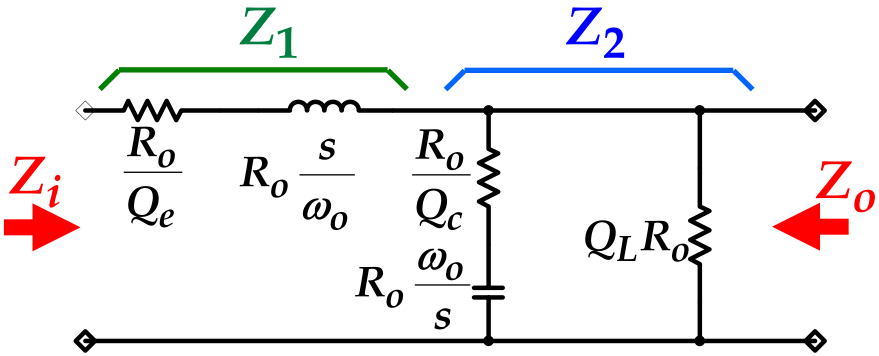
$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$

$$Z_i = Z_1 + Z_2$$

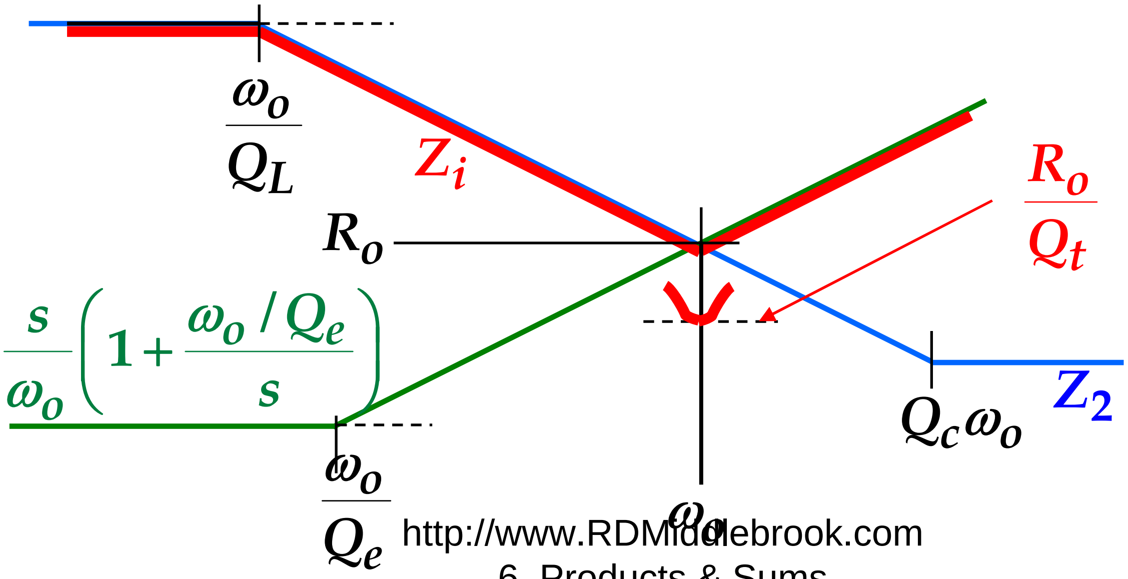
$$\frac{1}{Z_o} = \frac{1}{Z_1} + \frac{1}{Z_2}$$

For the triple-damped LC filter, draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the output impedance $Z_o = Z_1 \| Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

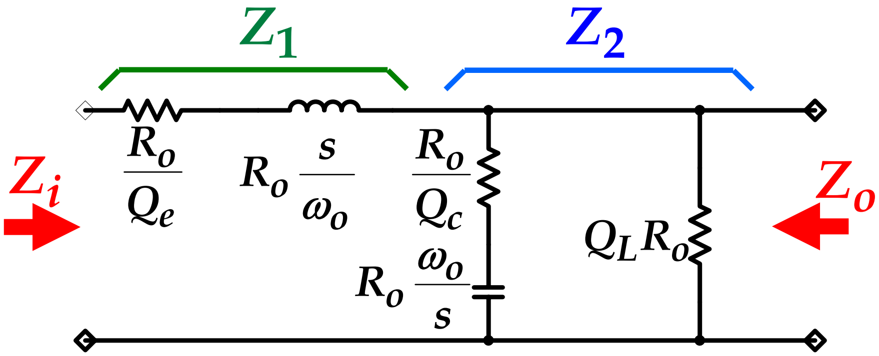
$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$



$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

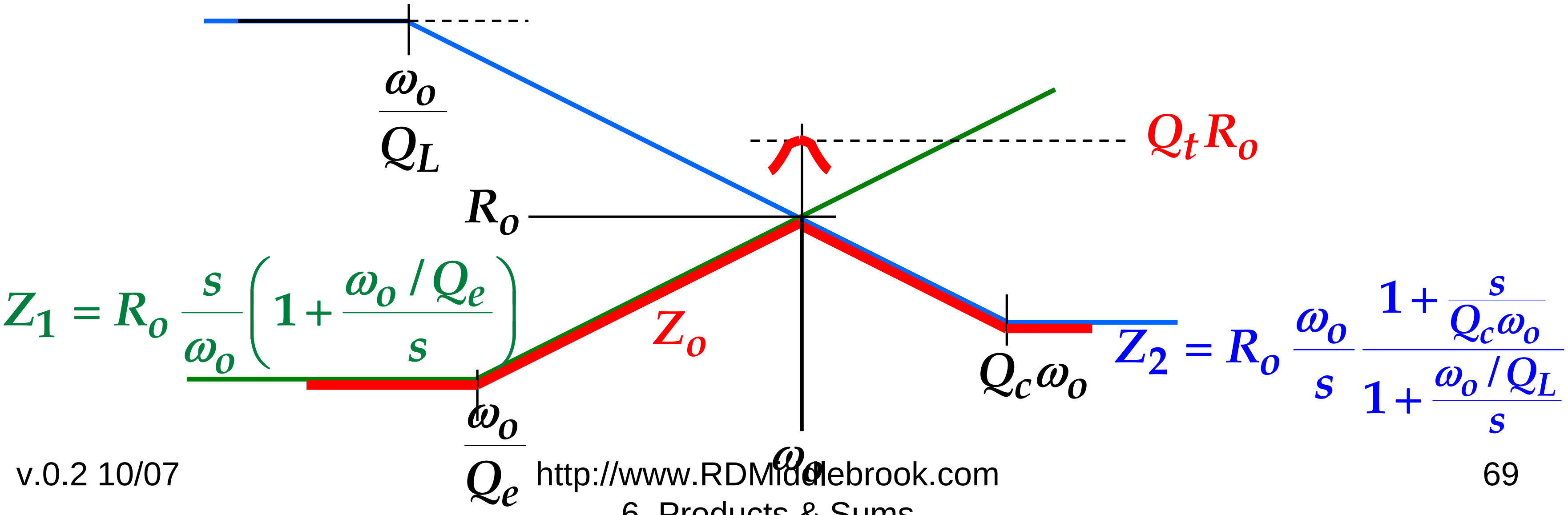
$$Z_2 = R_o \frac{\omega_o}{s} \frac{1 + \frac{s}{Q_c \omega_o}}{1 + \frac{\omega_o / Q_L}{s}}$$

For the triple-damped LC filter, draw the asymptotes for Z_1 and Z_2. Construct the asymptotes for the output impedance $Z_o = Z_1 \| Z_2$, and find the Q_t of the quadratic in s. Neglect second-order effects.



$$\omega_o \equiv \frac{1}{\sqrt{LC}} \qquad R_o \equiv \sqrt{\frac{L}{C}}$$

$$Q_e \equiv \frac{R_o}{R_e} \qquad Q_c \equiv \frac{R_o}{R_c} \qquad Q_L \equiv \frac{R_L}{R_o}$$



$$Z_1 = R_o \frac{s}{\omega_o}\left(1 + \frac{\omega_o / Q_e}{s}\right)$$

$$Z_2 = R_o \frac{\omega_o}{s} \frac{1 + \frac{s}{Q_c \omega_o}}{1 + \frac{\omega_o / Q_L}{s}}$$

"Doing the algebra on the graph" applies to any transfer functions, whether they be voltage gains, current gains, impedances, or admittances.

"Doing the algebra on the graph" applies to any transfer functions, whether they be voltage gains, current gains, impedances, or admittances.

In particular, the last example of impedances in parallel also applies to reciprocal sums of voltage gains or current gains, which will be valuable in the later applications of the Dissection Theorem.

7. THE I/O IT:

The Input/Output Impedance Theorem

How to find them directly from the Gain, thereby saving almost two-thirds of the work

Why do we need to deal with input and output impedances?

1. They may be part of the specifications.

2. They describe the interaction between two system blocks, and are therefore components of the Divide and Conquer approach, specifically incorporated in the Chain Theorem.

Definitions of "input" and "output:"

Input and output impedances are transfer functions (TFs), just as is the gain.

A TF is a ratio of one signal in a circuit to another, so the most general definition of "input" and "output" is that the "input" is the signal in the denominator, and the "output" is the signal in the numerator:

$$\frac{\text{"output"}}{\text{"input"}} = \text{ transfer function TF}$$

If numerator and denominator are both voltages or currents, the TF is a voltage gain or a current gain; if the numerator is a voltage and the denominator is a current, the TF is a transimpedance (and vice versa for a transadmittance).

If the numerator is the voltage across the same port into which the denominator current flows, the TF is a self-impedance.

The denominator of a TF ***is not necessarily*** **an independent excitation; the independent excitation may be elsewhere.**

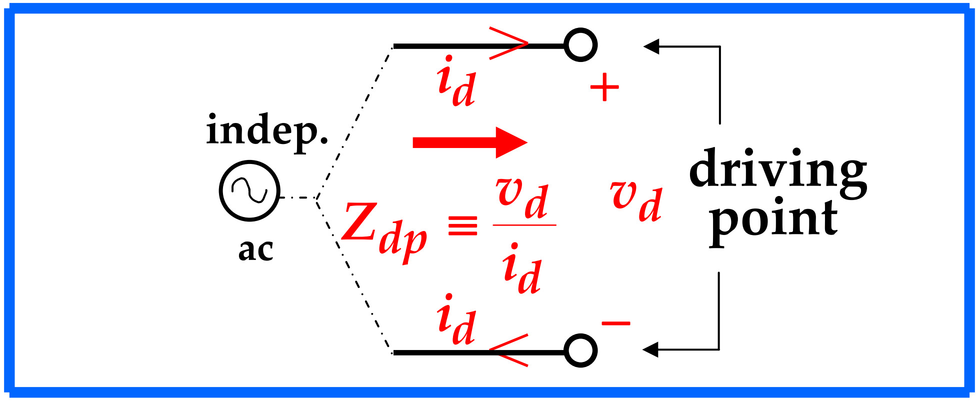
Thus, there are three kinds of "input":

1. **A signal at a port designated as "input"**
2. **An independent excitation**
3. **The denominator of a TF**

Driving Point Impedance

The port at which a circuit is driven is the driving point.

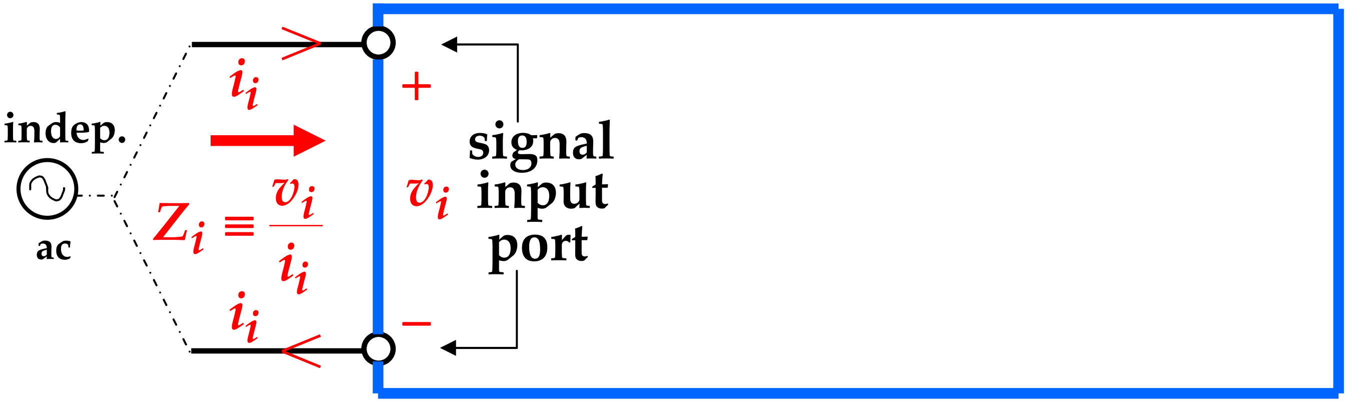
One of the many TFs of interest is the driving point impedance, which is the self-impedance "seen" at the driving point:



$$Z_{dp} \equiv \frac{v_d}{i_d}$$

A system usually has designated signal "input" and "output" ports:

Input Impedance

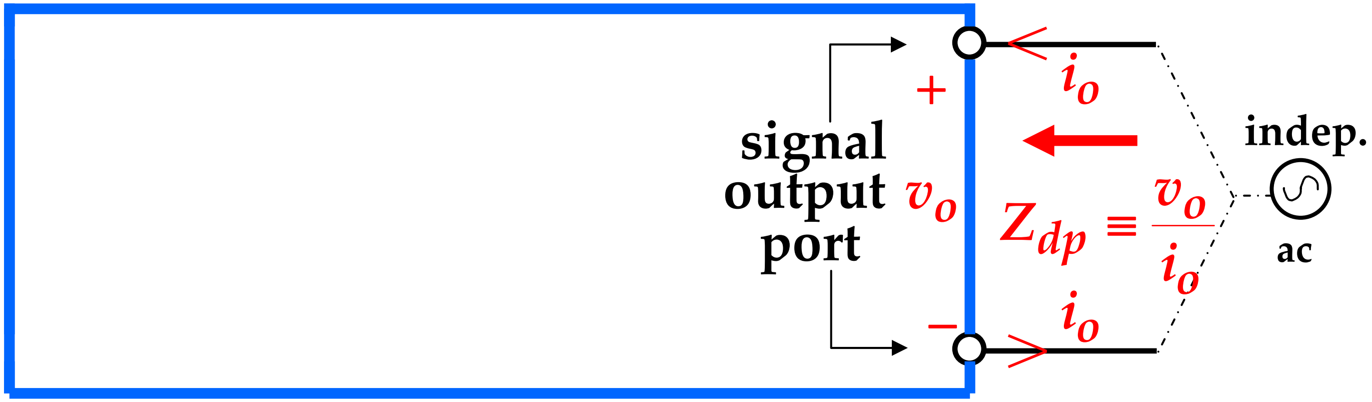


$$Z_{dp} \equiv \frac{v_i}{i_i} = Z_i \equiv \text{input impedance}$$

at the signal input port

Note: the "input" signal for the input impedance TF is i_i, although the input signal for the gain TF may be v_i or i_i, depending upon the definition of the gain.

Output Impedance



$$Z_{dp} \equiv \frac{v_o}{i_o} = Z_o \equiv \textbf{output impedance}$$

at the signal output port

Conventional Approach

Calculate the gain H

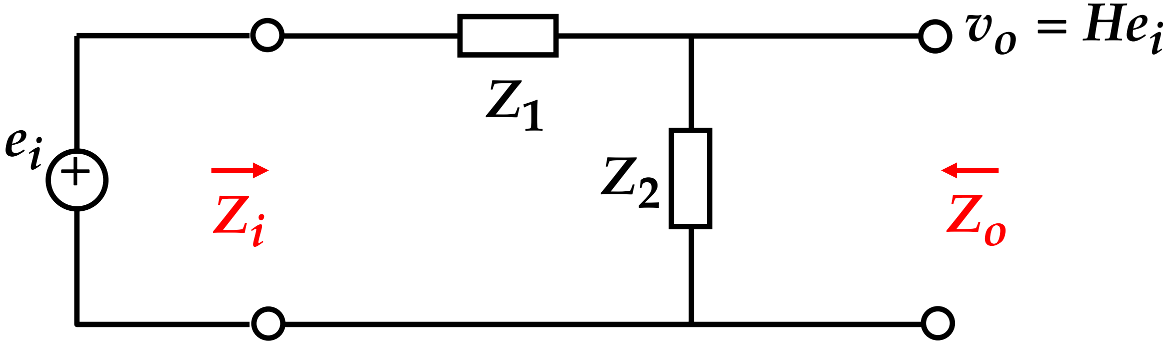
Calculate the output impedance Z_o

Calculate the input impedance Z_i

Usually these are done separately, each starting from scratch, and they may be equally lengthy analyses (especially if there is feedback present).

However, much of the analysis is the same in each case, so there is motivation to find a short cut that avoids the repetitions.

Consider the simple voltage divider:



The three analyses lead to:

$$H = \frac{Z_2}{Z_1 + Z_2} \qquad Z_i = Z_1 + Z_2 \qquad Z_o = Z_1 \| Z_2$$

The "hard part" in each case is calculation of $Z_1 + Z_2$.

However, Z_i and Z_o can be written in terms of H :

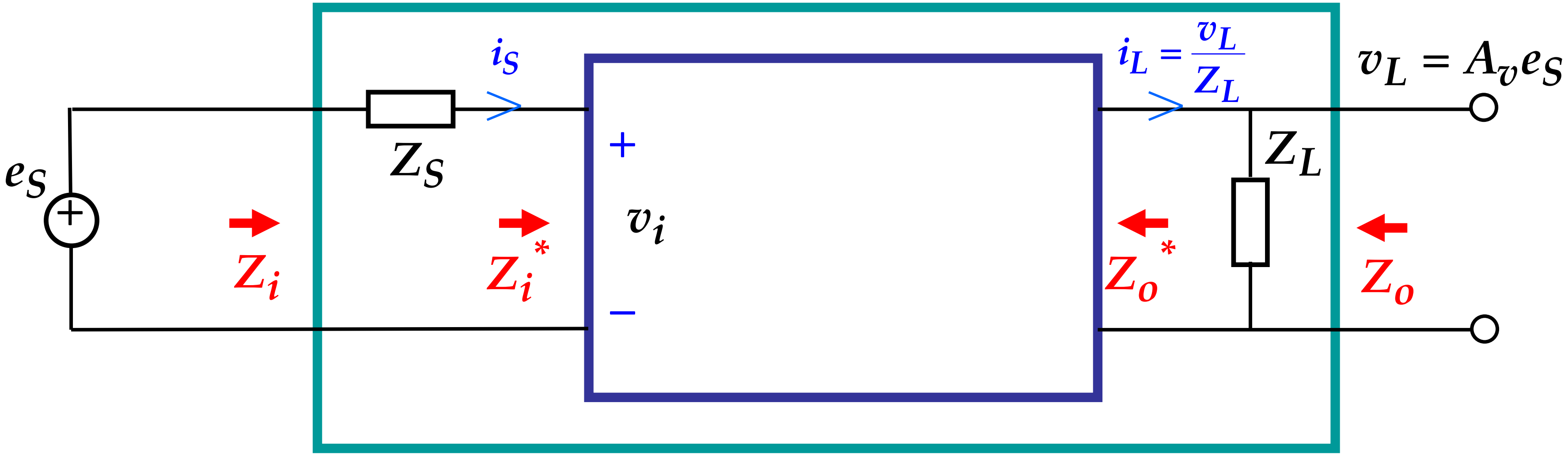
$$Z_i = \frac{Z_2}{H} \qquad Z_o = Z_1 H$$

Thus, the sum $Z_1 + Z_2$ need be calculated only once to find H, and then Z_i and $Z_o$can be found as products or quotients of H.

This trick doesn't work for more complex circuits, but there is still motivation to find a way to calculate Z_i and Z_o from H instead of starting from scratch.

Inner and Outer Input and Output Impedances

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$



There are two kinds of input and output impedances, depending on whether the system is defined to include the source and load impedances or not.

outer input impedance $\equiv Z_i$

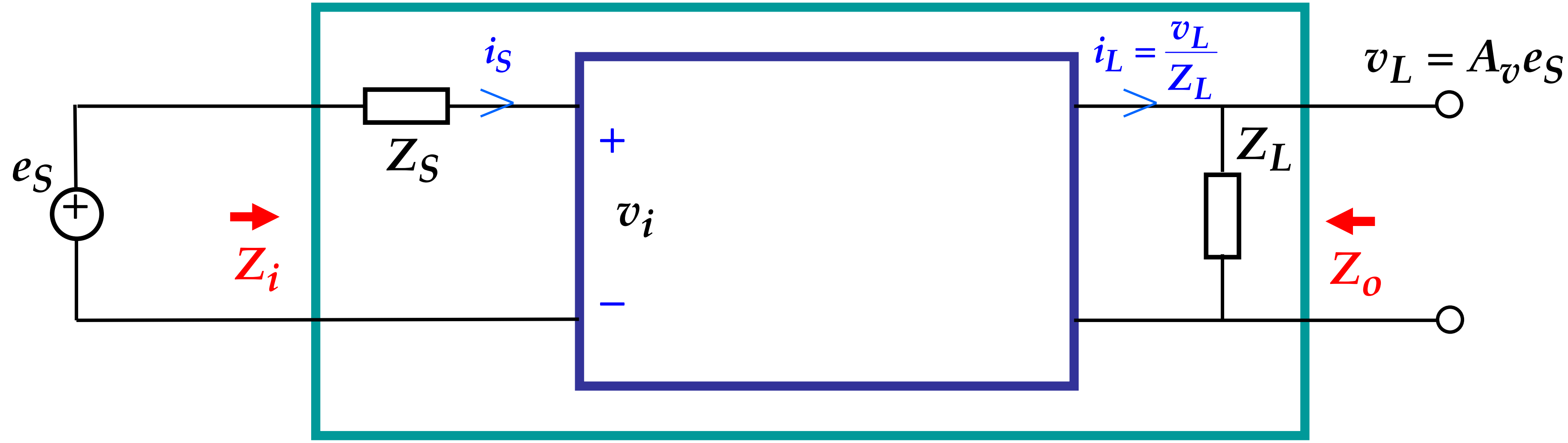
outer output impedance $\equiv Z_o$

inner input impedance $\equiv Z_i^*$

inner output impedance $\equiv Z_o^*$

Output Impedance Theorem

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$



Forward transadmittance gain $Y_t = \dfrac{i_L}{e_S} = \dfrac{v_L}{Z_L e_S} = \dfrac{A_v}{Z_L}$

Short-circuit forward transadmittance gain $Y_t^{sc} = \left.\dfrac{A_v}{Z_L}\right|_{Z_L \to 0}$

Output Impedance Theorem

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$

$$Z_o = \frac{\text{oc output voltage}}{\text{sc output current}} \quad \text{for the same } e_S \quad = \frac{A_v e_S}{Y_t^{sc} e_S}$$

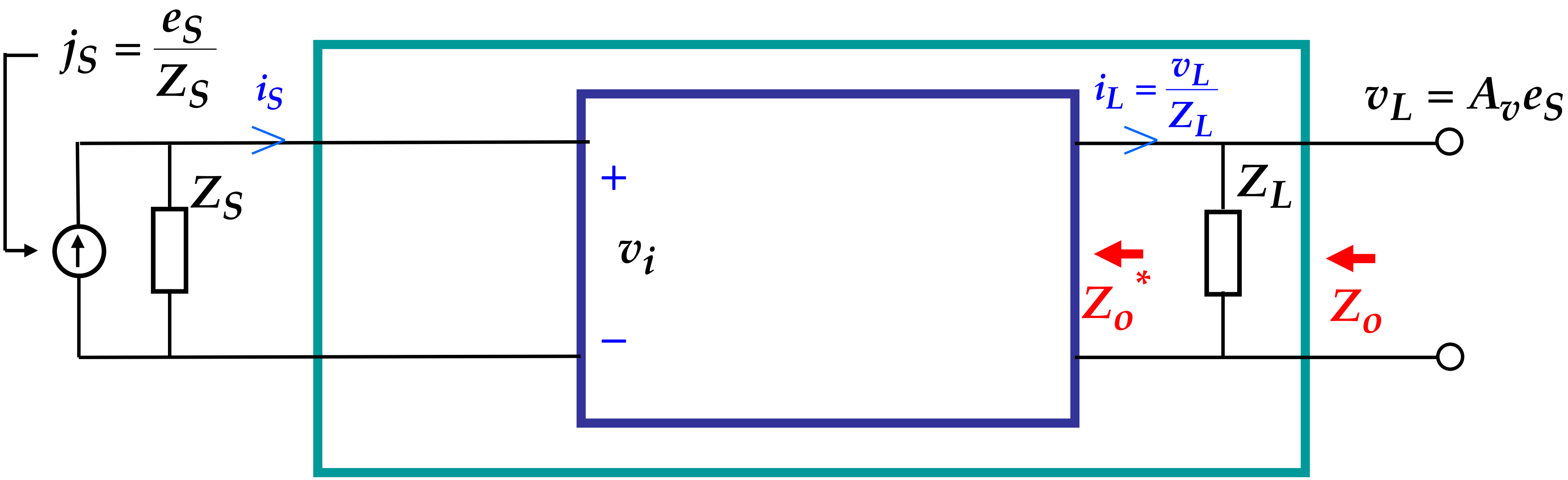
$$Z_o = \frac{A_v}{Y_t^{sc}} \quad = \frac{\text{fwd voltage gain}}{\text{sc fwd transadmittance}}$$

$$Z_o = \frac{A_v}{\left.\dfrac{A_v}{Z_L}\right|_{Z_L \to 0}}$$

This is the Output Impedance Theorem

Input Impedance Theorem

Convert the Thevenin independent source e_S, Z_S to a Norton equivalent:



Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{j_S\big|_{Z_S \to \infty}}$$

Input Impedance Theorem

Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{\left. j_S \right|_{Z_S \to \infty}} = \frac{v_L}{\left.\frac{e_S}{Z_S}\right|_{Z_S \to \infty}}$$

Input Impedance Theorem

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$

Forward transimpedance gain

$$Z_t = \frac{v_L}{i_S} = \frac{v_L}{j_S\big|_{Z_S \to \infty}} = \frac{v_L}{\dfrac{e_S}{Z_S}\Big|_{Z_S \to \infty}} = \frac{v_L}{\dfrac{v_L / A_v}{Z_S}\Big|_{Z_S \to \infty}} = Z_S A_v\big|_{Z_S \to \infty}$$

Input Impedance Theorem

Forward voltage gain $A_v = \dfrac{v_L}{e_S}$

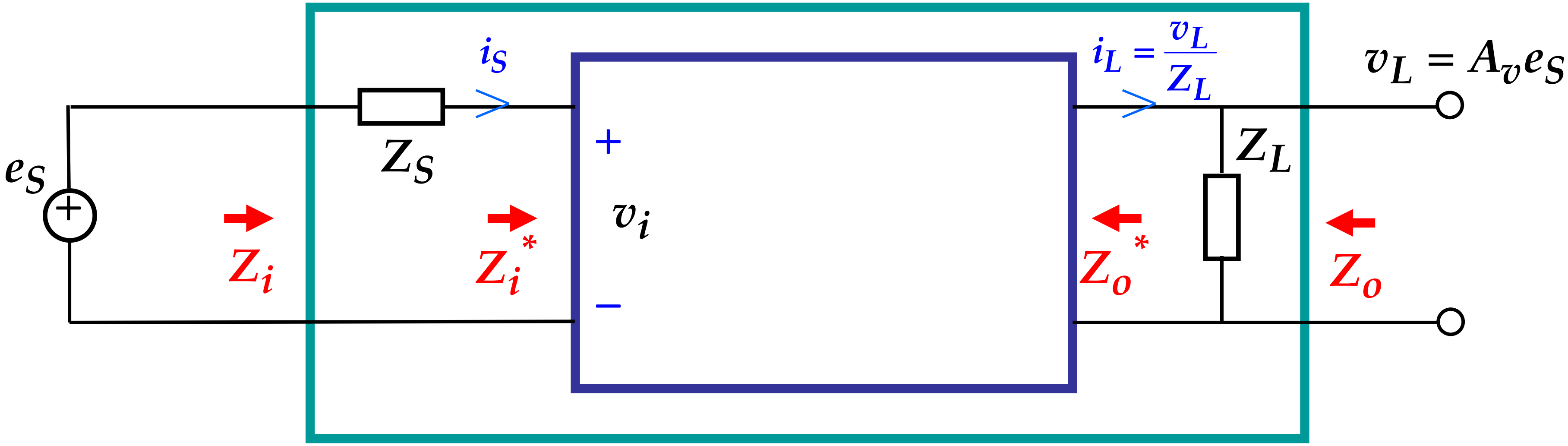
$$Z_i = \frac{\text{input voltage}}{\text{input current}} \quad \text{for the same } v_L \quad = \frac{v_L / A_v}{v_L / Z_t}$$

$$Z_i = \frac{Z_t}{A_v} = \frac{\text{fwd transimpedance}}{\text{fwd voltage gain}}$$

$$Z_i = \frac{Z_S A_v \big|_{Z_S \rightarrow \infty}}{A_v}$$

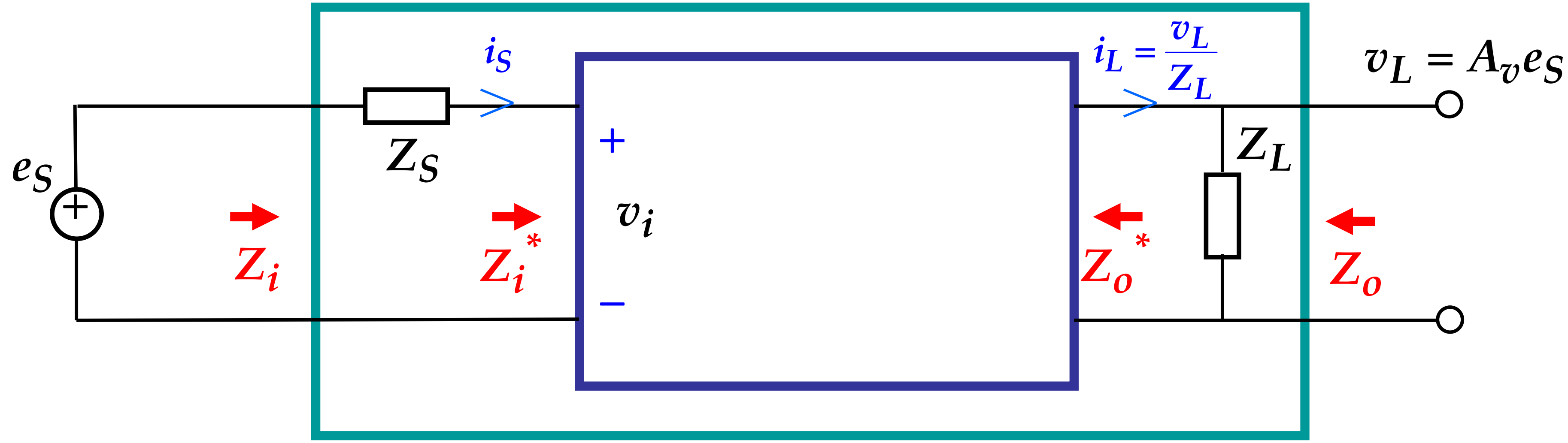
This is the Input Impedance Theorem

Inner and Outer Input and Output Impedances



The value of the formulas is that once the gain is known, only a simple limit with respect to either Z_L or Z_S need be calculated to find the outer output or input impedances Z_o or Z_i.

Inner Input and Output Impedances

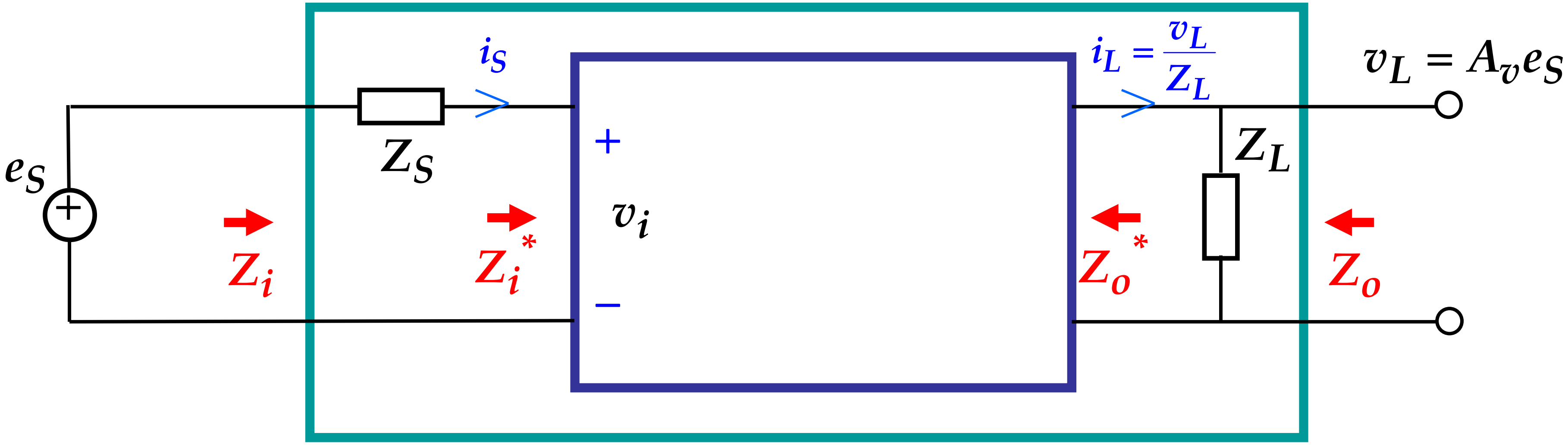


It is obvious that

$$Z_o^* = Z_o\Big|_{Z_L\to\infty} = \left.\frac{A_v}{\left.\frac{A_v}{Z_L}\right|_{Z_L\to 0}}\right|_{Z_L\to\infty} = \frac{A_v\big|_{Z_L\to\infty}}{\left.\frac{A_v}{Z_L}\right|_{Z_L\to 0}}$$

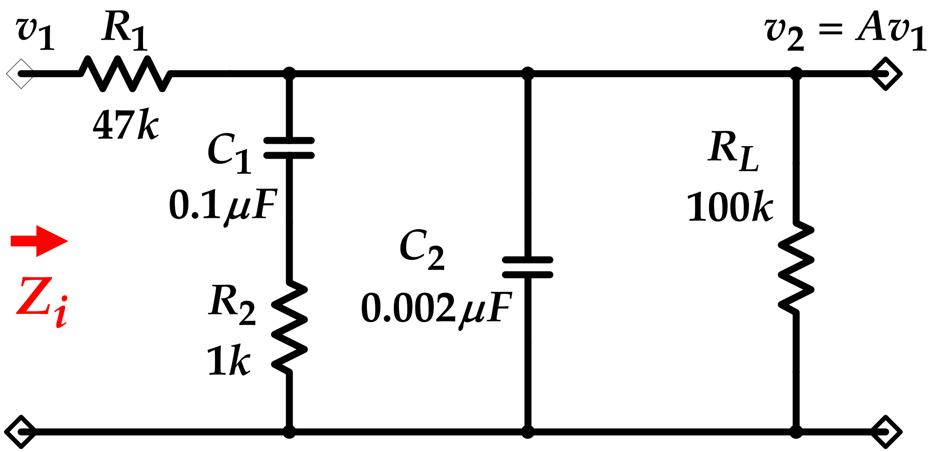
$$Z_i^* = Z_i\Big|_{Z_S\to 0} = \left.\frac{Z_S A_v\big|_{Z_S\to\infty}}{A_v}\right|_{Z_S\to 0} = \frac{Z_S A_v\big|_{Z_S\to\infty}}{A_v\big|_{Z_S\to 0}}$$

Inner Input and Output Impedances



Thus, all four input and output impedances can be found by taking simple limits upon the gain A_v.

This has been treated already. The result for the gain A is:



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1+R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2+R_1 \| R_L\right)}$$

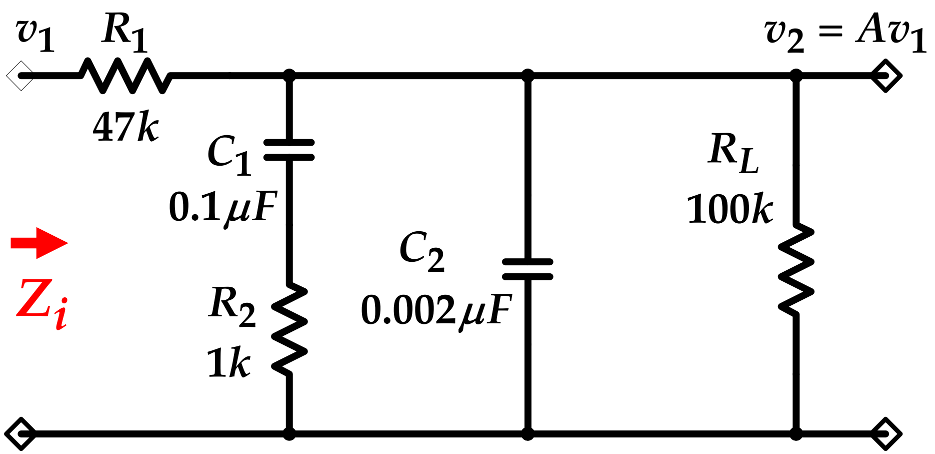
$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

The formula for the outer input impedance is

$$Z_i = \frac{Z_S A\big|_{Z_S \to \infty}}{A}$$

Since R_1 is a surrogate Z_S,

$$Z_i = \frac{R_1 A\big|_{R_1 \to \infty}}{A}$$



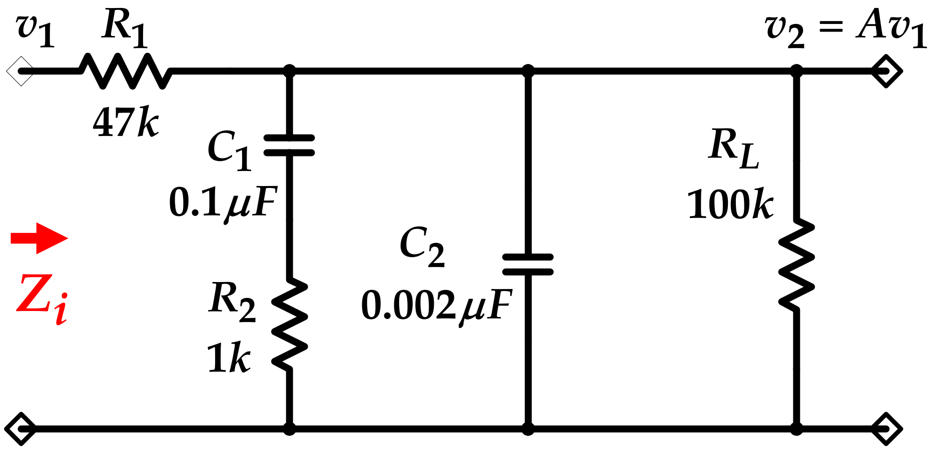
$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1+R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2+R_1\|R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1\|R_2\|R_L\right)}$$

The limit can be taken factor by factor:

$$Z_i = \frac{R_1A\big|_{R_1\to\infty}}{A} = \frac{R_1A_0\big|_{R_1\to\infty}}{A_0}\frac{\left(1+\frac{s}{\omega_z}\right)\Big|_{R_1\to\infty}}{\left(1+\frac{s}{\omega_z}\right)}\frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1}\right)\Big|_{R_1\to\infty}}\frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2}\right)\Big|_{R_1\to\infty}}$$



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

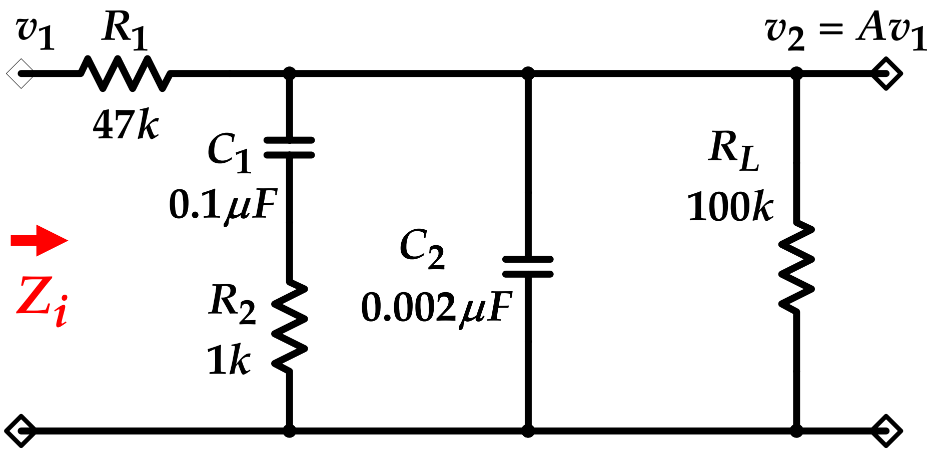
$$A_0 \equiv \frac{R_L}{R_1+R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2+R_1\|R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1\|R_2\|R_L\right)}$$

The limit can be taken factor by factor:

$$Z_i = \frac{R_1 A\big|_{R_1\to\infty}}{A} = \frac{R_1 A_0\big|_{R_1\to\infty}}{A_0} \frac{\left(1+\frac{s}{\omega_z}\right)\Big|_{R_1\to\infty}}{\left(1+\frac{s}{\omega_z}\right)} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1}\right)\Big|_{R_1\to\infty}} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2}\right)\Big|_{R_1\to\infty}}$$

A huge simplification emerges: any factor in A that does not contain R_1 is unaffected by the limit and therefore cancels.



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

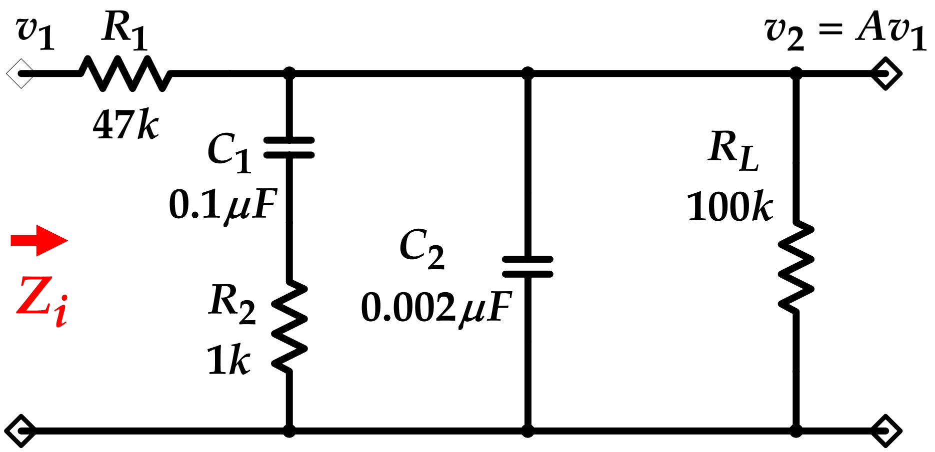
$$A_0 \equiv \frac{R_L}{R_1+R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2+R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

The limit can be taken factor by factor:

$$Z_i = \frac{R_1 A\big|_{R_1\to\infty}}{A} = \frac{R_1 A_0\big|_{R_1\to\infty}}{A_0} \frac{\cancel{\left(1+\frac{s}{\omega_z}\right)}\Big|_{R_1\to\infty}}{\cancel{\left(1+\frac{s}{\omega_z}\right)}} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1}\right)\Big|_{R_1\to\infty}} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2}\right)\Big|_{R_1\to\infty}}$$

A huge simplification emerges: any factor in A that does not contain R_1 is unaffected by the limit and therefore cancels.



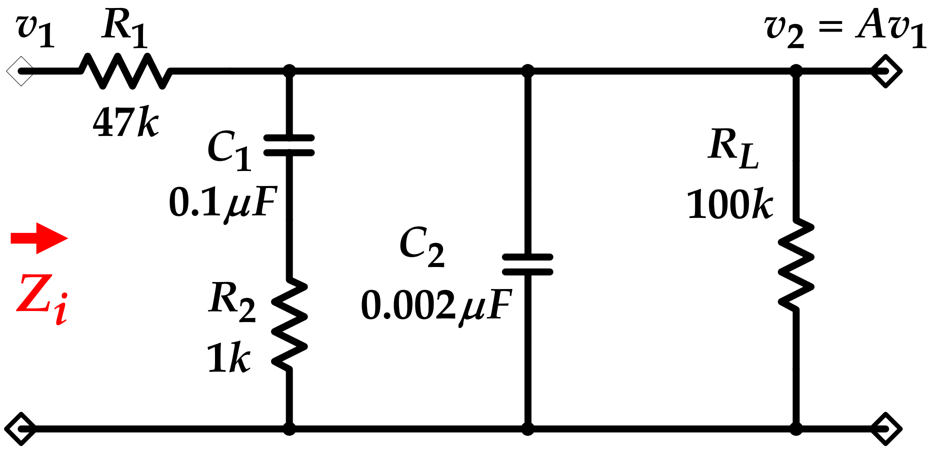
$$A = A_0 \frac{1 + \frac{s}{\omega_z}}{\left(1 + \frac{s}{\omega_1}\right)\left(1 + \frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1 + R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

The limit can be taken factor by factor:

$$Z_i = \frac{R_1 A\big|_{R_1 \to \infty}}{A} = \frac{R_1 A_0\big|_{R_1 \to \infty}}{A_0} \frac{\left(1 + \frac{s}{\omega_1}\right)}{\left(1 + \frac{s}{\omega_1}\right)\Big|_{R_1 \to \infty}} \frac{\left(1 + \frac{s}{\omega_2}\right)}{\left(1 + \frac{s}{\omega_2}\right)\Big|_{R_1 \to \infty}}$$



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1 + R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

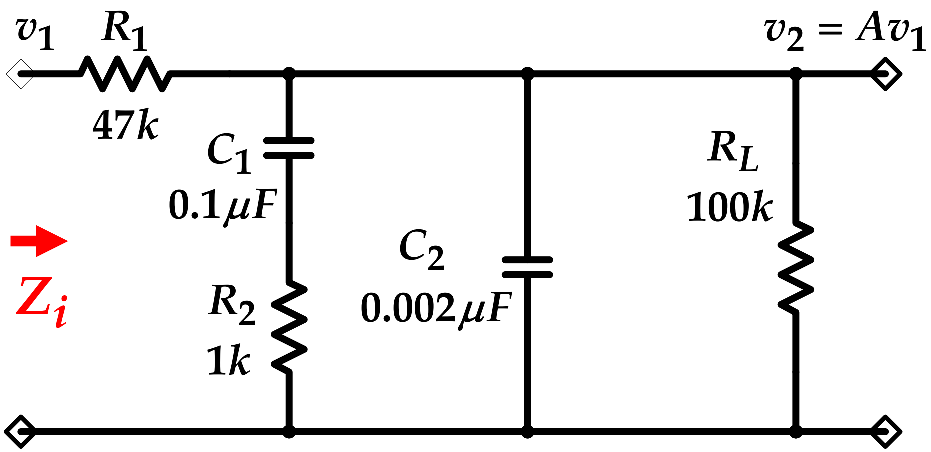
$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

The limit can be taken factor by factor:

$$Z_i = \frac{R_1 A\big|_{R_1\to\infty}}{A} = \frac{R_1 A_0\big|_{R_1\to\infty}}{A_0} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1}\right)\Big|_{R_1\to\infty}} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2}\right)\Big|_{R_1\to\infty}}$$

You can take advantage of this knowledge in advance, by highlighting R_1 in the parameter definitions and omitting these factors when substituting into the formula.

Rewrite the definitions highlighting R_1 :



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1 + R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

$$Z_i = R_{i0} \frac{\left(1+\frac{s}{\omega_1}\right)}{\left(1+\frac{s}{\omega_1\big|_{R_1\to\infty}}\right)} \frac{\left(1+\frac{s}{\omega_2}\right)}{\left(1+\frac{s}{\omega_2\big|_{R_1\to\infty}}\right)}$$

where

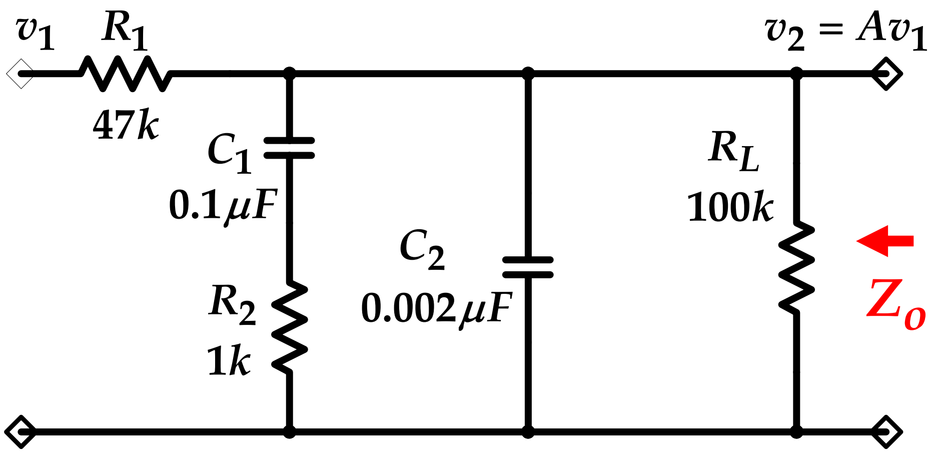
$$R_{i0} \equiv \frac{R_1 \left.\frac{1}{R_1+R_L}\right|_{R_1\to\infty}}{\frac{1}{R_1+R_L}} = R_1 + R_L$$

$$\omega_1\big|_{R_1\to\infty} \equiv \left.\frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}\right|_{R_1\to\infty} = \frac{1}{C_1\left(R_2 + R_L\right)} < \omega_1$$

$$\omega_2\big|_{R_1\to\infty} \equiv \left.\frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}\right|_{R_1\to\infty} = \frac{1}{C_2\left(R_2 \| R_L\right)} < \omega_2$$

Exercise 7.1

Find the outer output impedance Z_o



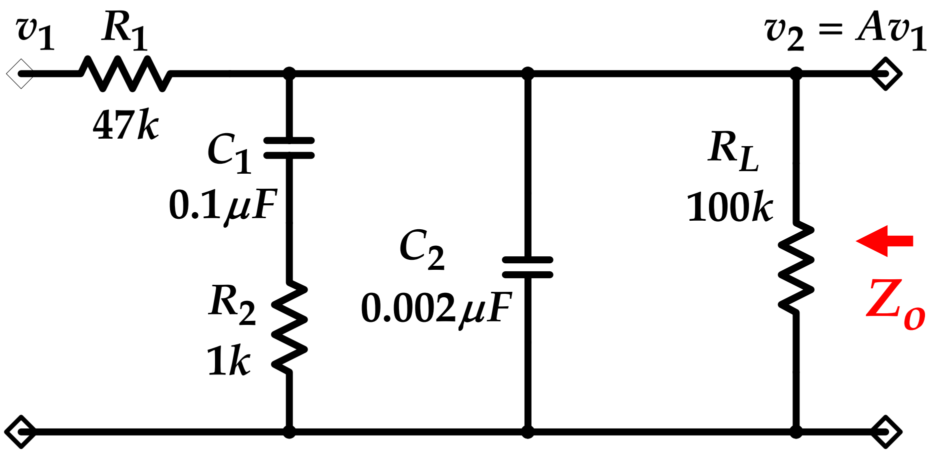
$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1+R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2+R_1\|R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1\|R_2\|R_L\right)}$$

Exercise 7.1 - Solution

Rewrite the definitions highlighting R_L :



$$A = A_0 \frac{1+\frac{s}{\omega_z}}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$A_0 \equiv \frac{R_L}{R_1 + R_L} \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

$$Z_o = \frac{A}{\left.\frac{A}{R_L}\right|_{R_L \to 0}} = R_{o0} \frac{\left(1+\frac{s}{\left.\omega_1\right|_{R_L\to 0}}\right)\left(1+\frac{s}{\left.\omega_2\right|_{R_L\to 0}}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

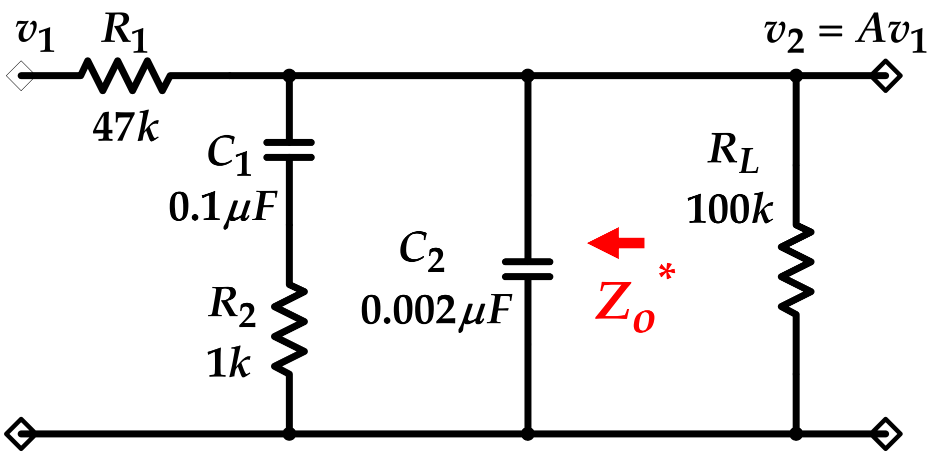
where

$$R_{o0} \equiv \frac{A_0}{\left.\frac{A_0}{R_L}\right|_{R_L\to 0}} = \frac{\frac{R_L}{R_1+R_L}}{\left.\frac{1}{R_L}\frac{R_L}{R_1+R_L}\right|_{R_L\to 0}} = R_1 \| R_L$$

$$\left.\omega_1\right|_{R_L\to 0} = \frac{1}{C_1 R_2} = \omega_z \qquad \left.\omega_2\right|_{R_L\to 0} = \infty$$

Exercise 7.2

Find the inner output impedance Z_o^*

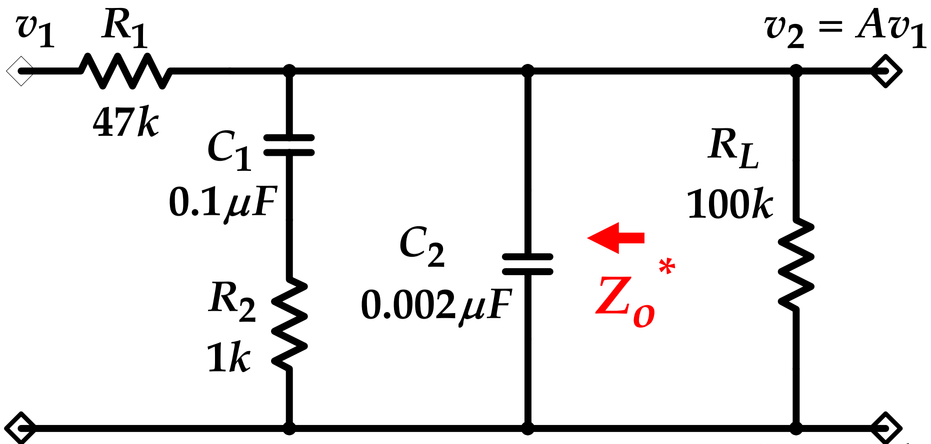


$$Z_o = R_{o0} \frac{\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$R_{o0} \equiv R_1 \| R_L \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

Exercise 7.2 - Solution



$$Z_o = R_{o0}\frac{\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_1}\right)\left(1+\frac{s}{\omega_2}\right)}$$

$$R_{o0} \equiv R_1 \| R_L \qquad \omega_1 \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)}$$

$$\omega_z \equiv \frac{1}{C_1 R_2} \qquad \omega_2 \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)}$$

$$Z_o^* = Z_o\Big|_{R_L\to\infty} = R_{o0}^* \frac{\left(1+\frac{s}{\omega_z}\right)}{\left(1+\frac{s}{\omega_1\big|_{R_L\to\infty}}\right)\left(1+\frac{s}{\omega_2\big|_{R_L\to\infty}}\right)}$$

where

$$R_{o0}^* \equiv R_{o0}\Big|_{R_L\to\infty} = R_1 \| R_L\Big|_{R_L\to\infty} = R_1$$

$$\omega_1\Big|_{R_L\to\infty} \equiv \frac{1}{C_1\left(R_2 + R_1 \| R_L\right)\Big|_{R_L\to\infty}} = \frac{1}{C_1\left(R_2 + R_1\right)} < \omega_1$$

$$\omega_2\Big|_{R_L\to\infty} \equiv \frac{1}{C_2\left(R_1 \| R_2 \| R_L\right)\Big|_{R_L\to\infty}} = \frac{1}{C_2\left(R_1 \| R_2\right)} < \omega_2$$

Bottom Line

The Input/Output Impedance Theorem allows you to find the input and output impedances of a circuit by taking simple limits upon the already known gain.

This saves almost two-thirds of the work required to obtain the three results separately.

Taking one limit upon the gain gives the outer impedances;

Taking two limits upon the gain gives the inner impedances.

A huge simplification occurs by anticipating that factors in the gain that do not contain the source or load impedance, do not appear in the result.

8. NDI AND THE EET:

Null Double Injection and the Extra Element Theorem

How to find the contribution of a particular element to the transfer function

Null Double Injection (ndi)

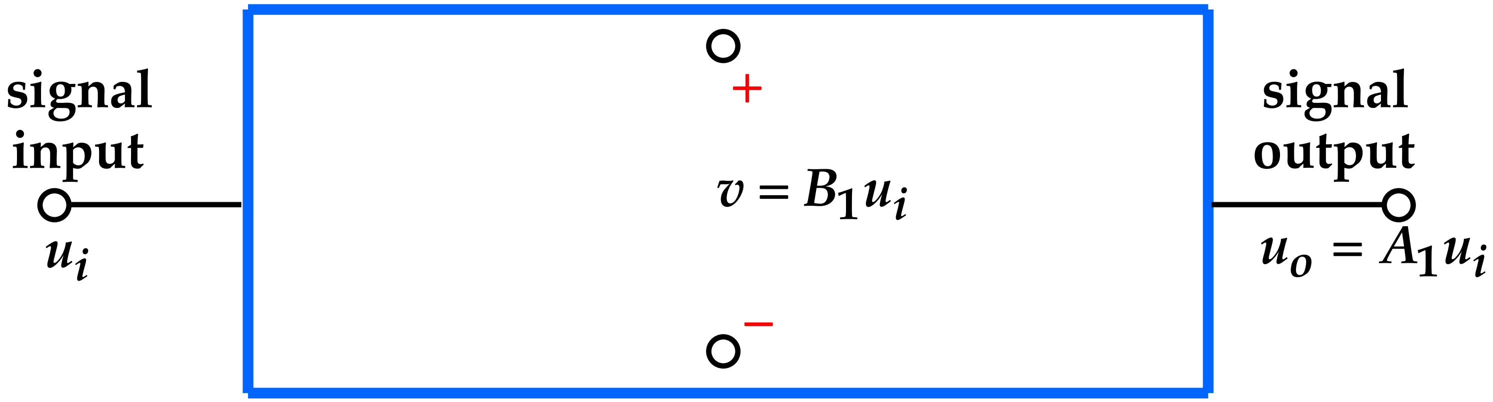
Usually, a transfer function (TF) is calculated as a response to a single independent excitation.

However, large analysis benefits accrue when certain constraints are imposed on several excitations present simultaneously.

The input is an independent signal, the output is a dependent signal.

The gain is A_1.

Consider a second dependent signal, a voltage v at some internal port :



signal input

u_i

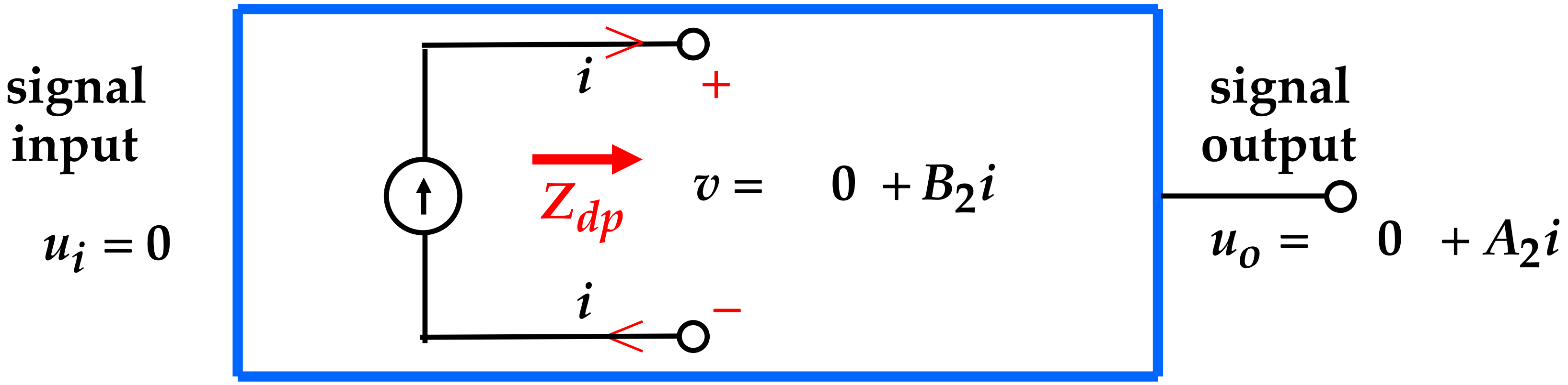
$v = B_1 u_i$

signal output

$u_o = A_1 u_i$

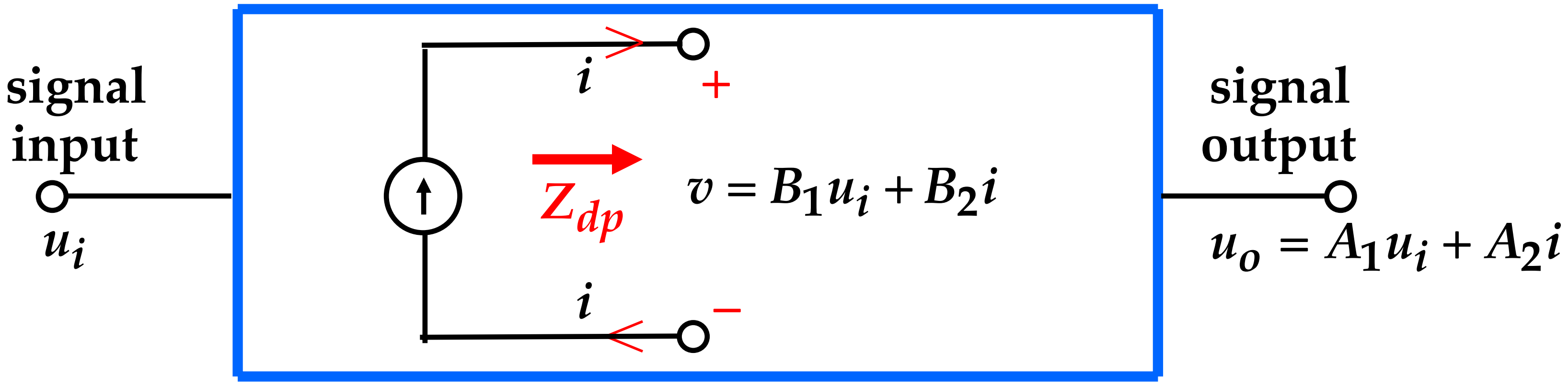
The gain from u_i to v is B_1.

Apply a second independent signal, a current i at the same internal port :



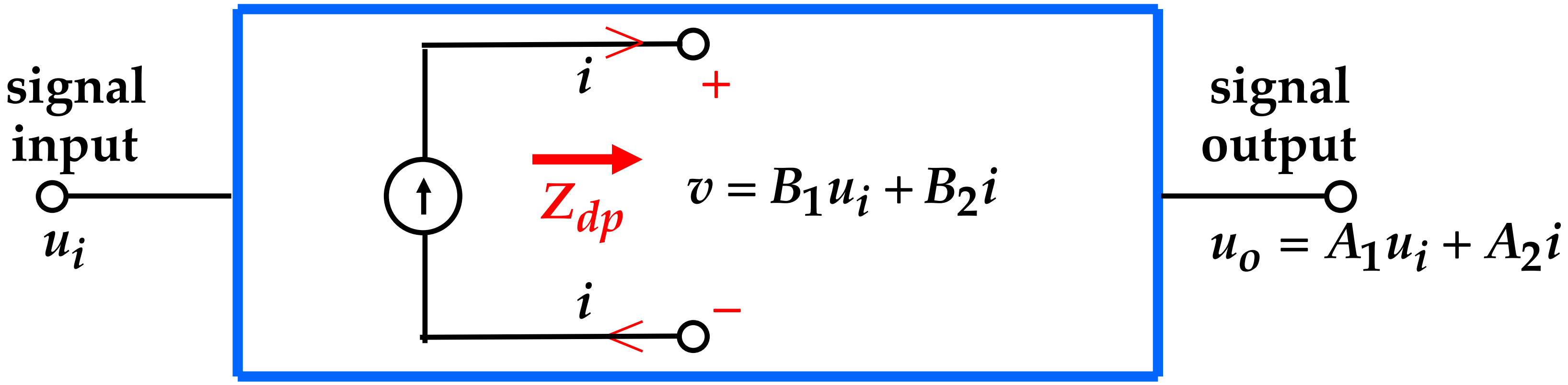
The "gain" from i to v is a driving point impedance B_2.

Apply both independent signals simultaneously:



For a linear system model, the two dependent signals are the superposition of the values they would have for each independent signal separately.

Apply both independent signals simultaneously:



$$v = B_1 u_i + B_2 i$$

$$u_o = A_1 u_i + A_2 i$$

For a linear system model, the two dependent signals are the superposition of the values they would have for each independent signal separately.

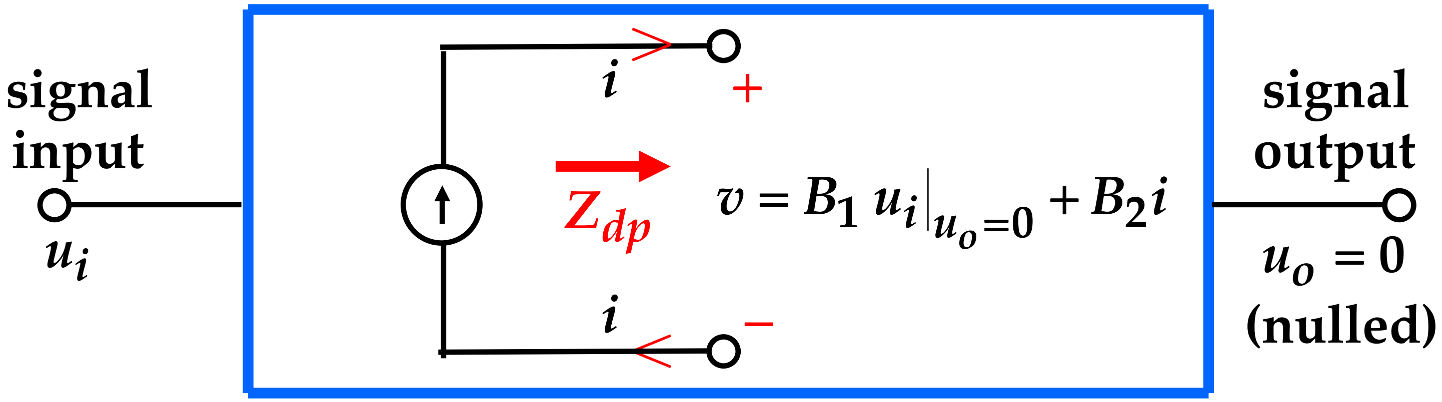
By adjustment of u_i and i, u_o can be made to have any value we like.

In particular:

u_o can be made zero by adjustment of the relative values of u_i and i, namely

$$u_i\big|_{u_o=0} = -\frac{A_2}{A_1} i$$

There is now a null double injection (ndi) condition:



The voltage at the internal port is

$$v = \left(B_2 - B_1 \frac{A_2}{A_1}\right) i$$

so the driving point impedance is

$$Z_{dp}\Big|_{u_o=0} = \left(B_2 - B_1 \frac{A_2}{A_1}\right)$$

Recap:

The driving point impedance (dpi) Z_{dp} at the internal port can have two different values, one when the input is zero, and another when the input is not zero, but is adjusted to null the output:

$$Z_{dp}\Big|_{u_i=0} = \underset{\uparrow}{B_2}$$

from i

$$Z_{dp}\Big|_{u_o=0} = \overset{\downarrow}{B_2} - B_1 \frac{A_2}{\underset{\uparrow}{A_1}}$$

from u_i

Recap:

The driving point impedance (dpi) Z_{dp} at the internal port can have two different values, one when the input is zero, and another when the input is not zero, but is adjusted to null the output:

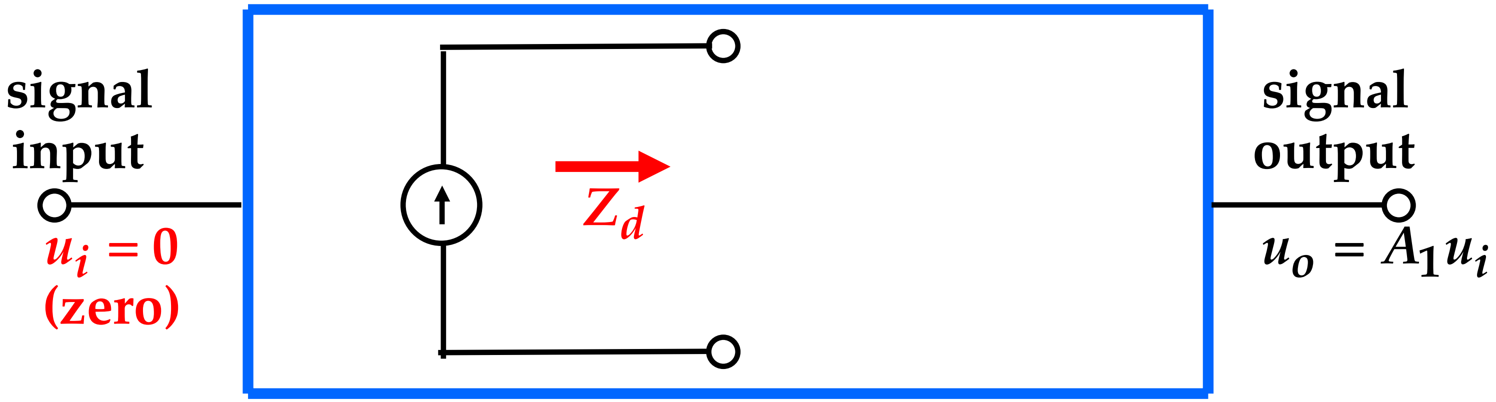
$$Z_{dp}\Big|_{u_i=0} = B_2 \qquad \equiv Z_d$$

from i

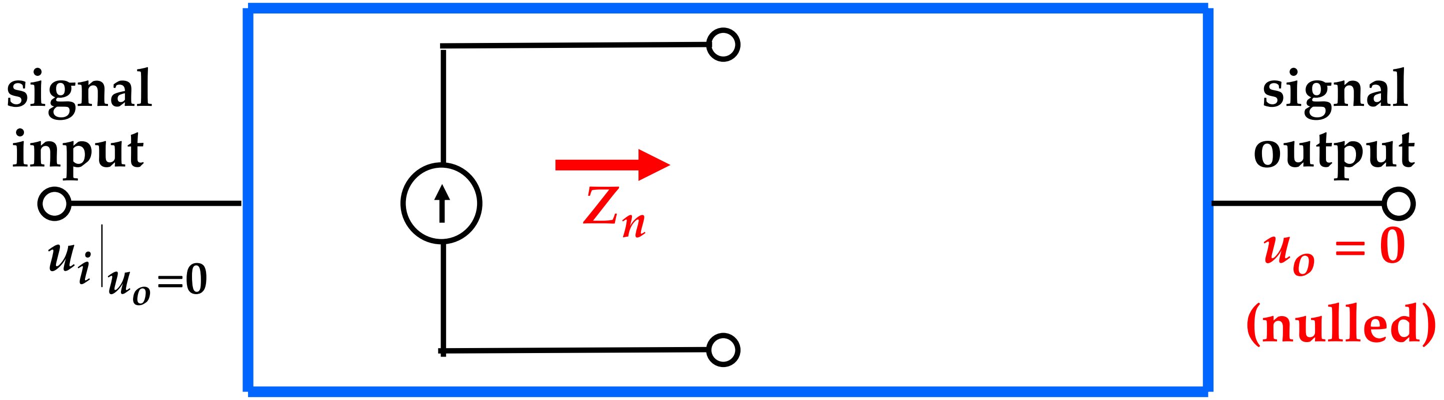
$$Z_{dp}\Big|_{u_o=0} = B_2 - B_1\frac{A_2}{A_1} \equiv Z_n$$

from u_i

The dpi Z_d is calculated under single injection (si) conditions:

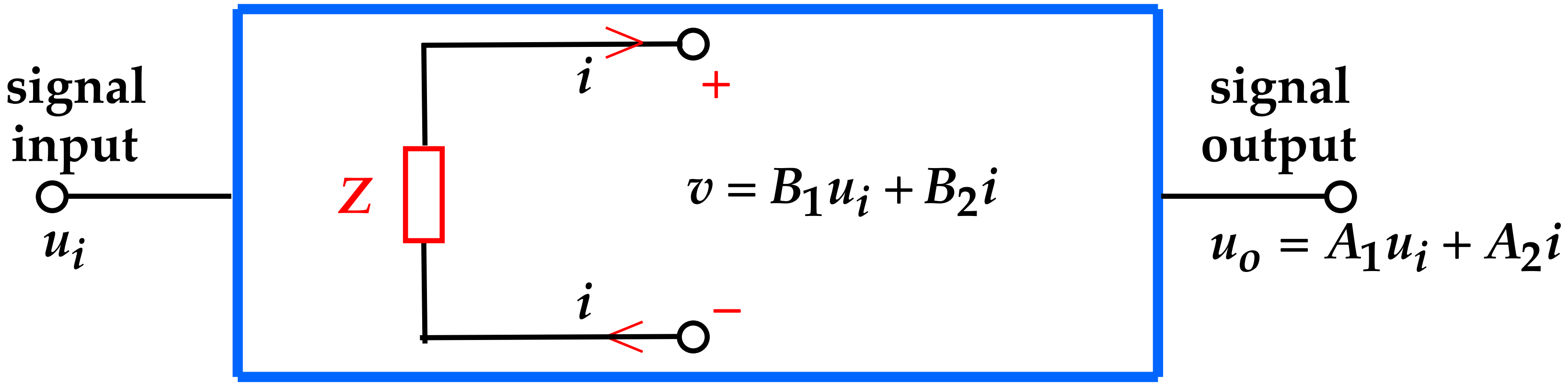


The dpi Z_n is calculated under null double injection (ndi) conditions:



The Extra Element Theorem (EET)

Replace the current source by an impedance Z :



The same linear superposition equations still apply to the rest of the circuit.

However, a relation between v and i is now enforced by Z, namely

$$v = -Zi$$

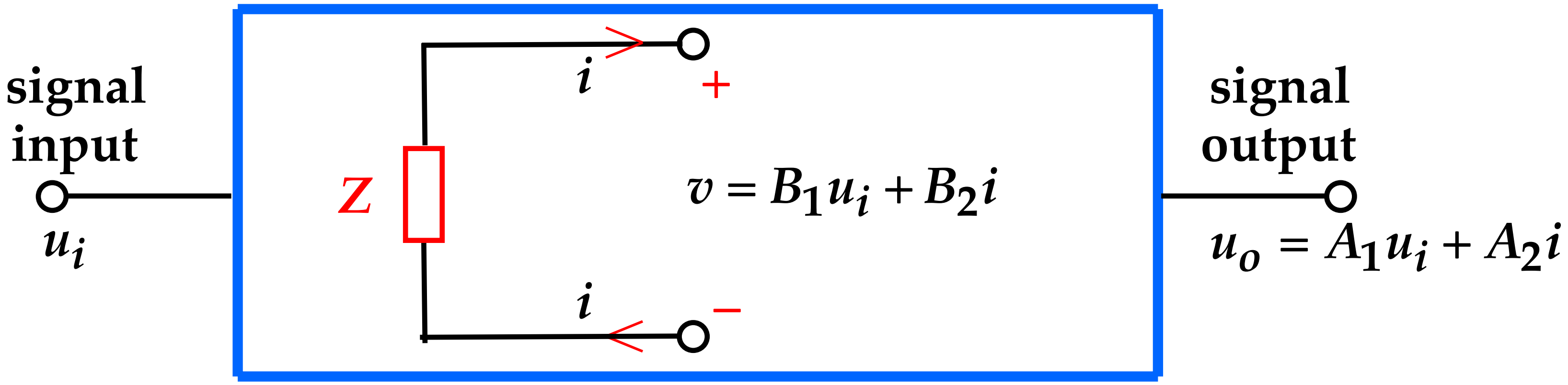
Substitute for v to find i in terms of u_i :

$$v = B_1 u_i + B_2 i = -Zi$$

so

$$i = -\frac{B_1}{B_2 + Z} u_i$$

The Extra Element Theorem (EET)



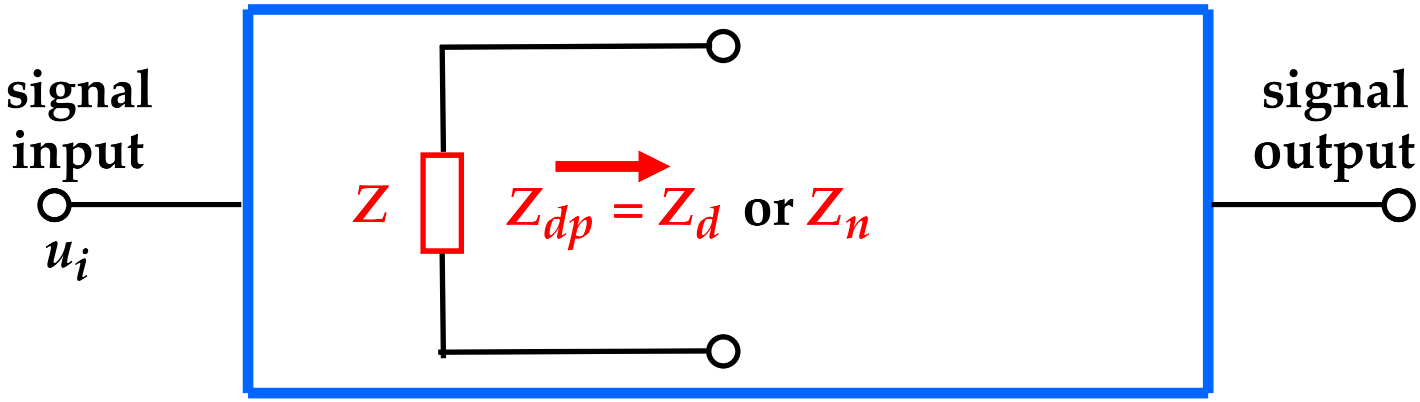
$$i = -\frac{B_1}{B_2 + Z} u_i$$

Now substitute for i in the equation for u_o to find u_o in terms of u_i :

$$u_o = A_1 u_i + A_2 i = A_1 u_i - A_2 \frac{B_1}{B_2 + Z} u_i$$

$$= A_1 \left(\frac{B_2 - B_1 \frac{A_2}{A_1} + Z}{B_2 + Z} \right) u_i = A_1 \left(\frac{1 + \frac{B_2 - B_1 \frac{A_2}{A_1}}{Z}}{1 + \frac{B_2}{Z}} \right) u_i$$

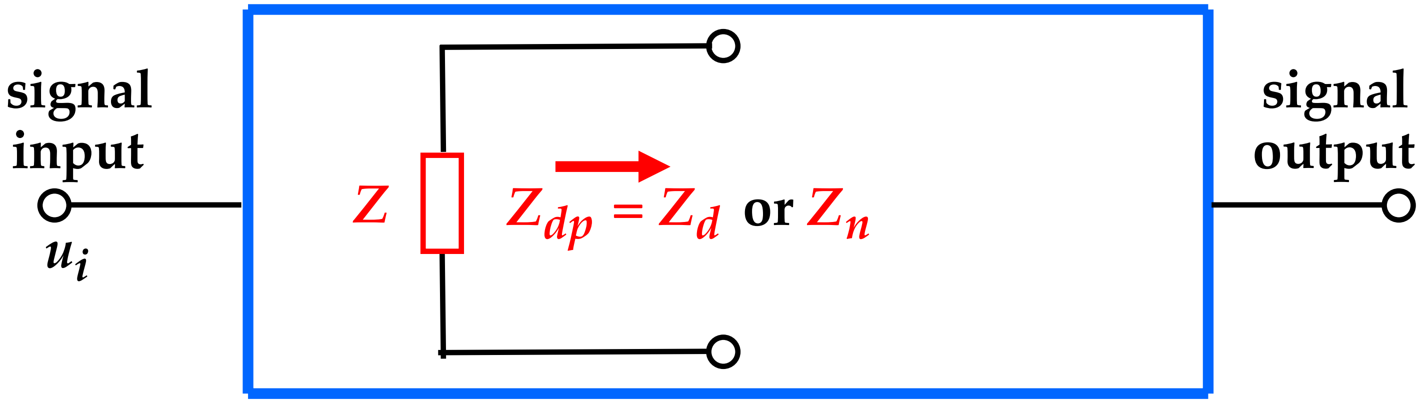
The Extra Element Theorem (EET)



The two combinations of the linear circuit parameters are precisely what have just been defined as Z_d and Z_n, so

$$u_o = A_1 \left(\frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \right) u_i$$

The Extra Element Theorem (EET)



signal input

u_i

Z

$Z_{dp} = Z_d$ or Z_n

signal output

The two combinations of the linear circuit parameters are precisely what have just been defined as Z_d and Z_n, so

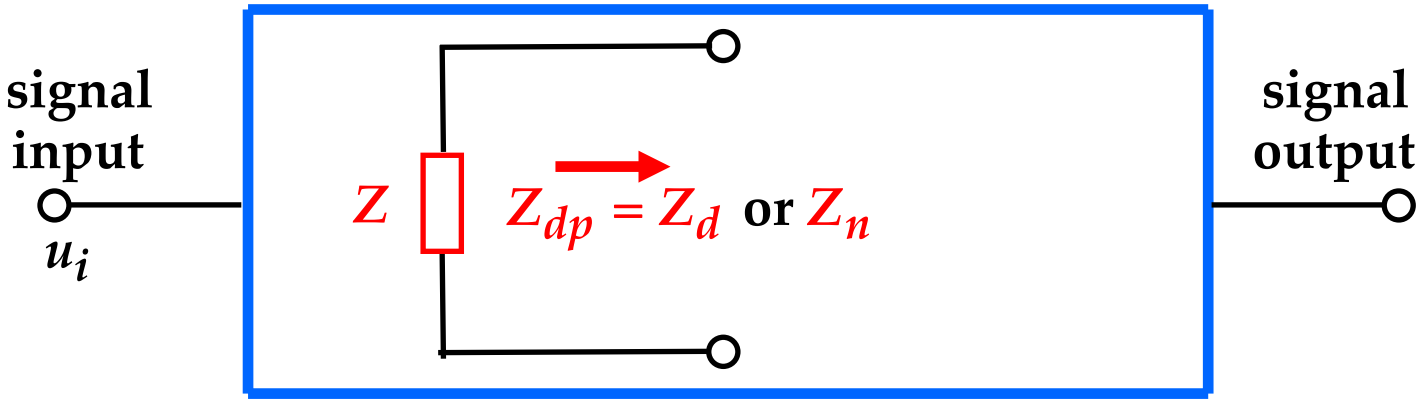
$$u_o = A_1 \left(\frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \right) u_i$$

However,

$\frac{u_o}{u_i}$ = gain in presence of Z

A_1 = gain when $Z = \infty$

The Extra Element Theorem (EET)



The two combinations of the linear circuit parameters are precisely what have just been defined as Z_d and Z_n, so

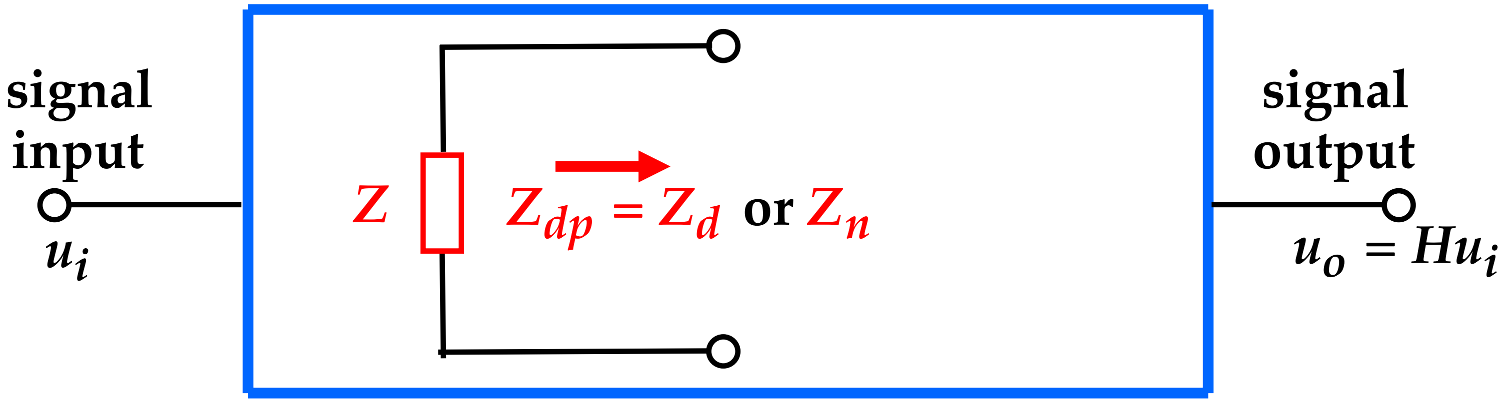
$$u_o = A_1 \left(\frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} \right) u_i$$

However,

$$\frac{u_o}{u_i} = \text{gain in presence of } Z \equiv H$$

$$A_1 = \text{gain when } Z = \infty \equiv H_\infty$$

The Extra Element Theorem (EET)



signal input

u_i

Z

$Z_{dp} = Z_d$ or Z_n

signal output

$u_o = Hu_i$

Hence, the Extra Element Theorem (EET) is:

$$H = H_\infty \left(\frac{1 + \dfrac{Z_n}{Z}}{1 + \dfrac{Z_d}{Z}} \right)$$

$H =$ **gain in presence of** Z $\qquad$ $H_\infty =$ **gain when** $Z = \infty$

$$Z_d = Z_{dp}\Big|_{u_i = 0} \qquad Z_n = Z_{dp}\Big|_{u_o = 0}$$

u_i = zero $\qquad$ **$\mathrm{u_i}$ ≠ zero, u_o nulled**

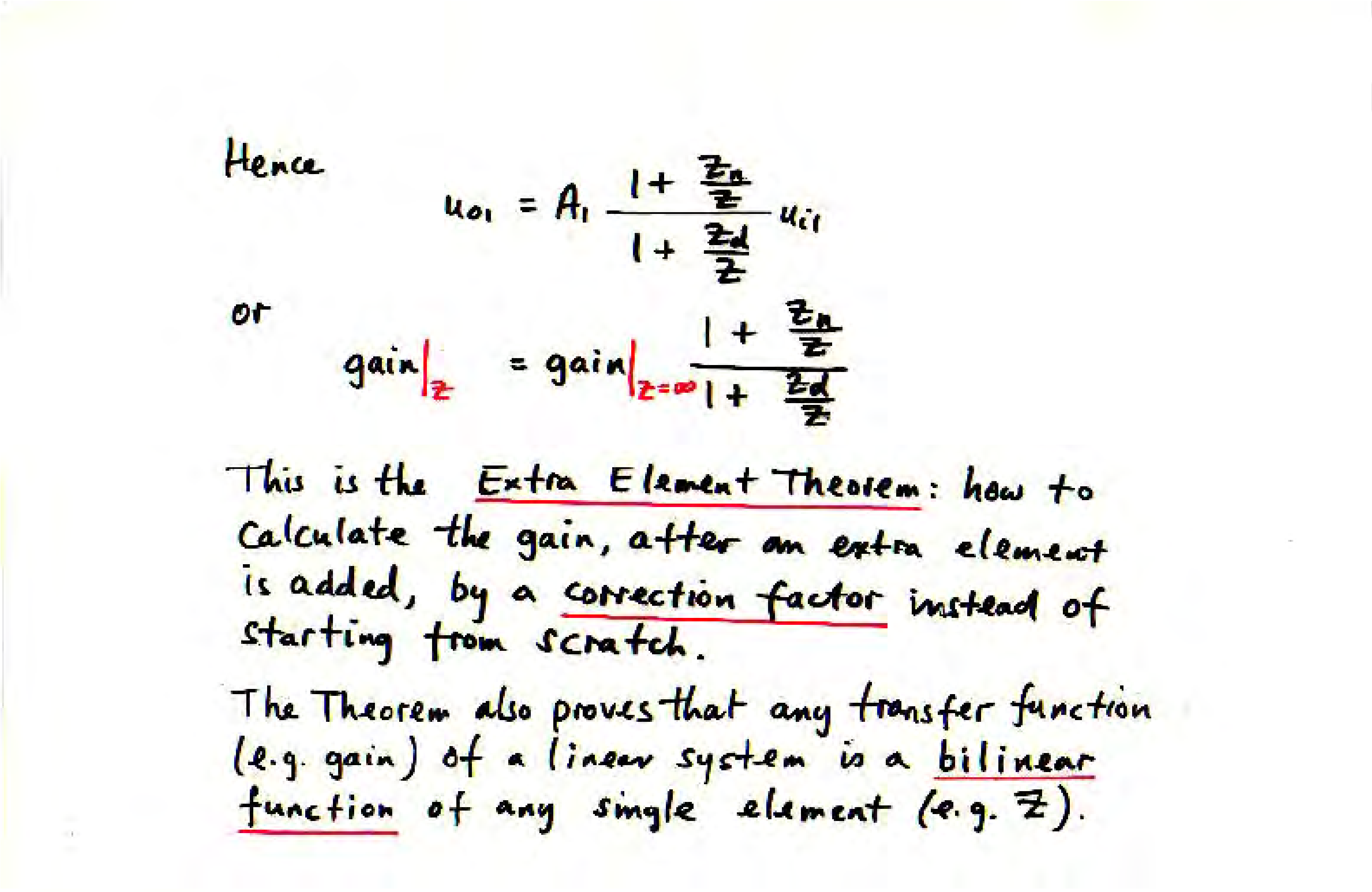
Hence

$$u_{o1} = A_1 \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} u_{i1}$$

or

$$\text{gain}\Big|_{Z} = \text{gain}\Big|_{Z=\infty} \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}}$$

This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.



Hence

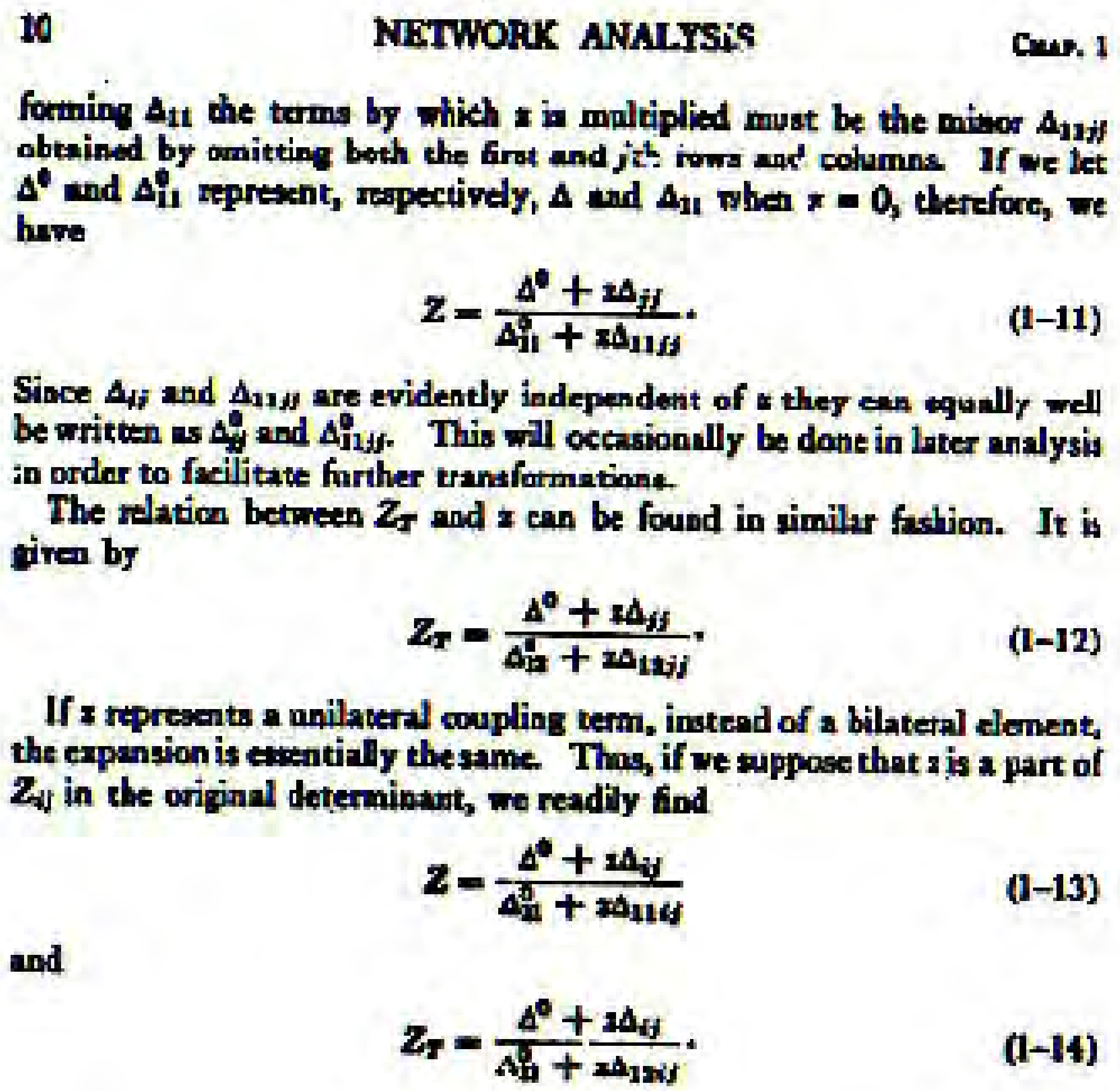
$$u_{o1} = A_1 \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} u_{i1}$$

or

$$\text{gain}\Big|_{Z} = \text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}}$$

This is the Extra Element Theorem: how to calculate the gain, after an extra element is added, by a correction factor instead of starting from scratch.

The Theorem also proves that any transfer function (e.g. gain) of a linear system is a bilinear function of any single element (e.g. Z).



forming Δ_{11} the terms by which z is multiplied must be the minor Δ_{11jj} obtained by omitting both the first and jth rows and columns. If we let Δ^0 and Δ^0_{11} represent, respectively, Δ and Δ_{11} when $z = 0$, therefore, we have

$$Z = \frac{\Delta^0 + z\Delta_{jj}}{\Delta^0_{11} + z\Delta_{11jj}}. \qquad (1\text{–}11)$$

Since Δ_{jj} and Δ_{11jj} are evidently independent of z they can equally well be written as Δ^0_{jj} and Δ^0_{11jj}. This will occasionally be done in later analysis in order to facilitate further transformations.

The relation between Z_T and z can be found in similar fashion. It is given by

$$Z_T = \frac{\Delta^0 + z\Delta_{jj}}{\Delta^0_{12} + z\Delta_{12jj}}. \qquad (1\text{–}12)$$

If z represents a unilateral coupling term, instead of a bilateral element, the expansion is essentially the same. Thus, if we suppose that z is a part of Z_{ij} in the original determinant, we readily find

$$Z = \frac{\Delta^0 + z\Delta_{ij}}{\Delta^0_{11} + z\Delta_{11ij}} \qquad (1\text{–}13)$$

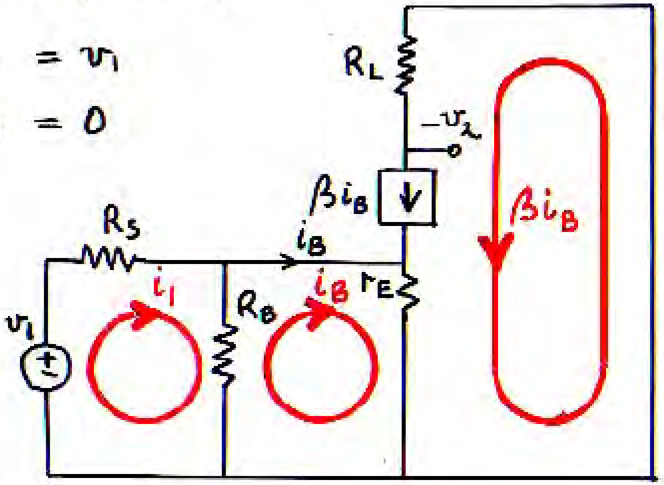
and

$$Z_T = \frac{\Delta^0 + z\Delta_{ij}}{\Delta^0_{12} + z\Delta_{12ij}}. \qquad (1\text{–}14)$$

The "brute-force" method: loop analysis

$$(R_s + R_B)\,i_1 \qquad - R_B\, i_B = v_1$$

$$-R_B\, i_1 + [R_B + (1+\beta) r_E]\, i_B = 0$$



$$i_B = \frac{\begin{vmatrix} R_s + R_B & v_1 \\ -R_B & 0 \end{vmatrix}}{\begin{vmatrix} R_s + R_B & -R_B \\ -R_B & R_B + (1+\beta) r_E \end{vmatrix}}$$

$$= \frac{R_B v_1}{(R_s + R_B)[R_B + (1+\beta) r_E] - R_B^2}$$

$$= \frac{R_B v_1}{R_s R_B + (1+\beta) r_E R_s + R_B^2 + (1+\beta) r_E R_B - R_B^2}$$

Finally, $v_2 = R_L \beta i_B$

which leads to:

$$A_m \equiv \frac{v_2}{v_1} = \frac{\beta R_B R_L}{(1+\beta) r_E R_s + (1+\beta) r_E R_B + R_s R_B}$$

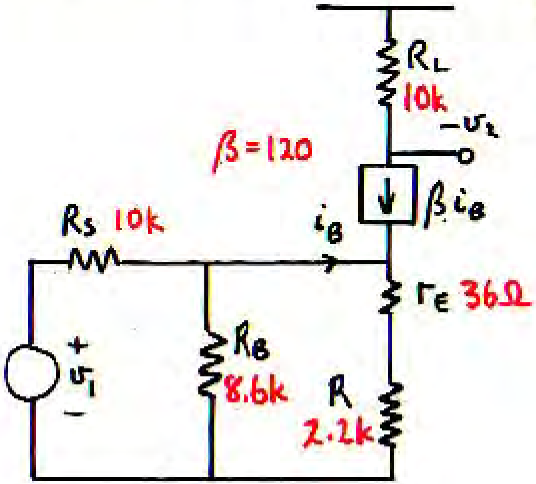
Implementation:

All that is needed is to calculate the driving point impedance across the terminals to which the extra element is to be added, under two conditions:

$$Z_d \equiv Z_{dp}\Big|_{u_{i1}=0} \quad \text{(original input zero)}$$

$$Z_n \equiv Z_{dp}\Big|_{u_{o1}=0} \quad \text{(original output nulled)}$$

Example: The previously designed CE amplifier
Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \rightarrow 1/sC_2$ is desired.

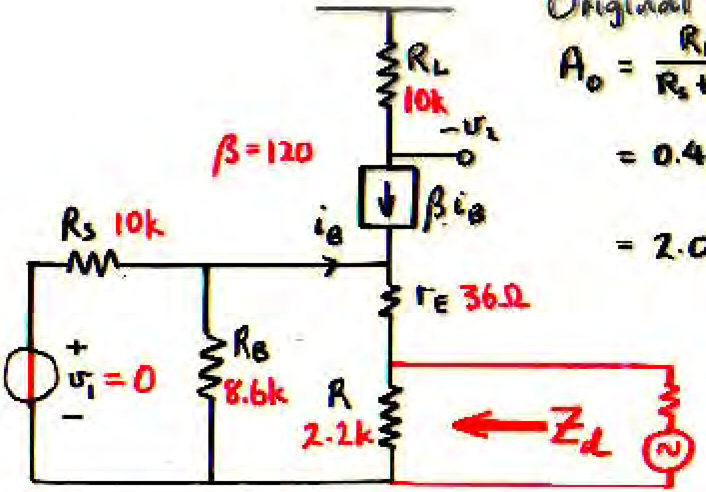


Original gain:

$$A_o = \frac{R_B}{R_s+R_B} \frac{\alpha R_L}{R+r_E+(R_s\|R_B)/(1+\beta)}$$

$$= 0.46 \frac{10}{2.2+0.036+0.039}$$

$$= 2.0 \Rightarrow 6dB$$



Original gain:

$$A_o = \frac{R_B}{R_s + R_B} \frac{\alpha R_L}{R + r_E + (R_s \| R_B)/(1+\beta)}$$

$$= 0.46 \frac{10}{2.2 + 0.036 + 0.039}$$

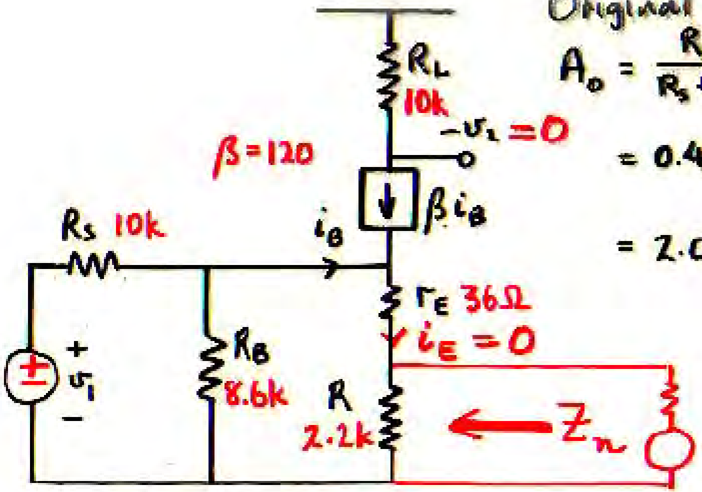
$$= 2.0 \Rightarrow 6\,\text{dB}$$

Step 1. Calculate Z_d by shorting $u_{i1} = v_1$, and applying a second injected signal across R:

$$Z_d = R_d = R \| [r_E + (R_s \| R_B)/(1+\beta)]$$

$$= 2.2 \| [0.036 + \underbrace{(10 \| 8.6)/120}_{0.039}]$$

$$= 75\,\Omega$$



Original gain:

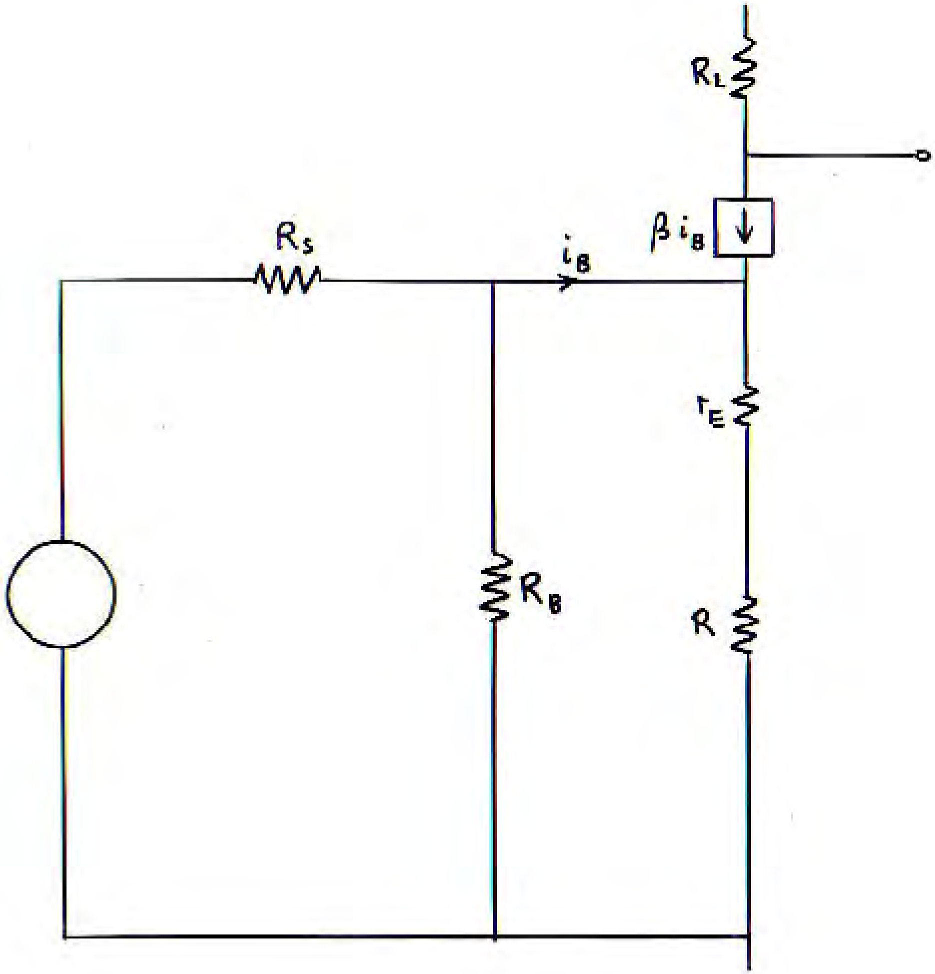
$$A_o = \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{R+r_e+(R_s\|R_B)/(1+\beta)}$$

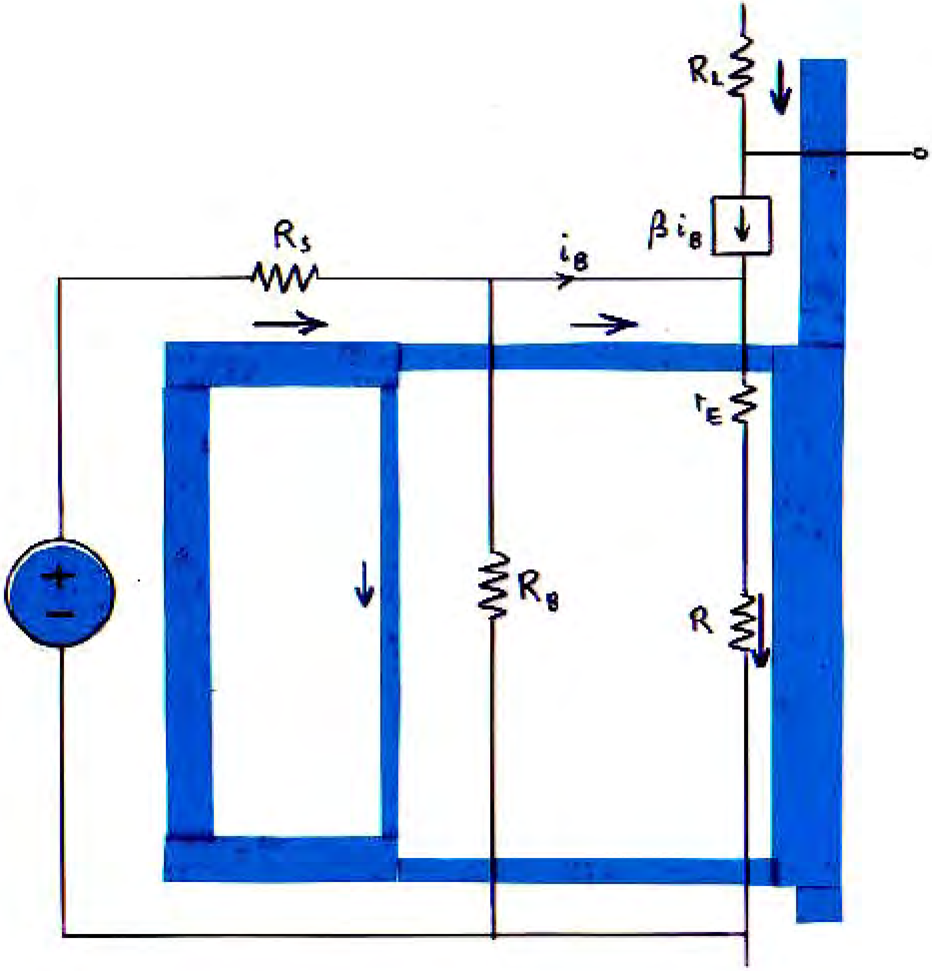
$$= 0.46\,\frac{10}{2.2+0.036+0.039}$$

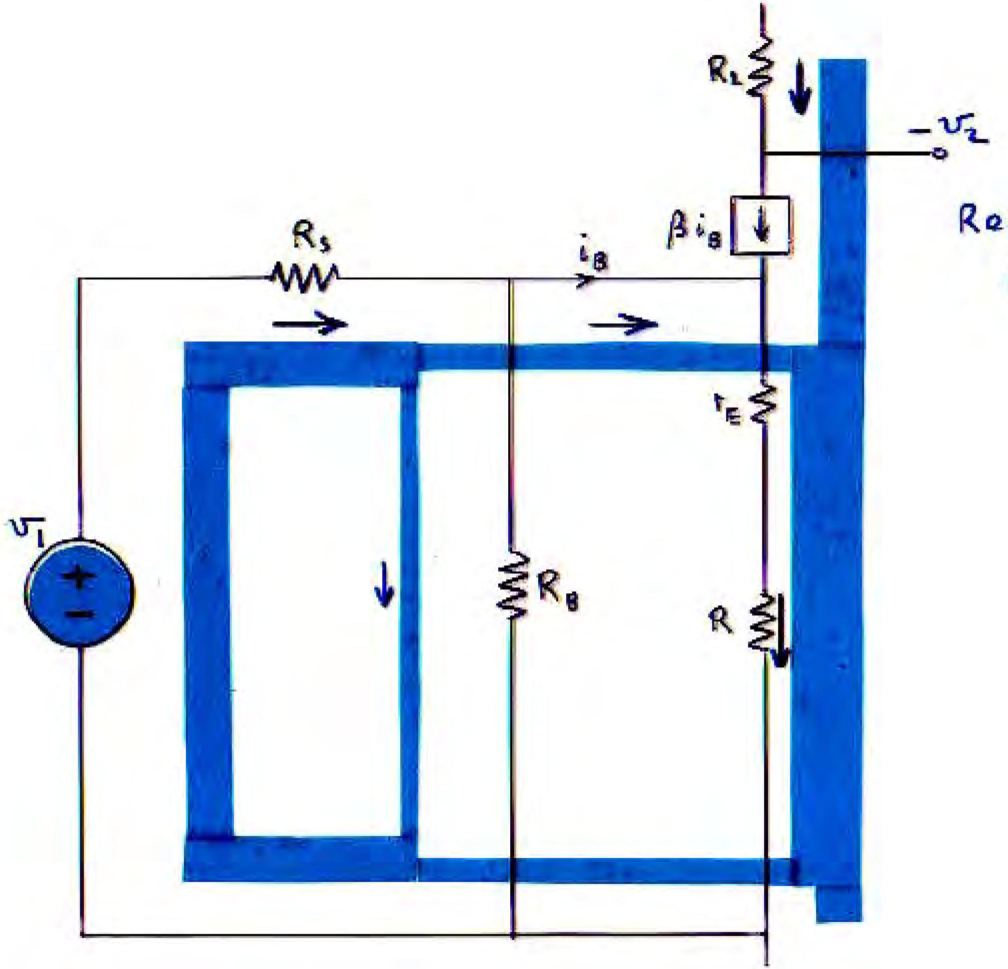
$$= 2.0 \Rightarrow 6dB$$

Step 2. Calculate Z_n by applying a second injected signal across R, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$. Then, since $v_2 = 0$, $i_E = 0$, hence:

$$Z_n = R_n = R = 2.2k$$

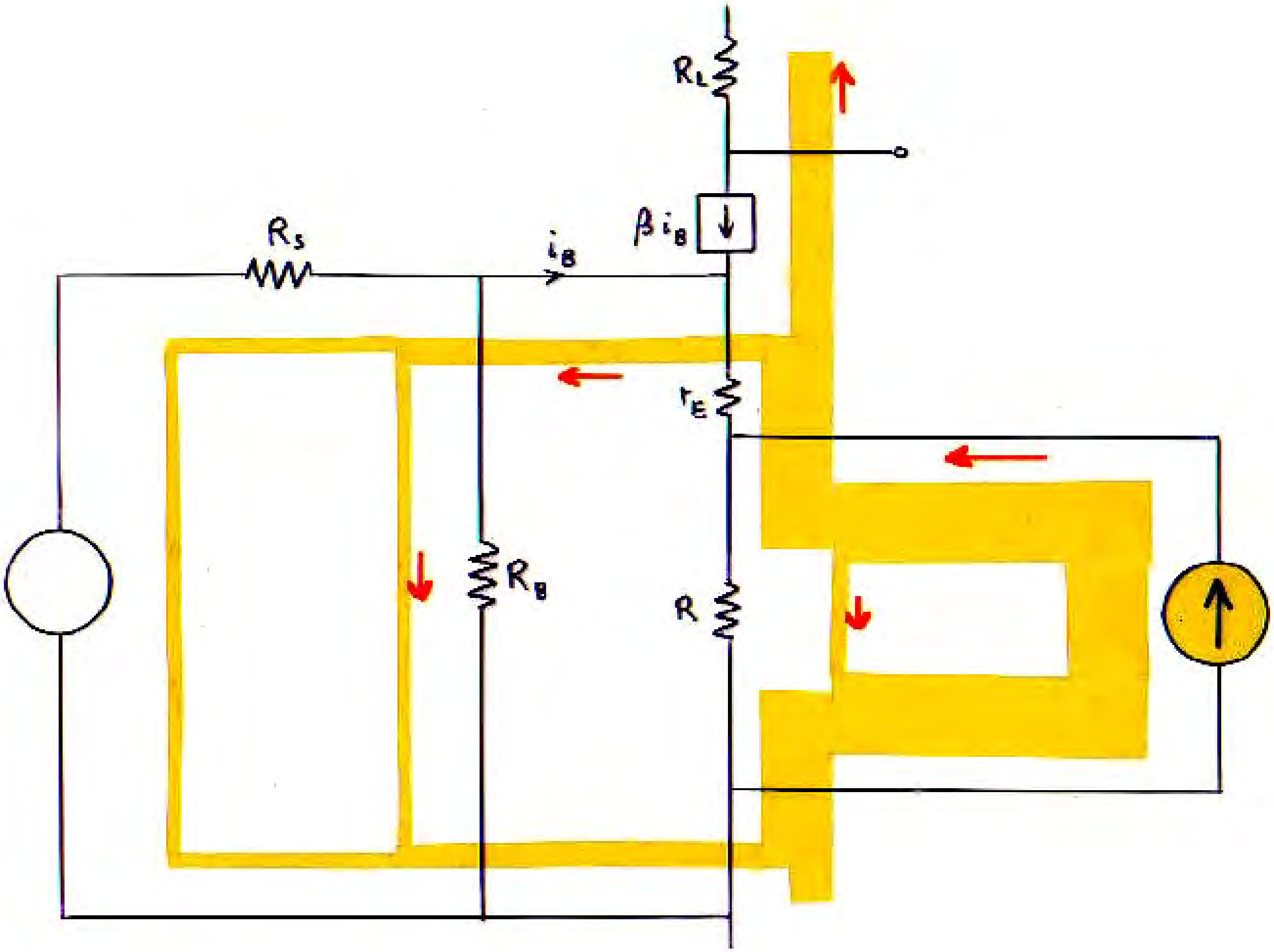

R_L
βi_B
R_s
i_B
r_E
R_B
R


R_L
βi_B
R_s
i_B
r_E
R_B
R
+
−

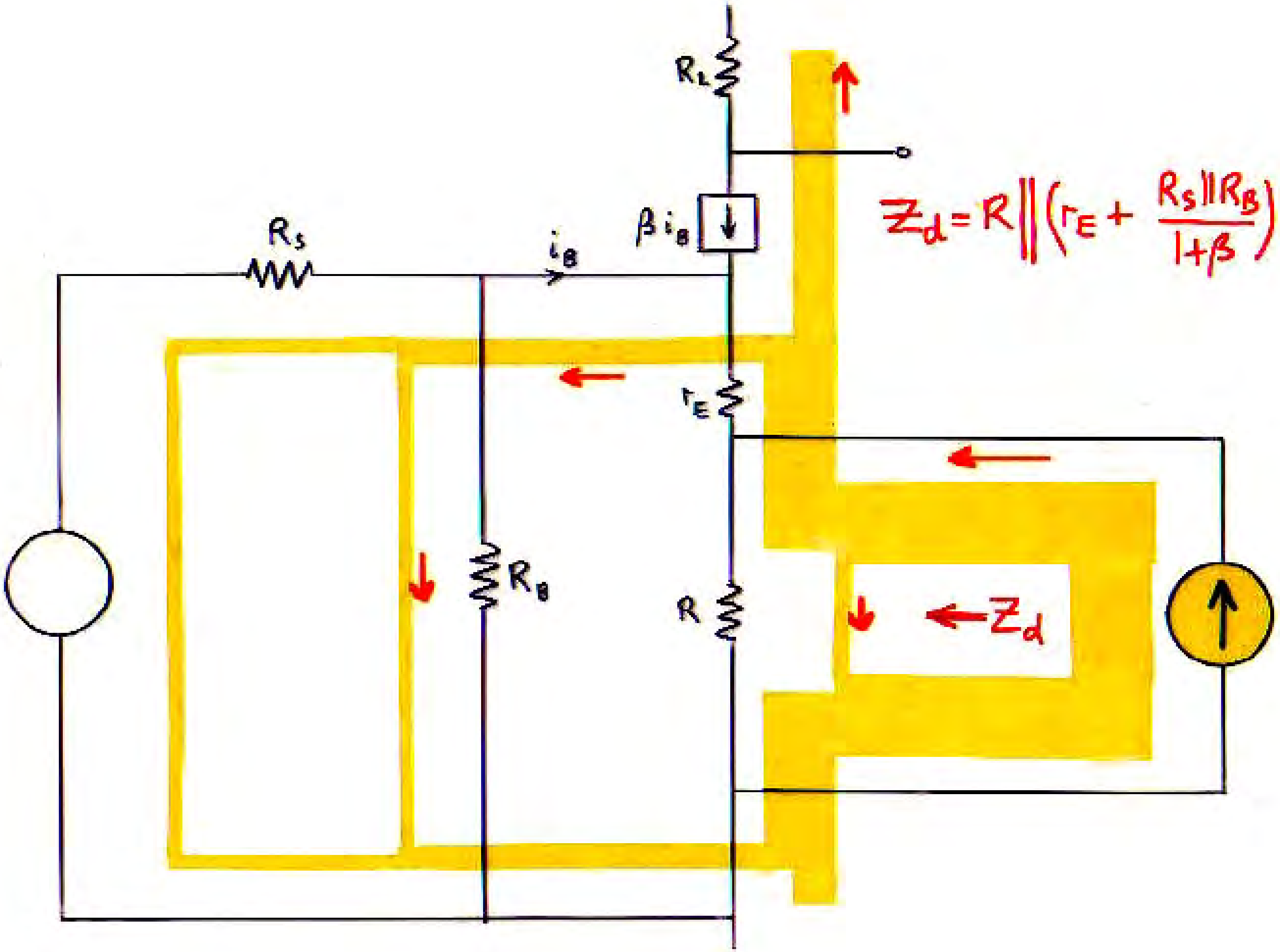


Reference gain:

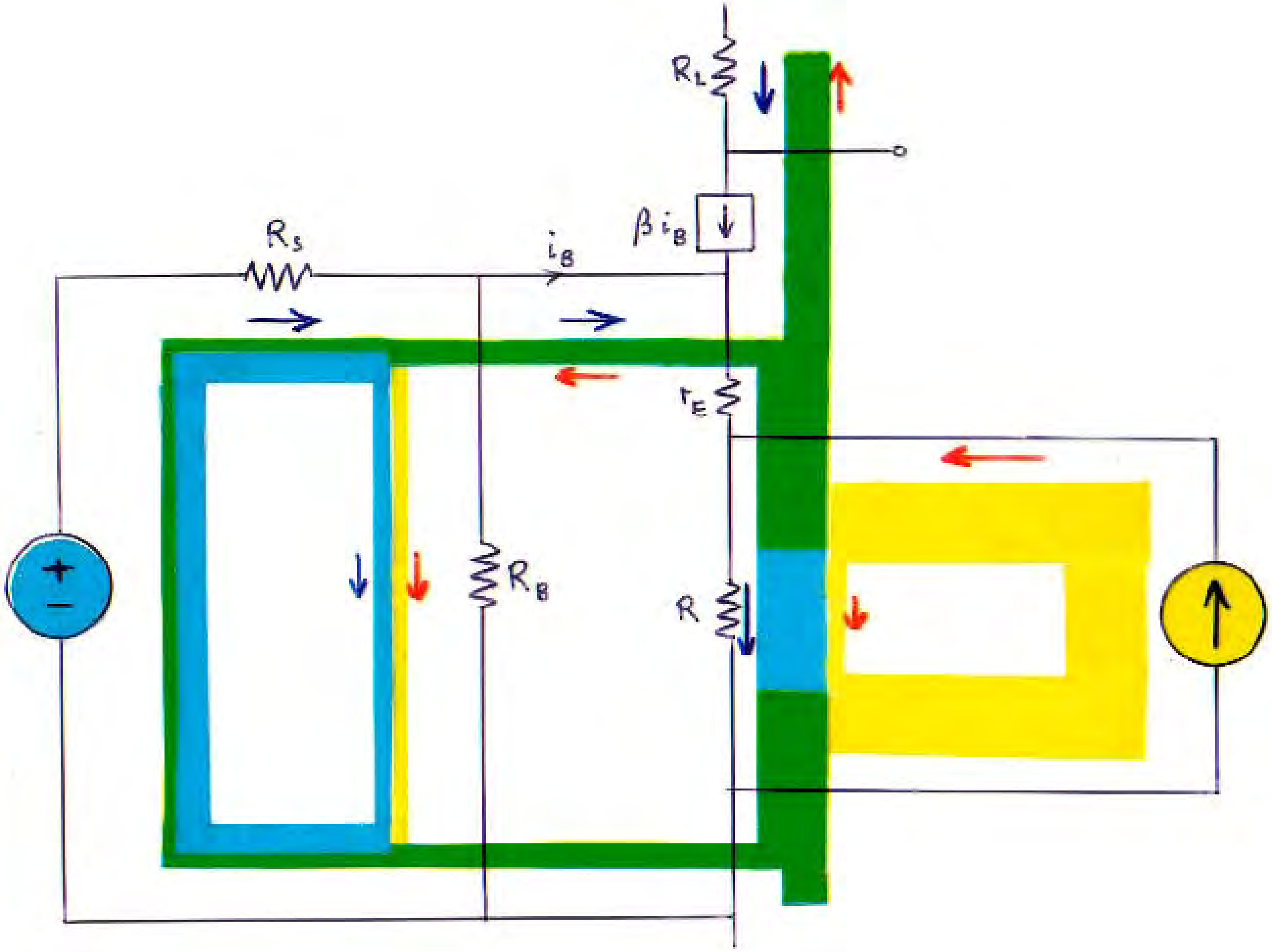
$$A_o = \frac{v_2}{v_1}$$

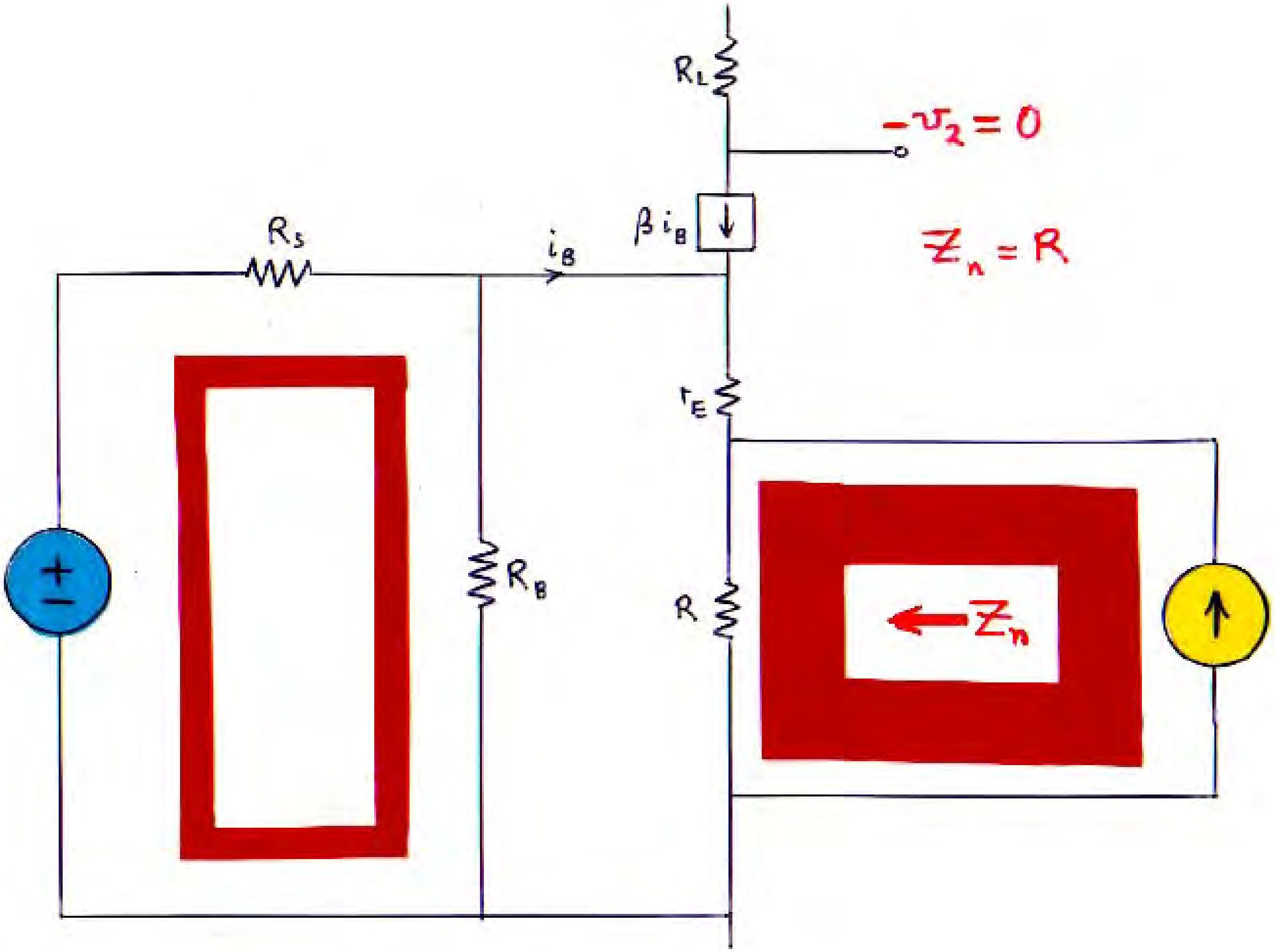


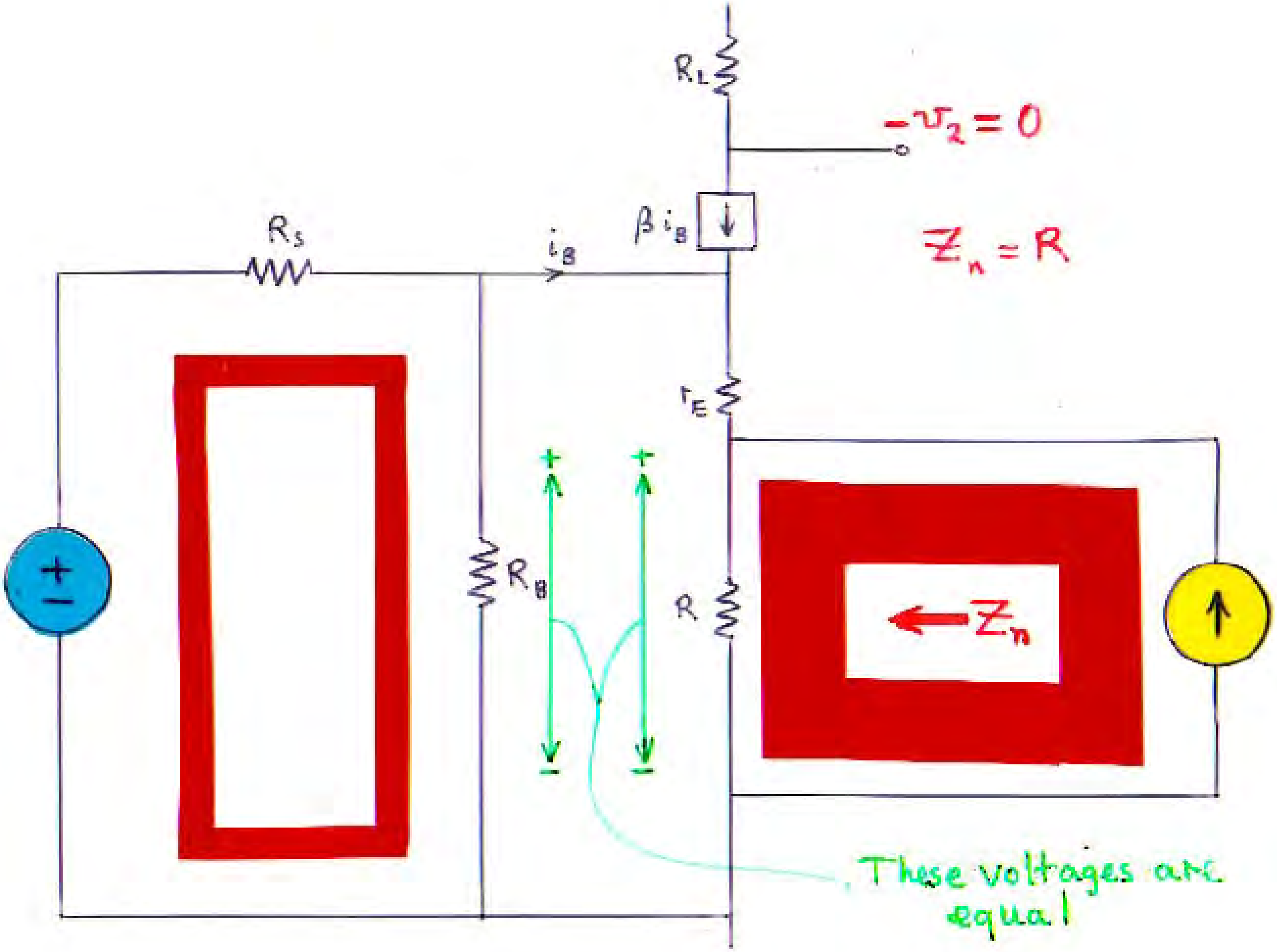
R_L
βi_B
R_s
i_B
r_E
R_B
R



$$Z_d = R \| \left(r_E + \frac{R_s \| R_B}{1+\beta}\right)$$

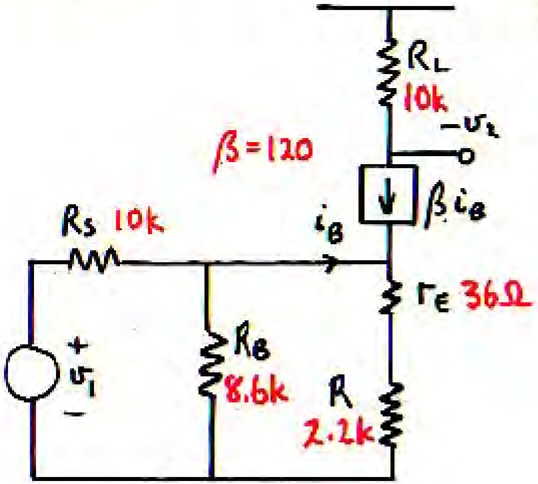

R_L
βi_B
R_s
i_B
r_E
R_B
R
+
−


R_L
$-v_2 = 0$
βi_B
$Z_n = R$
R_s
i_B
r_E
R_B
R
Z_n
±


R_L
$-v_2 = 0$
βi_B
$Z_n = R$
R_s
i_B
r_E
R_B
R
Z_n
These voltages are equal

Example: The previously designed CE amplifier

Suppose the gain has been calculated without the emitter bypass capacitance, and the correction factor resulting from addition of the extra element $Z \rightarrow 1/sC_2$ is desired.

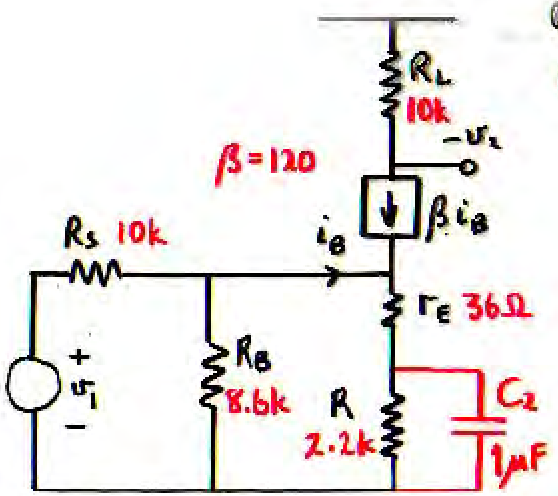


Original gain:

$$A_o = \frac{R_B}{R_s + R_B} \, \frac{\alpha R_L}{R + r_E + (R_s || R_B)/(1+\beta)}$$

$$= 0.46 \, \frac{10}{2.2 + 0.036 + 0.039}$$

$$= 2.0 \Rightarrow 6\text{dB}$$



Original gain:

$$A_o = \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{R+r_e+(R_s\|R_B)/(1+\beta)}$$

$$= 0.46\,\frac{10}{2.2+0.036+0.039}$$

$$= 2.0 \Rightarrow 6\text{dB}$$

Hence, corrected gain in presence of $C_2 = 1\mu F$ bypass capacitance is:

$$A = A_o\,\frac{1+\frac{R_n}{Z}}{1+\frac{R_d}{Z}} = A_o\,\frac{1+sC_2R_n}{1+sC_2R_d} = A_o\,\frac{1+\frac{s}{\omega_1}}{1+\frac{s}{\omega_2}} = A_m\,\frac{1+\frac{\omega_1}{s}}{1+\frac{\omega_2}{s}}$$

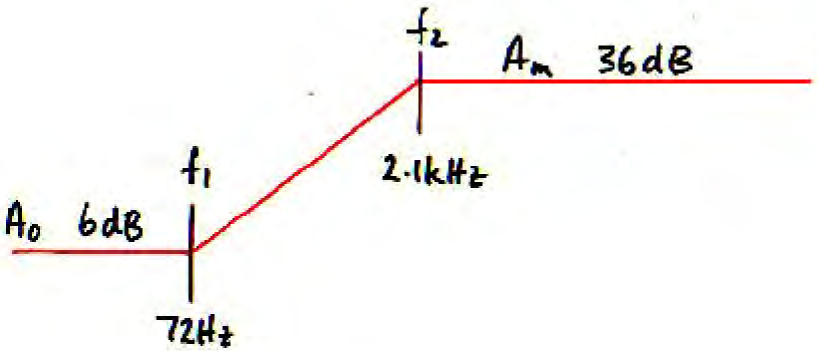
where

$$\omega_1 \equiv \frac{1}{C_2R_n} \quad f_1 = \frac{159}{1\times 2.2} = 72\text{Hz} \quad \omega_2 \equiv \frac{1}{C_2R_d} = \frac{159}{1\times 0.075} = 2.1\text{kHz}$$

$$A_m = A_o\,\frac{\omega_2}{\omega_1} = A_o\,\frac{R_n}{R_d}$$

$$= \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{R+r_E+(R_s\|R_B)/(1+\beta)}\,\frac{R[R+r_E+(R_s\|R_B)/(1+\beta)]}{R[r_E+(R_s\|R_B)/(1+\beta)]}$$

$$= \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{r_E+(R_s\|R_B)/(1+\beta)} = 62 \Rightarrow 36\text{dB}$$


f_2
A_m 36dB
f_1
2.1kHz
A_0 6dB
72Hz

NOTE: <u>Nulling</u> a voltage is <u>not</u> the same as <u>shorting</u> it!

NOTE: the null double injection <u>calculation</u> is easier than the single injection <u>calculation</u>!

Dual forms of the EET

The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element.

There is a corresponding form to find the correction factor resulting from an extra series element:

$$\text{gain}\Big|_{Z} = \overset{\text{reference gain}}{\overset{\downarrow}{\text{gain}\Big|_{Z=\infty}}} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\Big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

$$= \frac{Z_n}{Z_d}\,\text{gain}\Big|_{Z=\infty} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

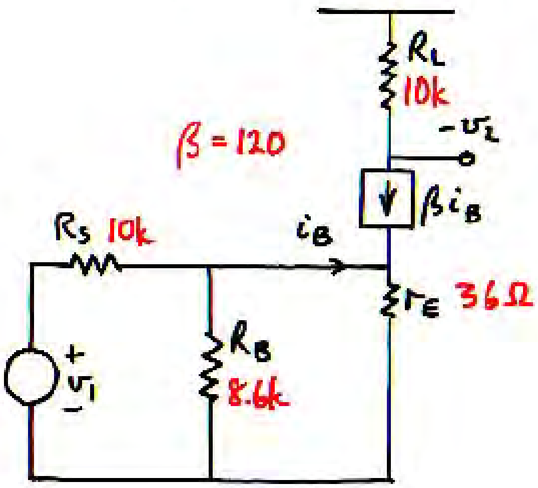
The Extra Element Theorem as derived applies to the correction factor resulting from an extra shunt element.

There is a corresponding form to find the correction factor resulting from an extra series element:

$$\text{gain}\Big|_{Z} = \overset{\text{reference gain}}{\overset{\downarrow}{\text{gain}\Big|_{Z=\infty}}}\;\frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} \qquad = \text{gain}\Big|_{Z=\infty}\;\frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}}\;\frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

$$= \overset{\text{reference gain}}{\overset{\downarrow}{\text{gain}\Big|_{Z=0}}}\;\frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}} \qquad = \underbrace{\boxed{\frac{Z_n}{Z_d}\,\text{gain}\Big|_{Z=\infty}}}_{\text{This must be the gain when } Z=0}\;\frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

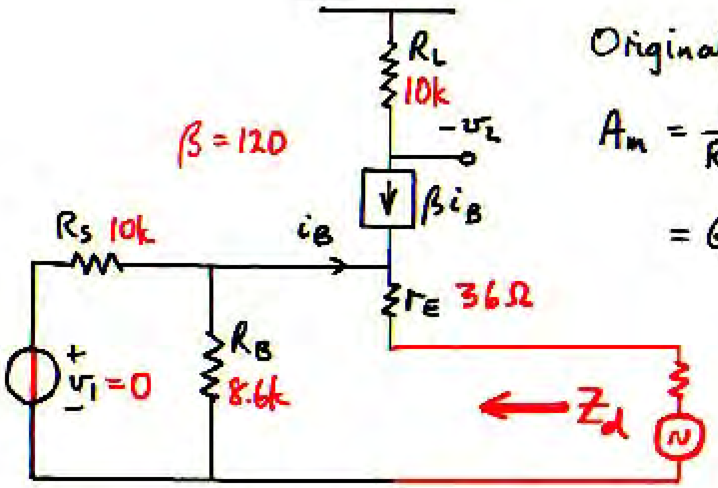
Example: An alternative to the method of the previous example is to find the correction factor to the midband gain A_m resulting from addition of the series "extra element" $Z \rightarrow R \| 1/sC$.



Original gain:

$$A_m = \frac{R_B}{R_S + R_B} \frac{\alpha R_L}{r_E + (R_S \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\text{dB}$$



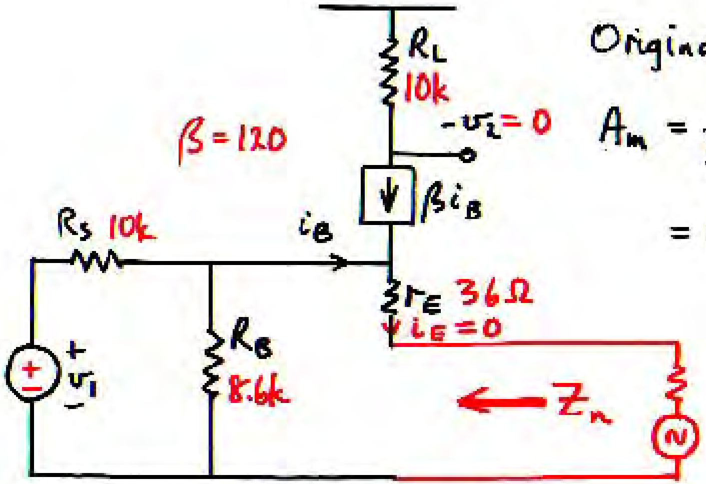
Original gain:

$$A_m = \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{r_E + (R_s||R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\text{dB}$$

Step 1. Calculate Z_d by shorting $u_{i1} = v_i$ and applying a second injected signal in series with r_E:

$$Z_d \equiv R_d' = r_E + (R_s||R_B)/(1+\beta)$$



Original gain:

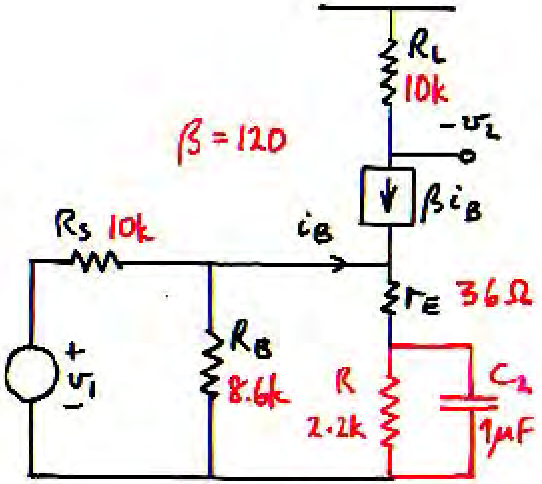
$$A_m = \frac{R_B}{R_s+R_B}\frac{\alpha R_L}{r_E + (R_s \| R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36\text{dB}$$

Step 2. Calculate Z_n by applying a second injected signal in series with r_E, and adjusting it with respect to v_1 to null $u_{o1} = v_2 = 0$.

Then, since $v_2 = 0$, $i_E = 0$, hence

$$Z_n \equiv R_n' = \infty$$



Original gain:

$$A_m = \frac{R_B}{R_s+R_B}\,\frac{\alpha R_L}{r_E + (R_s\|R_B)/(1+\beta)}$$

$$= 62 \Rightarrow 36dB$$

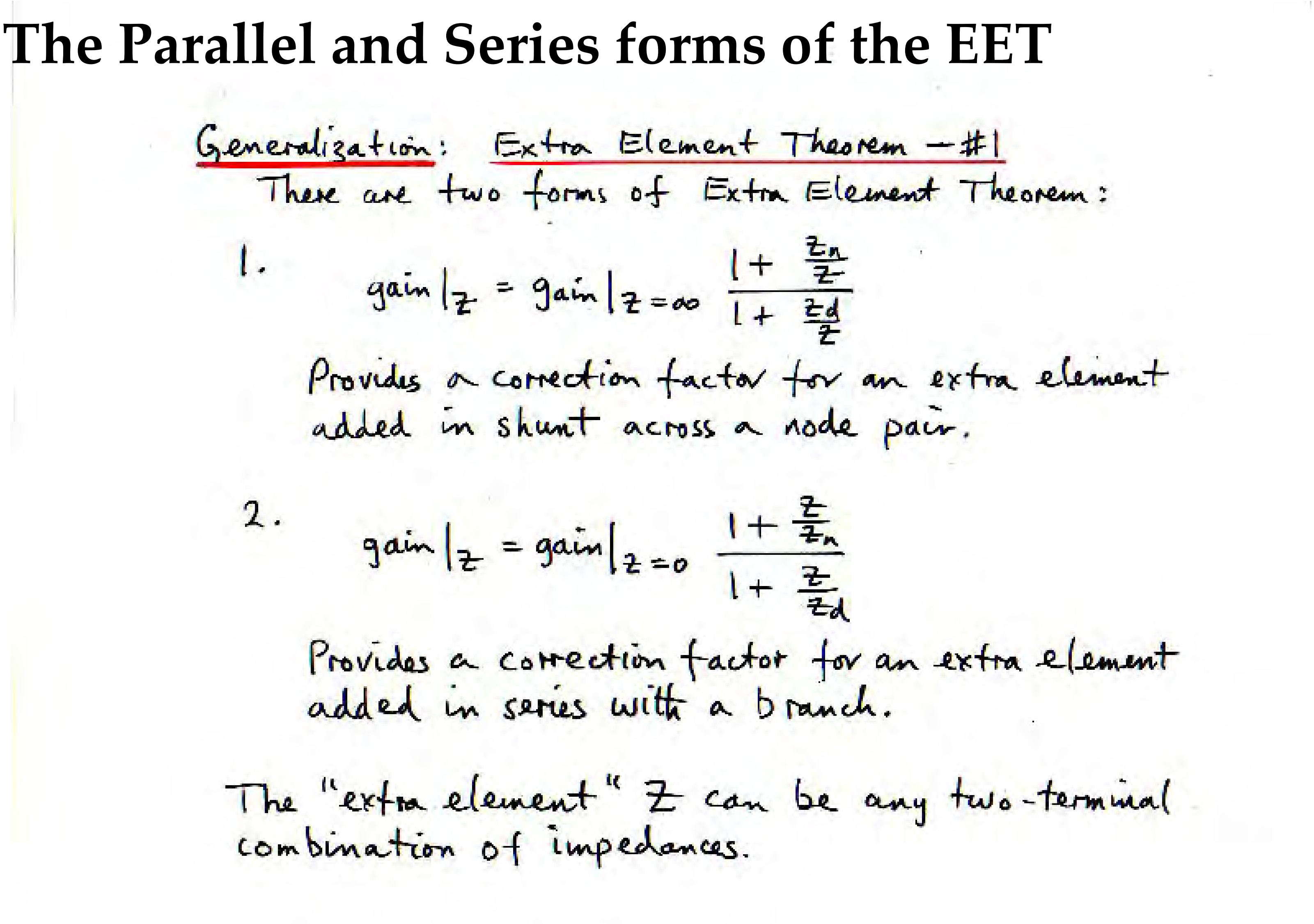
Hence, corrected gain in presence of $R\|1/sC_2$ is:

$$A = A_m\frac{1+\frac{Z}{R_n'}}{1+\frac{Z}{R_d'}} = A_m\frac{1}{1+\frac{R}{R_d'}\,\frac{1}{1+sC_2R}} = A_m\frac{1+1/sC_2R}{1+1/sC_2(R\|R_d')}$$

However, $R\|R_d' = R_d$, so

$$A = A_m\frac{1+1/sC_2R}{1+1/sC_2R_d} \longrightarrow$$ same result as before

The Parallel and Series forms of the EET



Generalization: Extra Element Theorem — #1

There are two forms of Extra Element Theorem:

1.
$$gain|_Z = gain|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}}$$

Provides a correction factor for an extra element added in shunt across a node pair.

2.
$$gain|_Z = gain|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

Provides a correction factor for an extra element added in series with a branch.

The "extra element" Z can be any two-terminal combination of impedances.

Note that in all cases the null double injection calculation is easier than the single injection calculation.

This results from use of the null condition (which makes several other quantities zero);

and

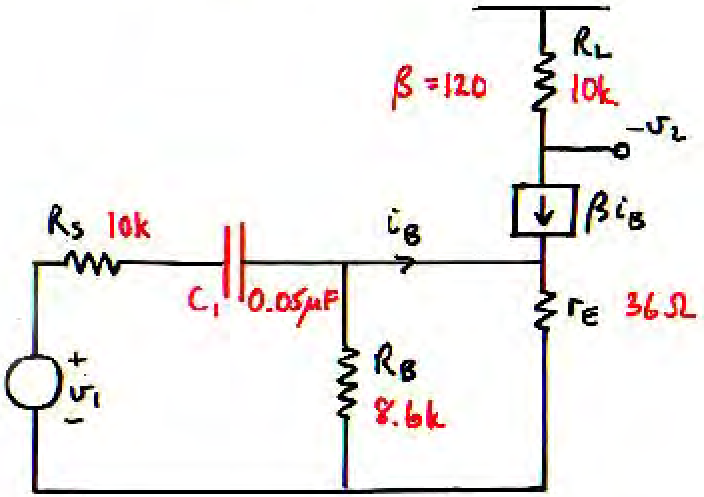
because the relation between u_{i1} and u_{i2} to produce the null is never needed — only the null itself is used.

Exercise 8.1

Insert C_1 by the EET

Exercise

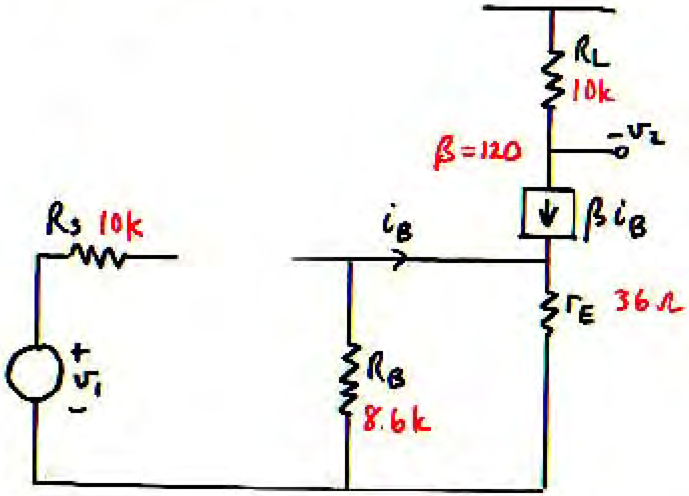
In the CE amplifier stage, find the correction factor to the midband gain A_m resulting from inclusion of the coupling capacitance $C_1 = 0.05\mu F$:



$A_m = 36dB$

Exercise 8.1 - Solution

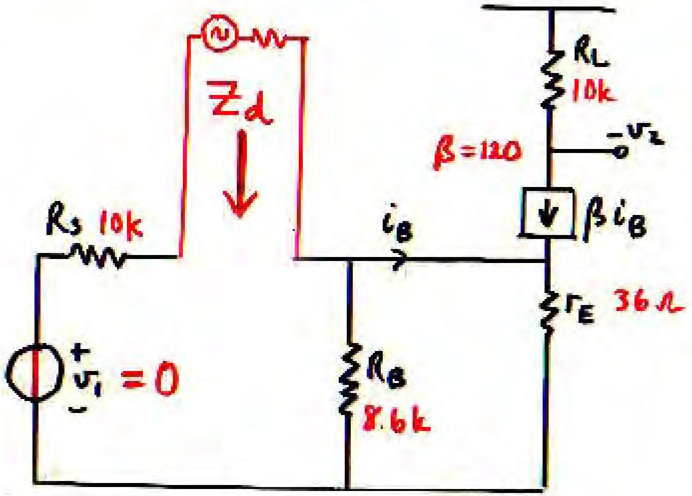
Exercise Solution



$A_m = 36\text{dB}$

Exercise 8.1 - Solution

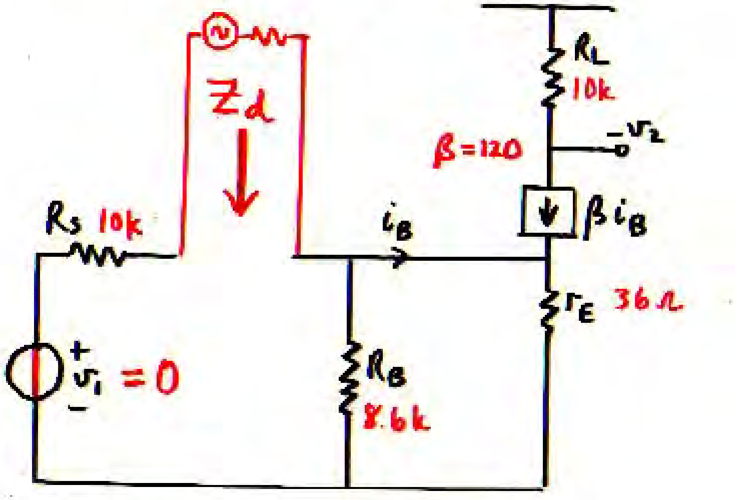
Exercise Solution



$A_m = 36dB$

Exercise 8.1 - Solution

Exercise Solution



$A_m = 36\text{dB}$

Step 1.

$$Z_d = R_d = R_s + R_B \| (1+\beta) r_E$$

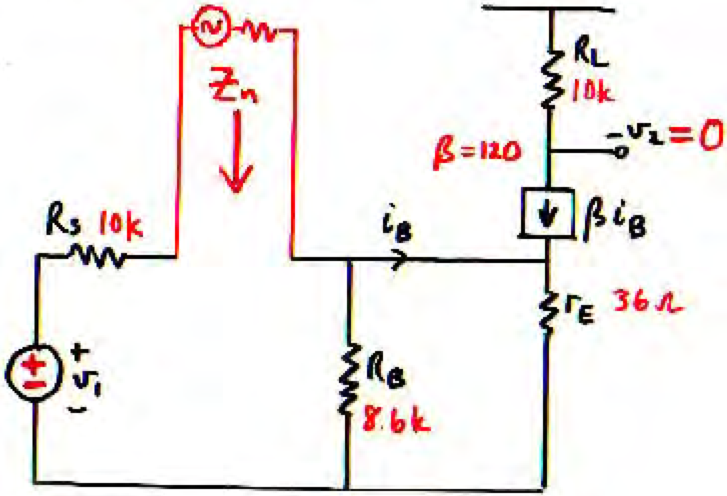
$$= 10 + 8.6 \| (120 \times 0.036)$$

$$= 10 + 8.6 \| 4.3$$

$$= 13\,k$$

Exercise 8.1 - Solution

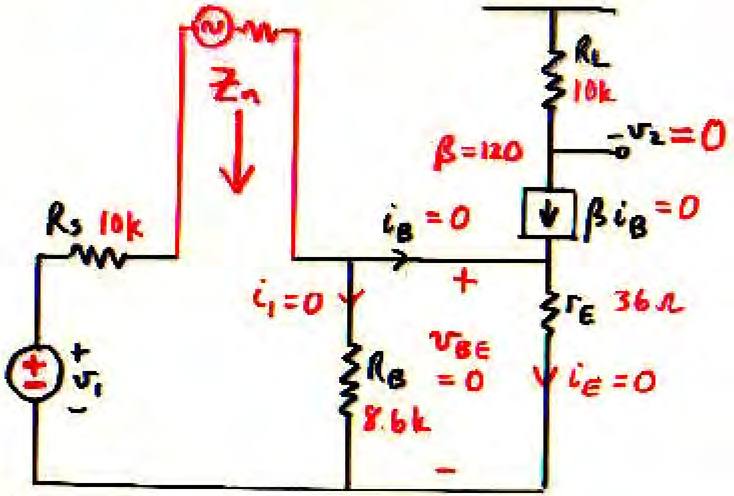
Exercise Solution



$A_m = 36dB$

Exercise 8.1 - Solution

Exercise Solution



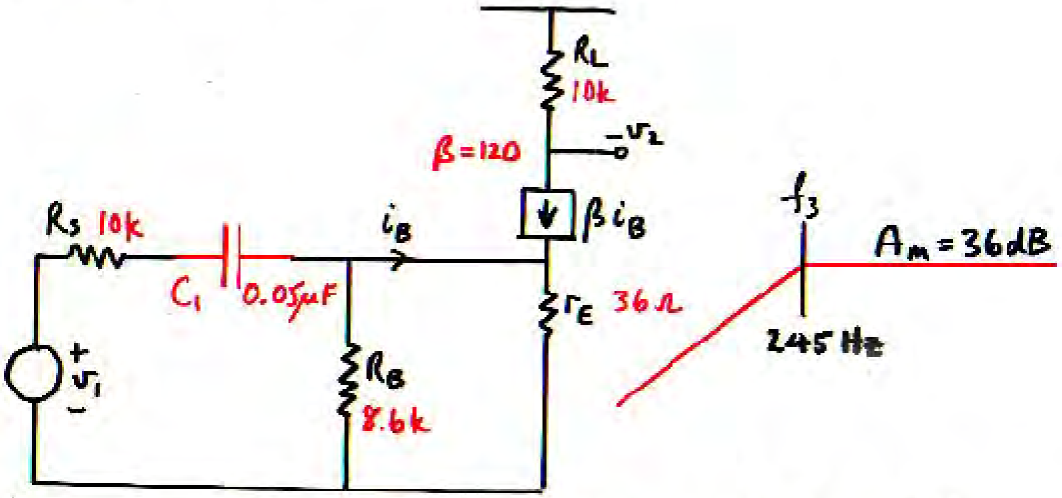
$A_m = 36\text{dB}$

Step 2.

$$Z_n = R_n = \infty$$

Exercise 8.1 - Solution

Exercise Solution



Hence corrected gain in the presence of $Z \rightarrow 1/sC_1$ is

$$A = A_m \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}} = A_m \frac{1}{1+\frac{1}{sC_1R_d}} = A_m \frac{1}{1+\frac{\omega_3}{s}}$$

where

$$\omega_3 \equiv \frac{1}{C_1R_d} \qquad f_3 = \frac{159}{0.05 \times 13} = 245\ \text{Hz}$$

Generalization: Extra Element Theorem — #2

If the reference circuit is purely resistive, $Z_d = R_d$ and $Z_n = R_n$ are pure resistances.

If, also, the extra element is a pure reactance, the Extra Element Theorem correction factor gives the corner frequencies directly.

Generalization: Extra Element Theorem —#3

The Extra Element Theorem can profitably be used to divide the analysis of a complicated circuit into successive simpler steps:

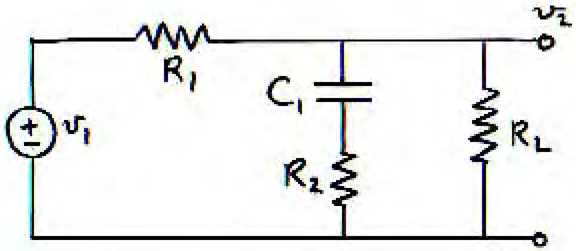
Designate one element as "extra," and the circuit without the element as the "reference circuit." Calculate the gain of the (simpler) reference circuit, then restore the omitted element by the Extra Element Theorem correction factor.

This is a particularly useful approach when the designated "extra" element is a reactance and the reference circuit is purely resistive.

Exercise 8.2

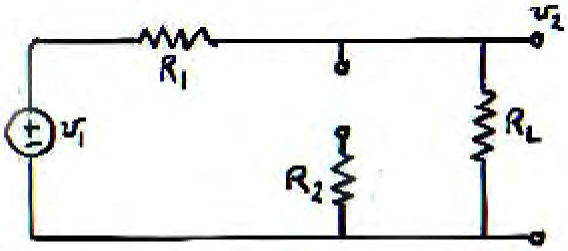
Lag-lead network: Find A by designating C_1 as an extra element.

Exercise : Lag-lead network



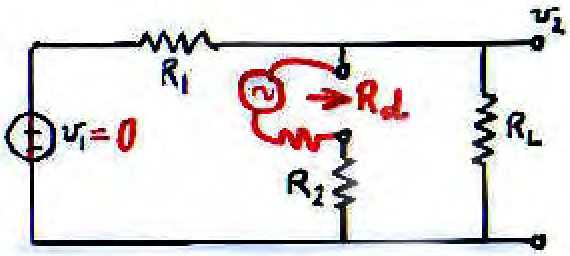
Find the transfer function $A \equiv v_2/v_1$ by designating C_1 as an "extra" element.

Exercise 8.2 - Solution



$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

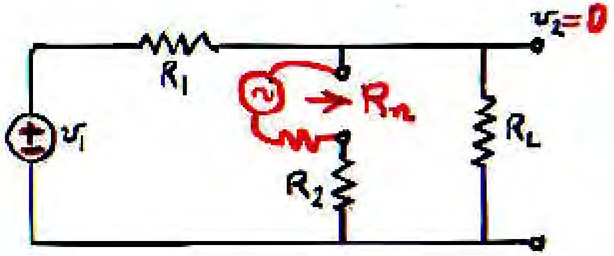
Exercise 8.2 - Solution



$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

$$R_d = R_2 + R_1 \| R_L$$

Exercise 8.2 - Solution

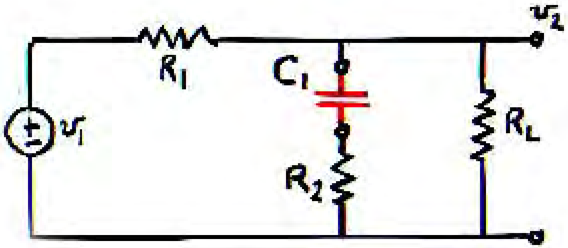


$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

$$R_d = R_2 + R_1 \| R_L$$

$$R_n = R_2$$

Exercise 8.2 - Solution



$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L}$$

$$R_d = R_2 + R_1 \| R_L$$

$$R_n = R_2$$

$$\frac{v_2}{v_1} = \frac{R_L}{R_1 + R_L} \frac{1 + sC_1 R_2}{1 + sC_1 (R_2 + R_1 \| R_L)}$$

Special case: The EET for a self-impedance

The Extra Element Theorem may be used to find an extra element correction factor for any transfer function of a linear circuit. It is necessary merely to identify the "input" and "output" signals; Z_d and Z_n are then calculated as the driving point impedance seen by the extra element with the "input" zero and with the "output" nulled, respectively.

Examples of transfer functions:

$$\frac{\text{"output"} \rightarrow \text{current drawn from power supply}}{\text{"input"} \rightarrow \text{input voltage}}$$

(a transadmittance)

$$\frac{\text{"output"} \rightarrow \text{output voltage ripple component}}{\text{"input"} \rightarrow \text{power supply ripple voltage}}$$

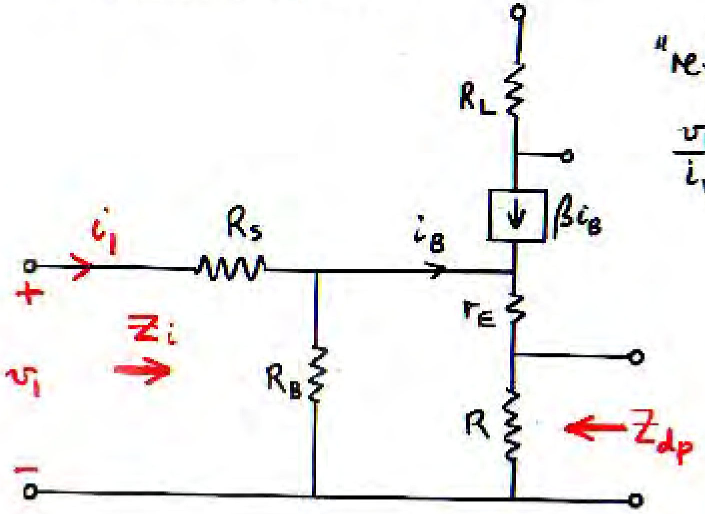
(a voltage gain; audio susceptibility of a power supply)

$$\frac{\text{"output"} \rightarrow \text{corresponding driving voltage}}{\text{"input"} \rightarrow \text{any driving current}}$$

(a self-impedance, e.g. input or output impedance)

Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



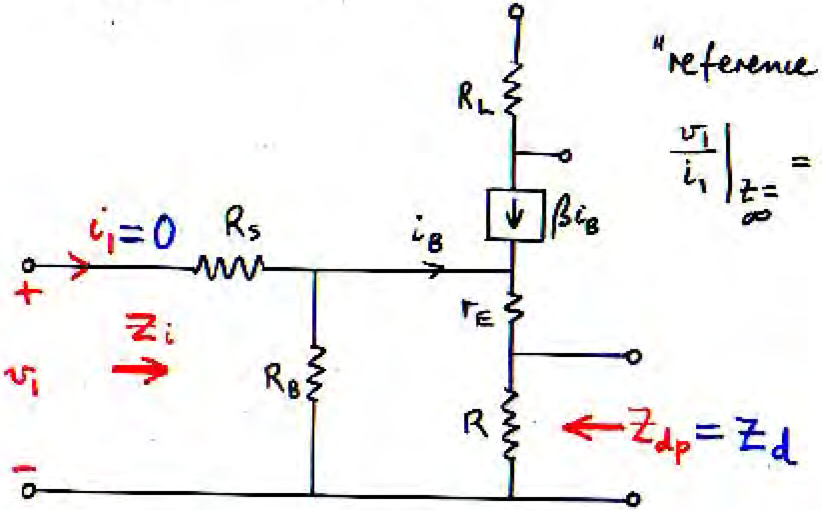
"reference transfer function":

$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E+R)$$

$$Z = \frac{1}{sC_2}$$

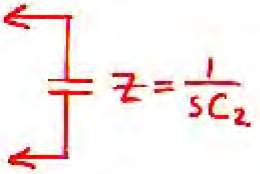
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



"reference transfer function":

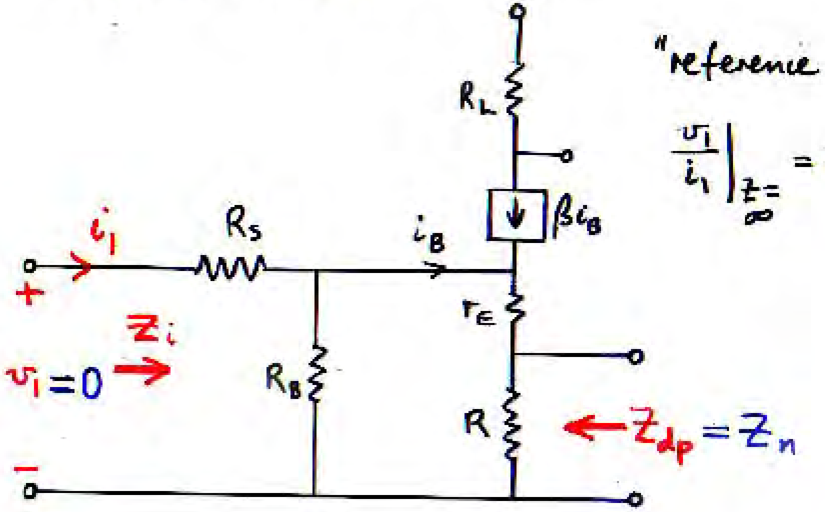
$$\frac{v_1}{i_1}\Big|_{Z=\infty} = Z_i\Big|_{Z=\infty} = R_s + R_B \| (1+\beta)(r_E + R)$$



$$Z_d = Z_{dp}\Big|_{\text{"input" zero}} = Z_{dp}\Big|_{i_1=0} = R_d = R \| \left(r_E + \frac{R_B}{1+\beta}\right)$$

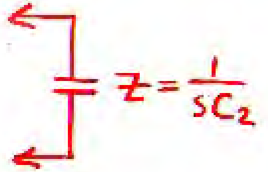
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



"reference transfer function":

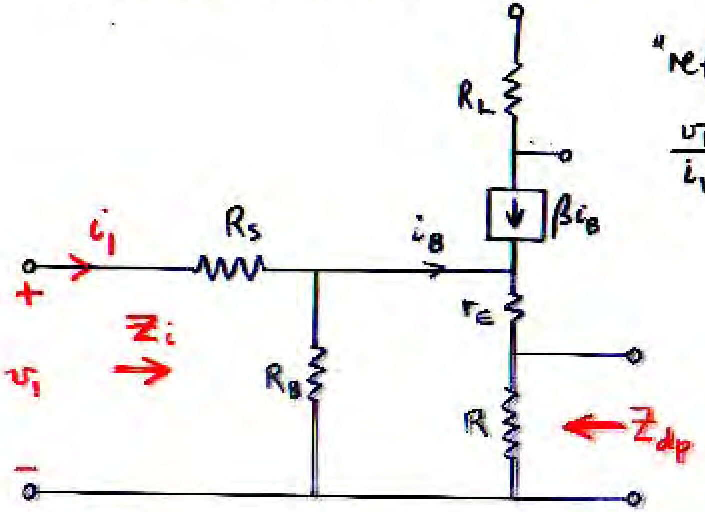
$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E+R)$$



$$Z_n = \left.Z_{dp}\right|_{\substack{\text{"output"}\\ \text{nulled}}} = \left.Z_{dp}\right|_{v_1=0} = R_n = R \| \left(r_E + \frac{R_S \| R_B}{1+\beta}\right)$$

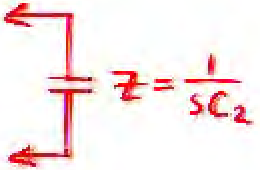
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



"reference transfer function":

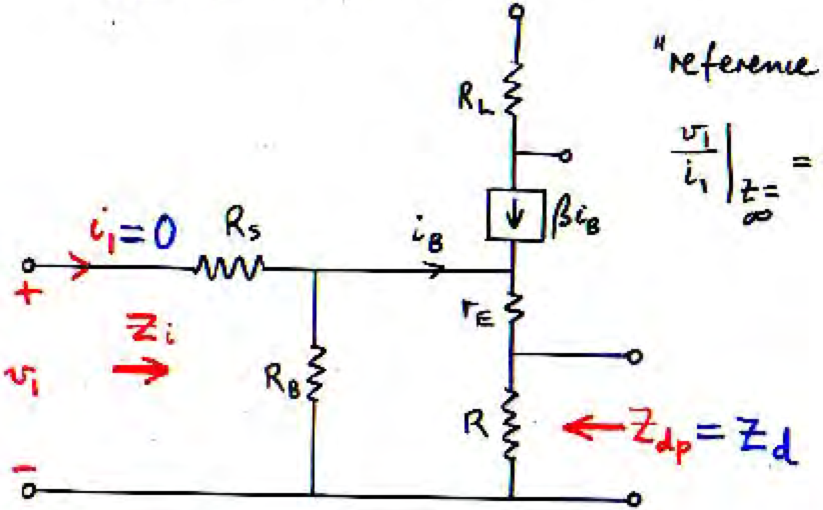
$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_s + R_B \| (1+\beta)(r_E+R)$$



Hence, $Z_i = \left[R_s + R_B \| (1+\beta)(r_E+R)\right] \dfrac{1 + sC_2R_n}{1 + sC_2R_d}$

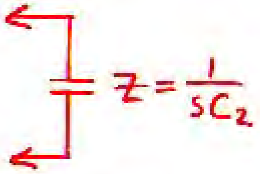
Example: Input impedance Z_i of a CE amplifier stage with emitter bypass capacitance as "extra" element.

"Reference circuit":



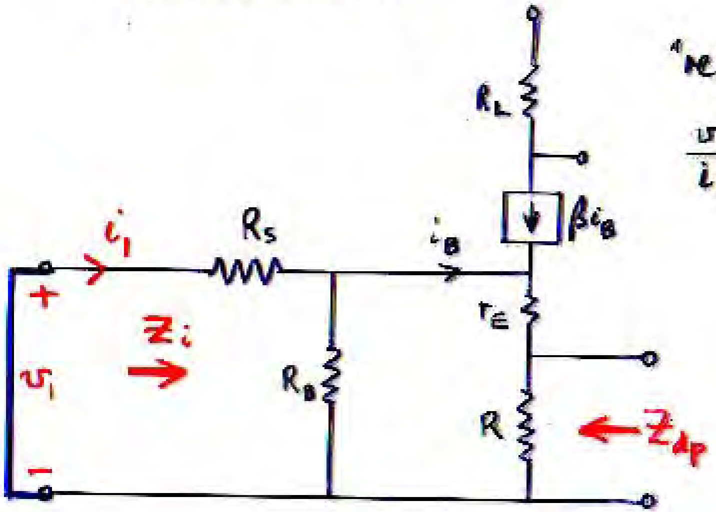
"reference transfer function":

$$\left.\frac{v_1}{i_1}\right|_{Z=\infty} = \left.Z_i\right|_{Z=\infty} = R_s + R_B \| (1+\beta)(r_E+R)$$



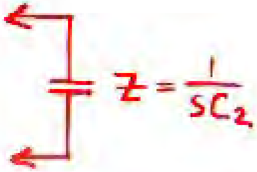
$$Z_d = \left.Z_{dp}\right|_{\text{"input" zero}} = \left.Z_{dp}\right|_{i_1=0} = R_d = R \| \left(r_E + \frac{R_B}{1+\beta}\right)$$

"Reference circuit":



"reference transfer function":

$$\frac{v_1}{i_1}\bigg|_{Z=\infty} = Z_i\bigg|_{Z=\infty} = R_S + R_B \| (1+\beta)(r_E+R)$$



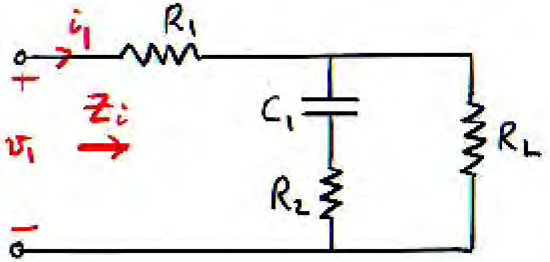
$$Z = \frac{1}{sC_2}$$

NOTE: In the special case of a self-impedance, nulling the "output" voltage is the same as shorting the "input" current, because the "output" and "input" are at the same node pair.

Exercise 8.3

Lag-lead network: Find Z_i by designating C_1 as an extra element.

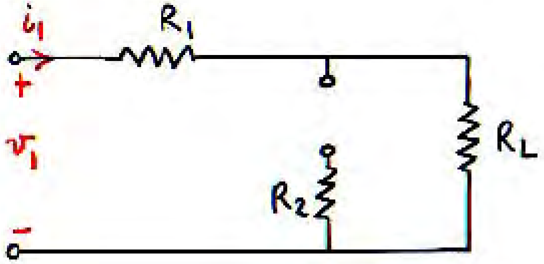
Example: Lag-lead network



Find the input impedance $Z_i = v_1 / i_1$ by designating C_1 as an "extra" element.

Exercise 8.3 - Solution

"Reference circuit:"

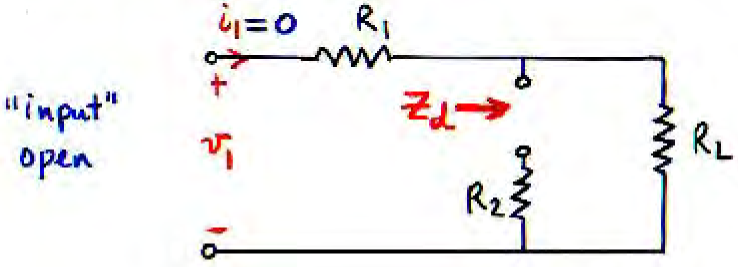


"Reference" input impedance:

$$Z_i\Big|_{Z=\infty} = \frac{v_1}{i_1}\Big|_{Z=\infty} = R_1 + R_L$$

Exercise 8.3 - Solution

"Reference circuit:"



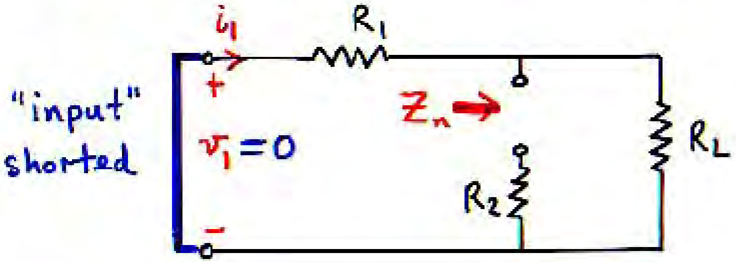
"Reference" input impedance:

$$Z_i\Big|_{Z=\infty} = \frac{v_1}{i_1}\Big|_{Z=\infty} = R_1+R_L$$

$$Z_d = R_d = R_2 + R_L$$

Exercise 8.3 - Solution

"Reference circuit:"



"Reference" input impedance:

$$Z_i\Big|_{Z=\infty} = \frac{v_1}{i_1}\Big|_{Z=\infty} = R_1 + R_L$$

$$Z_n = R_n = R_2 + R_1 \| R_L$$

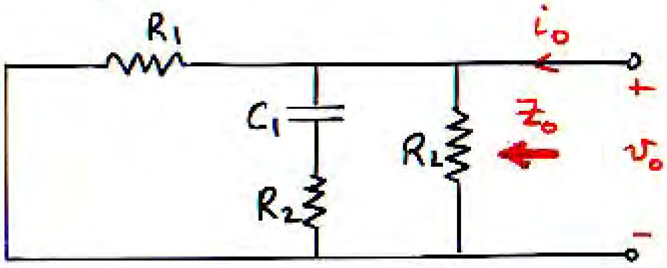
Hence:

$$Z_i = (R_1 + R_L)\frac{1 + sC_1R_n}{1 + sC_1R_d}$$

Exercise 8.4

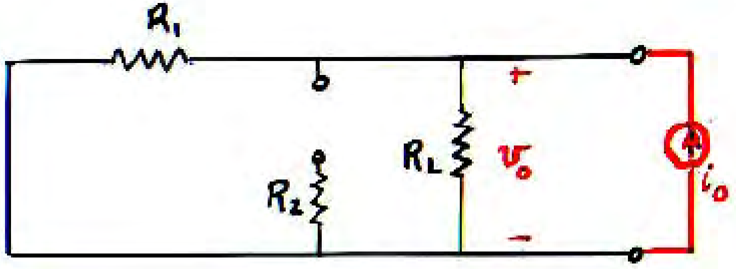
Lag-lead network: Find Z_o by designating C_1 as an extra element.

Exercise: Lag-lead network



Find the output impedance $Z_o = v_o / i_o$ by designating C_1 as an "extra" element.

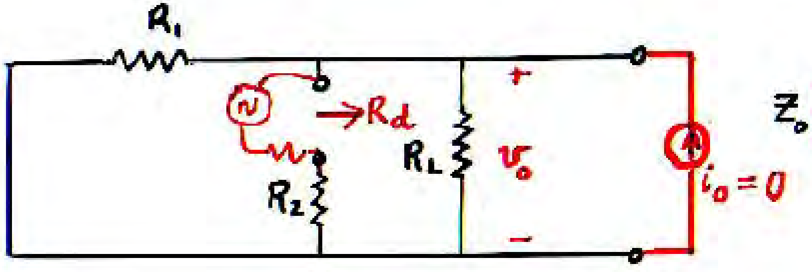
Exercise 8.4 - Solution



$$Z_o\Big|_{Z=\infty} \equiv \frac{v_o}{i_o}\Big|_{Z=\infty}$$

$$= R_1 \| R_L$$

Exercise 8.4 - Solution

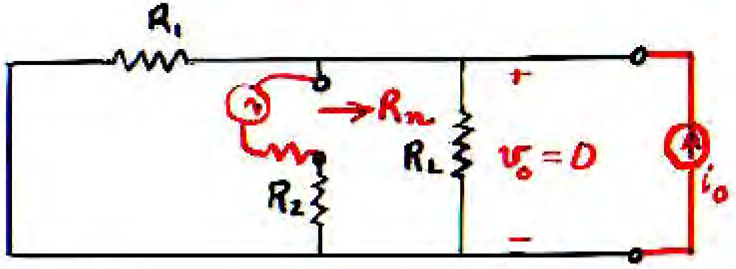


$$Z_o\Big|_{Z=\infty} \equiv \frac{v_o}{i_o}\Big|_{Z=\infty}$$

$$= R_1 \| R_L$$

$$R_d = R_2 + R_1 \| R_L$$

Exercise 8.4 - Solution



$$Z_o\Big|_{Z=\infty} \equiv \frac{v_o}{i_o}\Big|_{Z=\infty}$$

$$= R_1 \| R_L$$

$$R_d = R_2 + R_1 \| R_L$$

$$R_n = R_2$$

With C_1 replaced:

$$Z_o = R_1 \| R_L \frac{1 + sC_1R_2}{1 + sC_1(R_2 + R_1 \| R_2)}$$

Generalization: Extra Element Theorem - #4

The Extra Element theorem can be used to find an extra element correction factor for any transfer function; Z_d and Z_n are then the driving point impedances seen by the extra element with the "input" zero and with the "output" nulled, respectively.

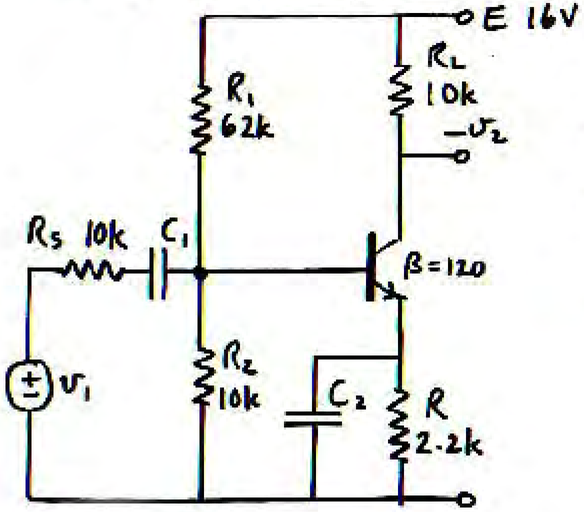
When the transfer function is a self-impedance, such as the input impedance Z_i or the output impedance Z_o, nulling the "output" is the same as shorting the "input," hence

$$Z_d = Z_{dp}\Big|_{\text{"input" zero}} = Z_{dp}\Big|_{\text{"input" open}}$$

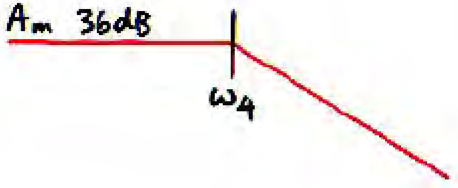
$$Z_n = Z_{dp}\Big|_{\text{"output" nulled}} = Z_{dp}\Big|_{\text{"input" shorted}}$$

1CE: The basic Common-Emitter amplifier stage

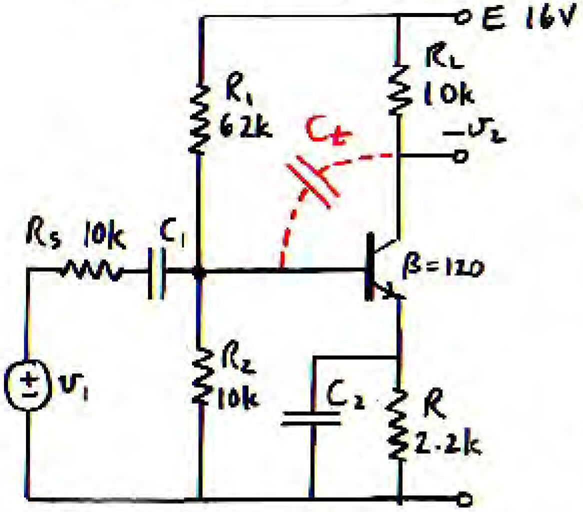
High-frequency properties of CE amplifier



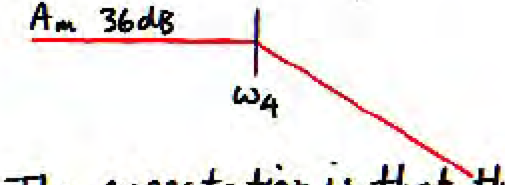
Measurement indicates that there is a high-frequency pole ω_4:



High-frequency properties of CE amplifier



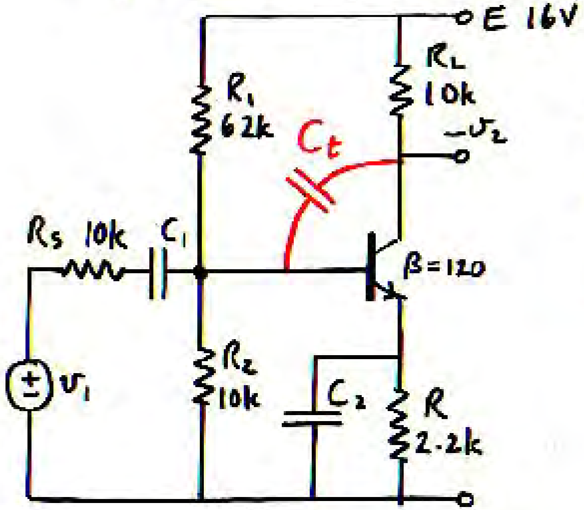
Measurement indicates that there is a high-frequency pole ω_4:



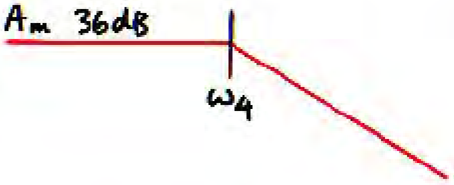
The expectation is that this is caused by the collector-base transition-layer capacitance C_t.

A typical value is $C_t = 5pF$. The resulting corner frequency with $R_L = 10k$ is $159/5\times 10^{-6}\times 10 = 3.2MHz$. Since the actual corner frequency is much lower, there must be a multiplying effect on C_t resulting from its connection to the transistor base instead of to ground.

High-frequency properties of CE amplifier

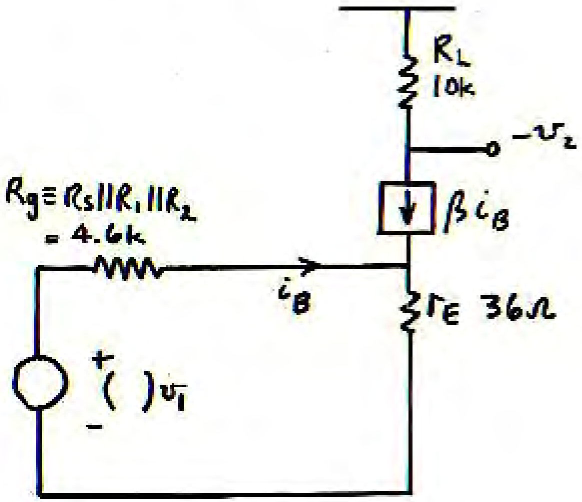


Measurement indicates that there is a high-frequency pole ω_4:

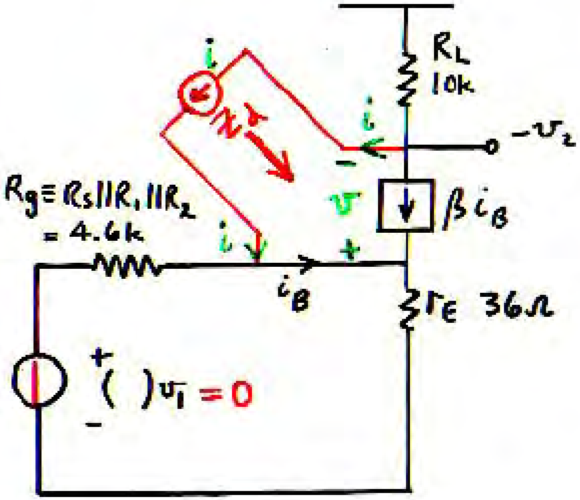


Since the midband gain $A_m = 36dB$ has already been determined, use the Extra Element Theorem to find the correction factor resulting from inclusion of $Z \rightarrow 1/sC_t$.

Midband model after Thevenin reduction of R_s, R_1, R_2:

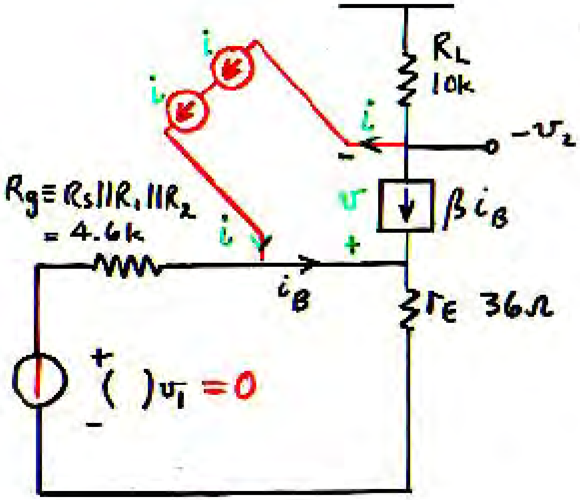


Midband model after Thevenin reduction of R_s, R_1, R_2:



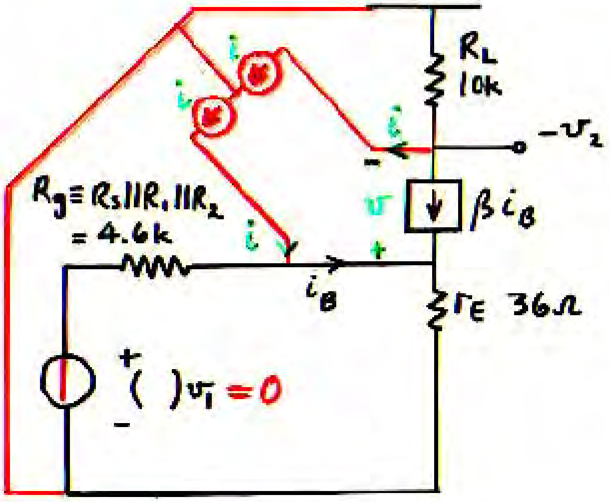
$$Z_d = \left.\frac{v}{i}\right|_{v_1 = 0}$$

Midband model after Thevenin reduction of R_s, R_1, R_2:



The current generator i can be divided into two equal current generators in series.

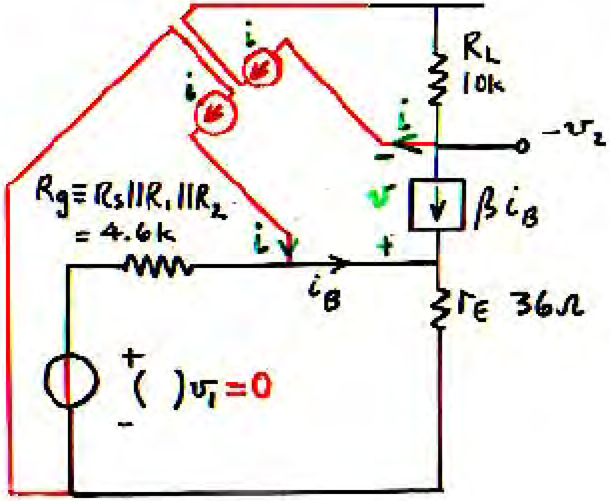
Midband model after Thevenin reduction of R_s, R_1, R_2:



The current generator i can be divided into two equal current generators in series.

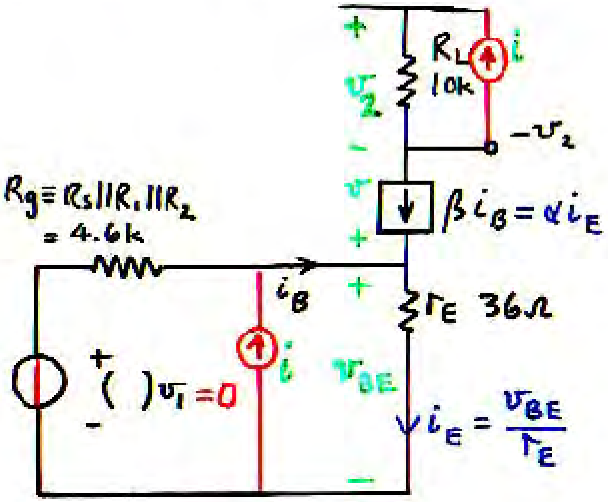
Since the voltage at the junction of the two current generators i is immaterial, the junction can be grounded.

Midband model after Thevenin reduction of R_s, R_1, R_2:



A separate ground can be identified for each current generator i.

Midband model after Thevenin reduction of R_s, R_1, R_2:



Rearranged diagram.

$$Z_d = R_d = \frac{v}{i} = \frac{v_{BE}}{i} + \frac{v_2}{i}$$

$$v_{BE} = \left[R_g \| (1+\beta) r_E\right] i$$

$$v_2 = R_L\left(\alpha i_E + i\right) = R_L\left(\frac{\alpha}{r_E} v_{BE} + i\right)$$

$$R_d = \frac{v_{BE}}{i} + R_L\left(\frac{\alpha}{r_E}\frac{v_{BE}}{i} + 1\right)$$

$$R_d = \left(1 + \frac{\alpha R_L}{r_E}\right)\left[R_g \| (1+\beta) r_E\right] + R_L = R_L\left[R_g \| (1+\beta) r_E\right]\left[\frac{1}{R_L} + \frac{\alpha}{r_E} + \frac{1}{R_g \| (1+\beta) r_E}\right]$$

$$= R_L\left[R_g \| (1+\beta) r_E\right]\left[\frac{1}{R_L} + \frac{\beta}{(1+\beta) r_E} + \frac{1}{R_g} + \frac{1}{(1+\beta) r_E}\right]$$

$$= \frac{R_g \| (1+\beta) r_E}{R_g \| r_E \| R_L} R_L = \frac{4.6 \| 4.3}{4.6 \| 0.036 \| 10} R_L = 62 R_L$$

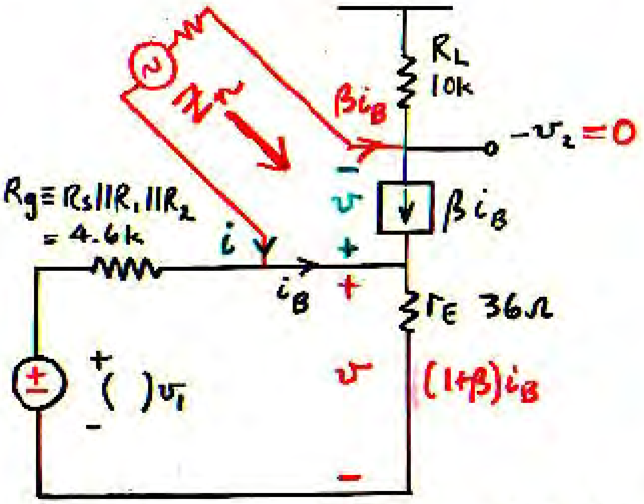
$$= 620k$$

Generalization: Floating Current Generator

A floating current generator can be replaced by two separate, equal, grounded current generators.

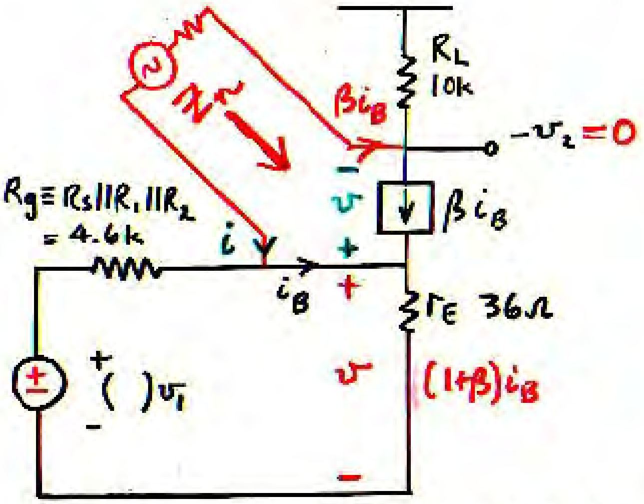
This is a useful technique in "doing the algebra on the circuit diagram."

Midband model after Thevenin reduction of R_s, R_1, R_2:



$$Z_n = R_n = \frac{v}{i} = \frac{(1+\beta)r_E\, i_B}{-\beta i_B} = -\frac{r_E}{\alpha} = -36\Omega$$

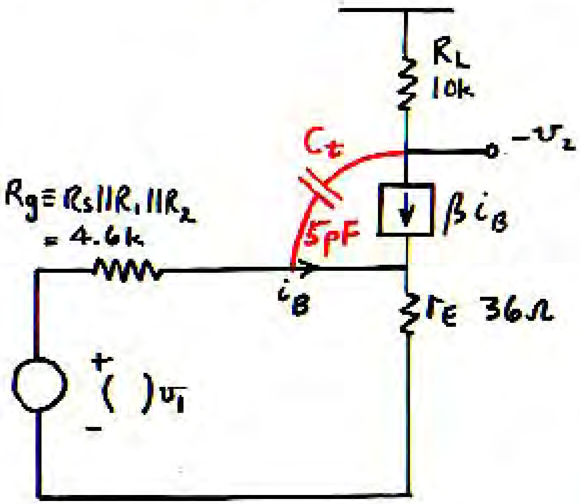
Midband model after Thevenin reduction of R_s, R_1, R_2:



$$Z_n = R_n = \frac{v}{i} = \frac{(1+\beta) r_E\, i_B}{-\beta i_B} = -\frac{r_E}{\alpha} = -36\Omega$$

Note that the Z_n calculation is much shorter and easier than the Z_d calculation!

Midband model after Thevenin reduction of R_s, R_1, R_2:



Hence corrected gain after inclusion of C_t is

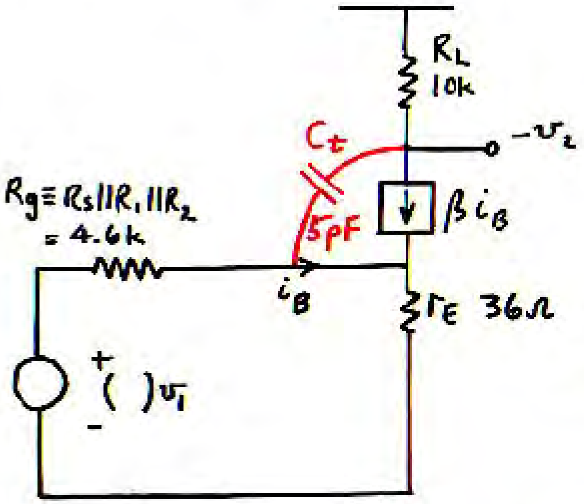
$$A = A_m \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} = A_m \frac{1 + sC_tR_n}{1 + sC_tR_d}$$

$$= A_m \frac{1 - \frac{s}{\omega_5}}{1 + \frac{s}{\omega_4}}$$ where

$$\omega_4 \equiv \frac{1}{C_tR_d} \qquad f_4 = \frac{159}{5\times10^{-6}\times620} = 51\text{kHz}$$

$$\omega_5 \equiv \frac{1}{C_tR_n} \qquad f_5 = \frac{159}{5\times10^{-6}\times0.036} = 880\text{MHz}$$

Midband model after Thevenin reduction of R_s, R_1, R_2:



Hence corrected gain after inclusion of C_t is

$$A = A_m \frac{1 + \frac{Z_n}{Z}}{1 + \frac{Z_d}{Z}} = A_m \frac{1 + sC_tR_n}{1 + sC_tR_d}$$

$$= A_m \frac{1 - \frac{s}{\omega_5}}{1 + \frac{s}{\omega_4}} \quad \text{where}$$

$$\omega_4 \equiv \frac{1}{C_tR_d} \qquad f_4 = \frac{159}{5 \times 10^{-6} \times 620} = 51\text{kHz}$$

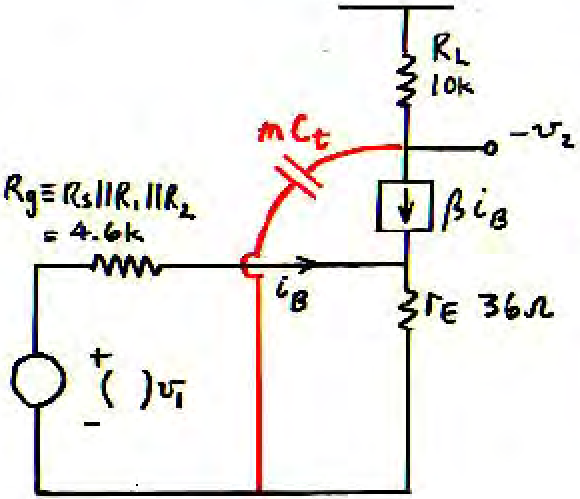
$$\omega_5 \equiv \frac{1}{C_tR_n} \qquad f_5 = \frac{159}{5 \times 10^{-6} \times 0.036} = 880\text{MHz}$$

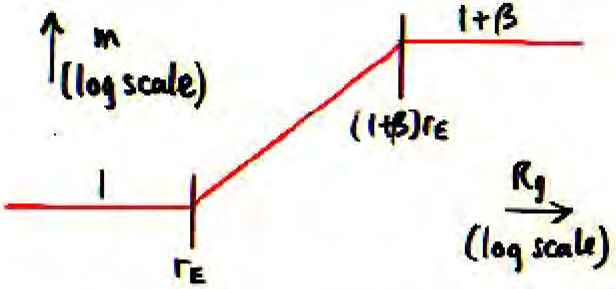
Note that the zero $\omega_5 = \frac{1}{C_tR_n} = \frac{\alpha}{C_tr_E}$ is negative (right half-plane), and is at a very high frequency unless there is substantial external emitter resistance and/or there is substantial external collector-base capacitance (as often exists).

Note that the pole $\omega_4 = \frac{1}{C_t R_d} = \frac{1}{C_t R_L} \frac{R_g \| r_E \| R_L}{R_g \| (1+\beta) r_E}$ is at a much lower frequency than $\omega_6 \equiv \frac{1}{C_t R_L}$, and can be ascribed to an effective multiplication of C_t by a factor

$$m \equiv \frac{R_g \| (1+\beta) r_E}{R_g \| r_E \| R_L} = \frac{R_g \| (1+\beta) r_E}{R_g \| r_E} \left(1 + \cancel{\frac{R_g \| r_E}{R_L}} \right) \begin{array}{l} \xrightarrow{R_g \gg (1+\beta) r_E} 1+\beta \\ \xrightarrow[R_g \ll r_E]{} 1 \end{array}$$

Midband model after Thevenin reduction of R_s, R_1, R_2:





This example: m = 62

Alternative method for calculation of Z_d

There are two forms of the Extra Element theorem:

$$A = A\big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = A\big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

where $$A\big|_{Z=0} = \frac{Z_n}{Z_d} A\big|_{Z=\infty}$$

Hence in general

$$\frac{A\big|_{Z=0}}{A\big|_{Z=\infty}} = \frac{Z_n}{Z_d}$$

It may be easier to find $A\big|_{Z=0}$, $A\big|_{Z=\infty}$, and Z_n than to find Z_d directly.

Example: Addition of collector-base capacitance C_t to the CE amplifier stage. $A\big|_{Z=\infty}$ and Z_n were easily found:

$$A\big|_{Z=\infty} = A_m = \frac{R_B}{R_S+R_B} \cdot \frac{\beta R_L}{R_g+(1+\beta) r_E} \qquad Z_n = R_n = -\frac{r_E}{\alpha}$$

The Extra Element Theorem as derived applies to the correction factor resulting from an extra **shunt** element.

There is a corresponding form to find the correction factor resulting from an extra **series** element:

$$\text{gain}\Big|_{Z} = \overset{\substack{\text{reference gain}\\ \downarrow}}{\text{gain}}\Big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = \text{gain}\Big|_{Z=\infty} \frac{\frac{Z_n}{Z}}{\frac{Z_d}{Z}} \frac{\frac{Z}{Z_n}+1}{\frac{Z}{Z_d}+1}$$

$$= \overset{\substack{\text{reference gain}\\ \downarrow}}{\text{gain}}\Big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}} = \boxed{\frac{Z_n}{Z_d}\,\text{gain}\Big|_{Z=\infty}} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

This must be the gain when $Z=0$

Alternative method for calculation of Z_d

There are two forms of the Extra Element theorem:

$$A = A\Big|_{Z=\infty} \frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}} = A\Big|_{Z=0} \frac{1+\frac{Z}{Z_n}}{1+\frac{Z}{Z_d}}$$

where $$A\Big|_{Z=0} = \frac{Z_n}{Z_d} A\Big|_{Z=\infty}$$

Hence in general

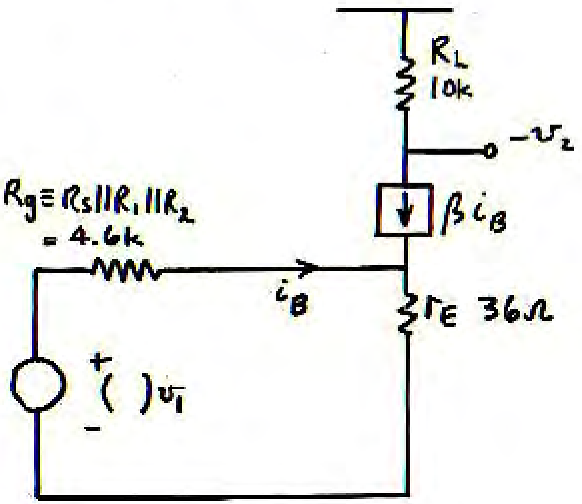
$$\frac{A\big|_{Z=0}}{A\big|_{Z=\infty}} = \frac{Z_n}{Z_d}$$

It may be easier to find $A\big|_{Z=0}$, $A\big|_{Z=\infty}$, and Z_n than to find Z_d directly.

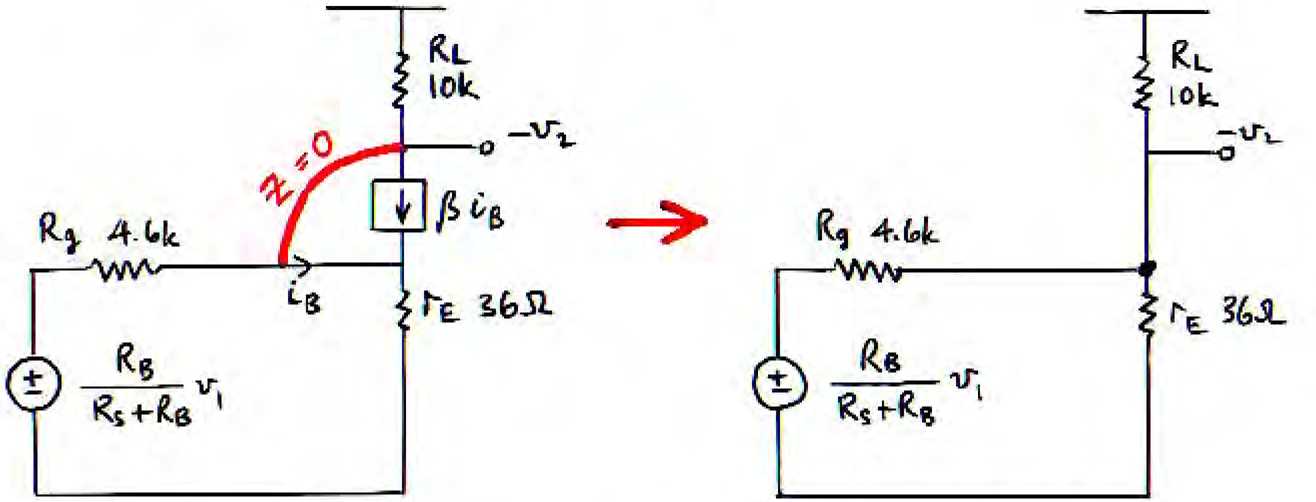
Example: Addition of collector-base capacitance C_t to the CE amplifier stage. $A\big|_{Z=\infty}$ and Z_n were easily found:

$$A\Big|_{Z=\infty} = A_m = \frac{R_B}{R_s+R_B} \cdot \frac{\beta R_L}{R_g+(1+\beta)r_E} \qquad Z_n = R_n = -\frac{r_E}{\alpha}$$

Midband model after Thevenin reduction of R_s, R_1, R_2:



Model for calculation of $A|_{Z=0}$



$$A|_{Z=0} = -\frac{R_B}{R_s+R_B}\,\frac{r_E \| R_L}{R_g + r_E \| R_L} = -\frac{R_B}{R_s+R_B}\,\frac{R_g \| r_E \| R_L}{R_g}$$

Hence: $$Z_d = R_d = R_n \frac{A|_{Z=\infty}}{A|_{Z=0}} = \frac{r_E}{\alpha}\,\frac{\beta R_L}{R_g+(1+\beta)r_E}\,\frac{R_g}{R_g \| r_E \| R_L} = \frac{R_g \| (1+\beta) r_E}{R_g \| r_E \| R_L} R_L$$

This is much easier than was the direct calculation of Z_d!

Generalization: Extra Element theorem — #5

The two reference gains and the two driving point impedances are related by:

$$\frac{A|_{Z=0}}{A|_{Z=\infty}} = \frac{Z_n}{Z_d}$$

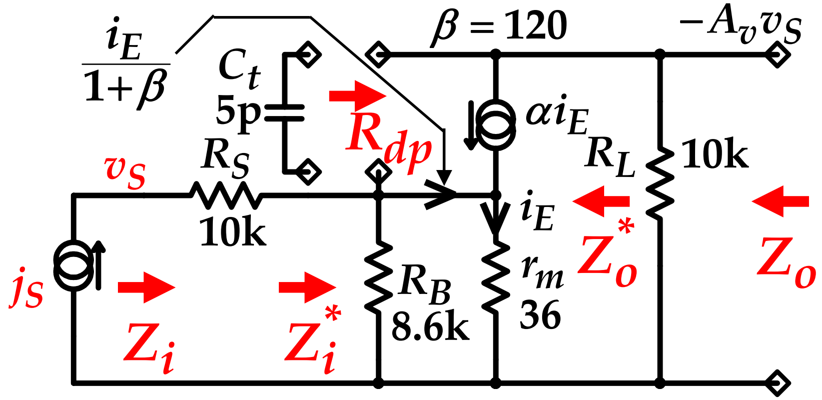
One reference gain is always known or is easily found.
Z_n is always easier to find than Z_d.

Therefore:

It is often easier to find the other reference gain and to use the above ratio relation for Z_d, than to find Z_d directly.

Common-emitter (1CE) amplifier stage

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



Previous results:

$$A_v = A_{vm}\frac{1 - s/\omega_z}{1 + s/\omega_p} = 36dB\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S + R_B}\frac{\alpha R_L}{r_m + \frac{R_S \| R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} = 62$$

Outer input impedance Z_i :

Use the parallel EET with reference value $Z \equiv 1/sC_t$ infinite:

$$Z_i \equiv \frac{v_S}{j_S} = R_{im}\frac{1 + sC_t R_{ni}}{1 + sC_t R_{di}}$$

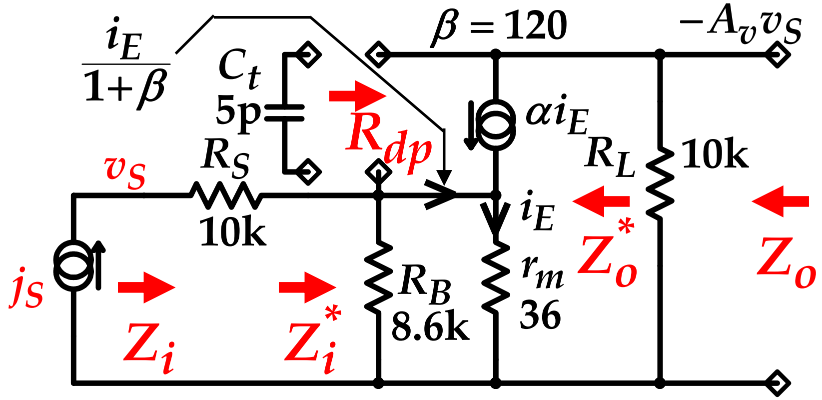
$$R_{im} \equiv R_S + R_B \| (1+\beta) r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega$$

$R_{ni} = R_{dp}$ **with output v_S nulled**

$= R_{dp}$ **with input j_S shorted**

$= R_{dp}$ **for the voltage gain A**

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



Previous results:

$$A_v = A_{vm}\frac{1-s/\omega_z}{1+s/\omega_p} = 36dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S+R_B}\frac{\alpha R_L}{r_m+\frac{R_S\|R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L} = 62$$

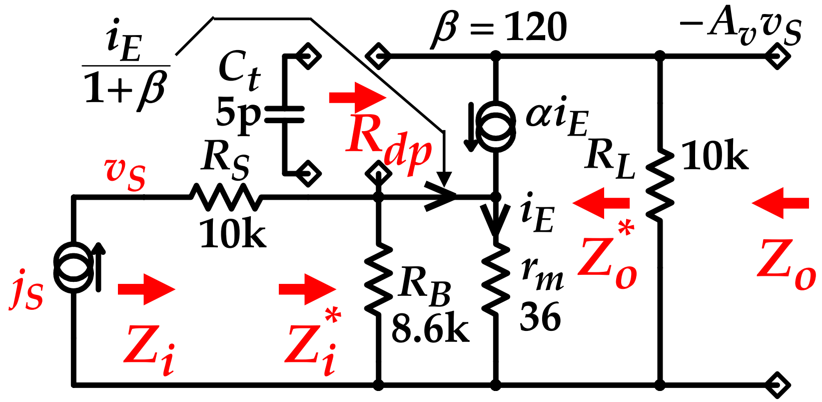
Outer input impedance Z_i :

$$Z_i \equiv \frac{v_S}{j_S} = R_{im}\frac{1+sC_tR_{ni}}{1+sC_tR_{di}}$$

$$R_{im} \equiv R_S + R_B\|(1+\beta)r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega$$

$$R_{ni} = mR_L \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L}R_L = 620k$$

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



Previous results:

$$A_v = A_{vm}\frac{1-s/\omega_z}{1+s/\omega_p} = 36dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S+R_B}\,\frac{\alpha R_L}{r_m+\frac{R_S\|R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L} = 62$$

Outer input impedance Z_i :

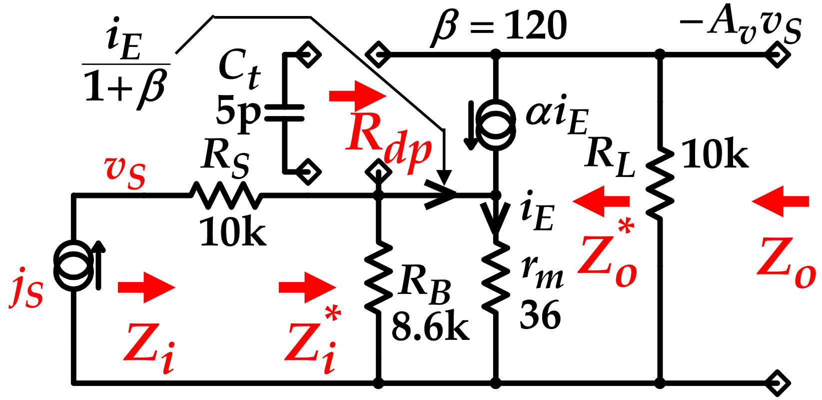
$$Z_i \equiv \frac{v_S}{j_S} = R_{im}\frac{1+sC_tR_{ni}}{1+sC_tR_{di}}$$

$$R_{im} \equiv R_S + R_B\|(1+\beta)r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega$$

$$R_{ni} = mR_L \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L}R_L = 620k$$

$R_{di} = R_{dp}$ with input j_S open

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



Previous results:

$$A_v = A_{vm}\frac{1 - s/\omega_z}{1 + s/\omega_p} = 36dB\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S + R_B}\frac{\alpha R_L}{r_m + \frac{R_S \| R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} = 62$$

Outer input impedance Z_i :

$$Z_i \equiv \frac{v_S}{j_S} = R_{im}\frac{1 + sC_t R_{ni}}{1 + sC_t R_{di}}$$

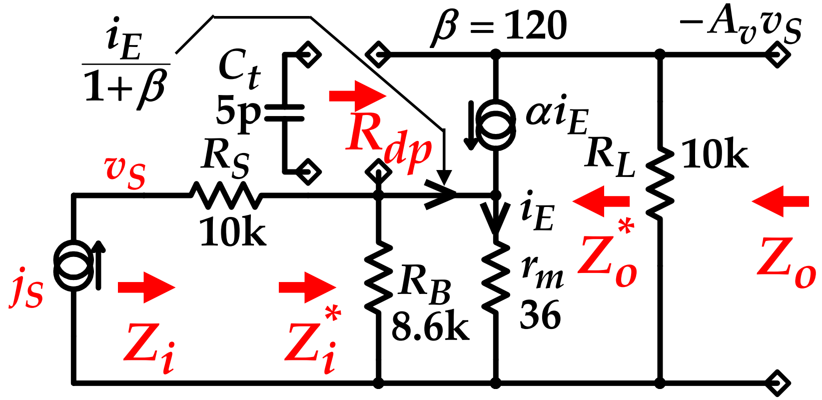
$$R_{im} \equiv R_S + R_B \| (1+\beta) r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega$$

$$R_{ni} = mR_L \equiv \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} R_L = 620k$$

$R_{di} = R_{dp}$ **with input j_S open**

$$= R_{ni}\Big|_{R_S \to \infty} = \frac{R_B \| (1+\beta) r_m}{R_B \| r_m \| R_L} R_L = 820k$$

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



Previous results:

$$A_v = A_{vm}\frac{1 - s/\omega_z}{1 + s/\omega_p} = 36dB\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{51kHz}}$$

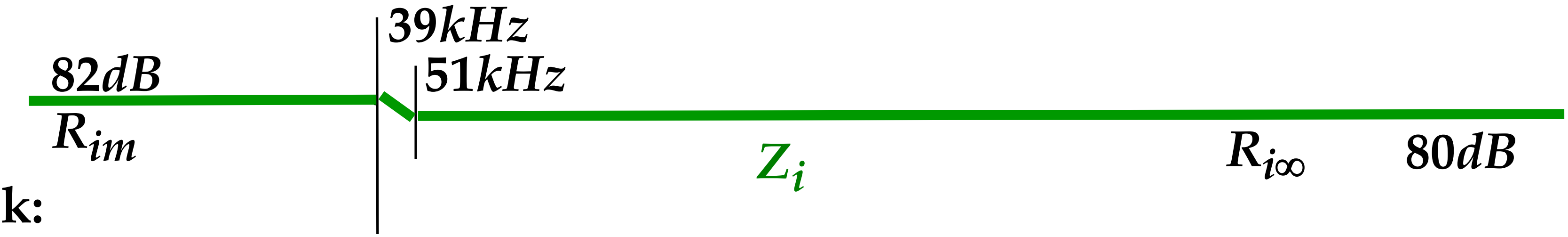
$$A_{vm} \equiv \frac{R_B}{R_S + R_B}\frac{\alpha R_L}{r_m + \frac{R_S \| R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} = 62$$

Outer input impedance Z_i :

$$Z_i \equiv \frac{v_S}{j_S} = R_{im}\frac{1 + sC_t R_{ni}}{1 + sC_t R_{di}} = 82dB\frac{1 + \frac{s/2\pi}{51kHz}}{1 + \frac{s/2\pi}{39kHz}}$$

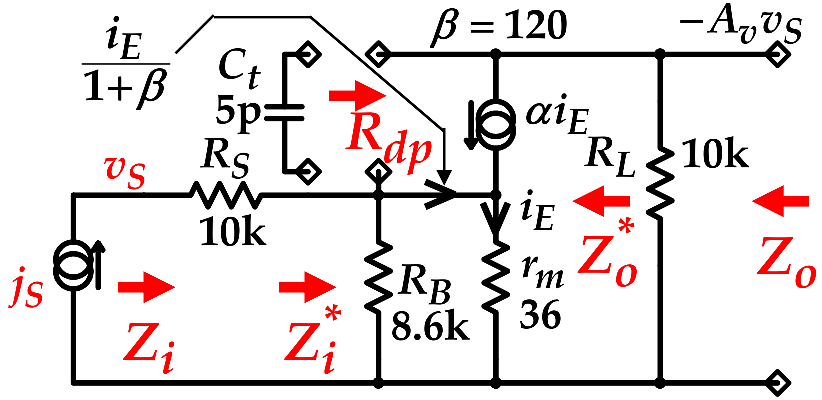


Check:

By GB trade-off: $R_{i\infty} = R_{im}\frac{R_n}{R_d} = 13k\frac{620k}{820k} = 10k \Rightarrow 80dB$

By inspection: $R_{i\infty} = R_S + R_B \| r_m \| R_L = 10k \Rightarrow 80dB$

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



Previous results:

$$A_v = A_{vm}\frac{1-s/\omega_z}{1+s/\omega_p} = 36dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S+R_B}\frac{\alpha R_L}{r_m+\frac{R_S\|R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L} = 62$$

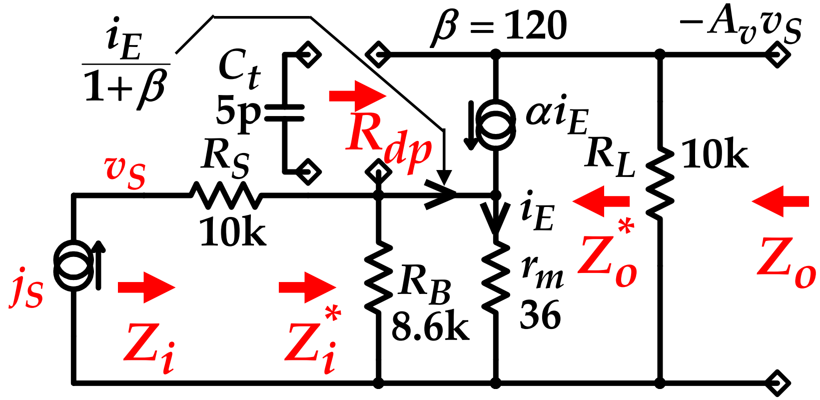
Inner input impedance Z_i^* :

$$Z_i^* = Z_i\big|_{R_S\to 0} = R_{im}\frac{1+sC_tR_{ni}}{1+sC_tR_{di}}\bigg|_{R_S\to 0} = R_{im}^*\frac{1+sC_tR_{ni}^*}{1+sC_tR_{di}}$$

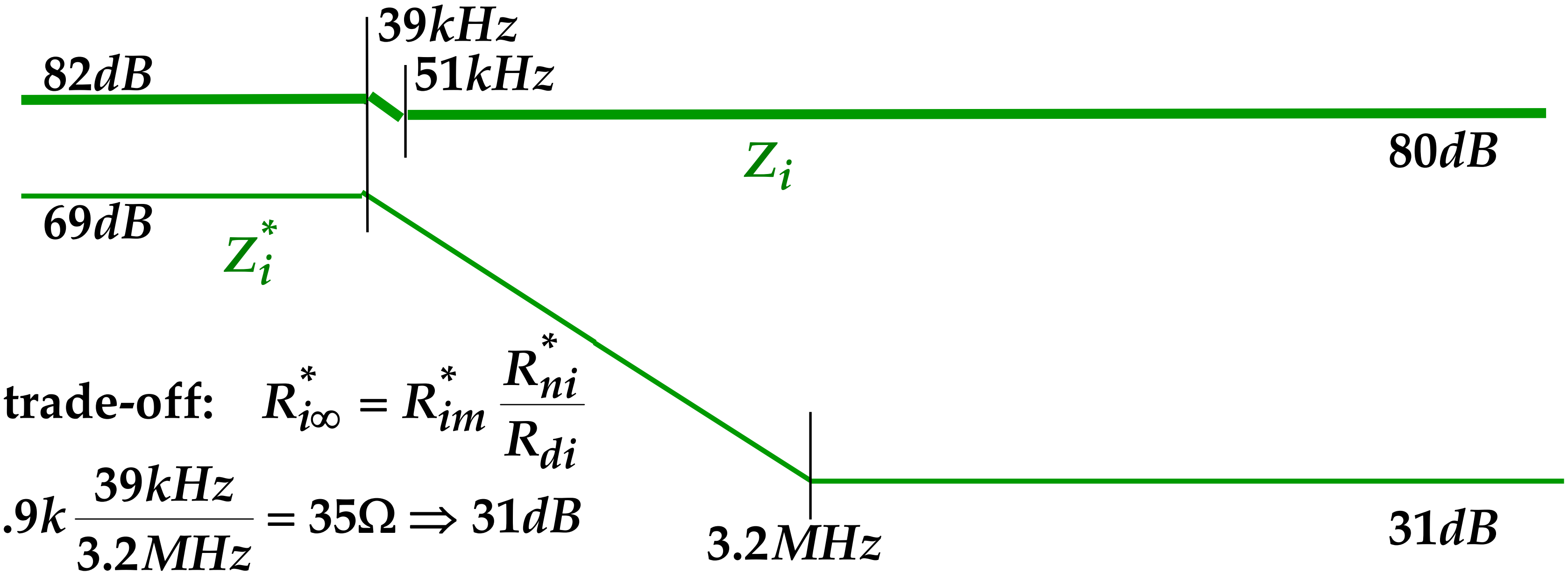
$$R_{im}^* = R_{im}\big|_{R_S\to 0} = R_S + R_B\|(1+\beta)r_m\big|_{R_S\to 0} = R_B\|(1+\beta)r_m = 2.9k \Rightarrow 69dB$$

$$R_{ni}^* = R_{ni}\big|_{R_S\to 0} = \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L}R_L\bigg|_{R_S\to 0} = R_L = 10k \Rightarrow 80dB$$

Use the EET to find the outer and inner input impedances Z_i and Z_i^*



$$Z_i^* = 69dB \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{39kHz}}$$



Check:

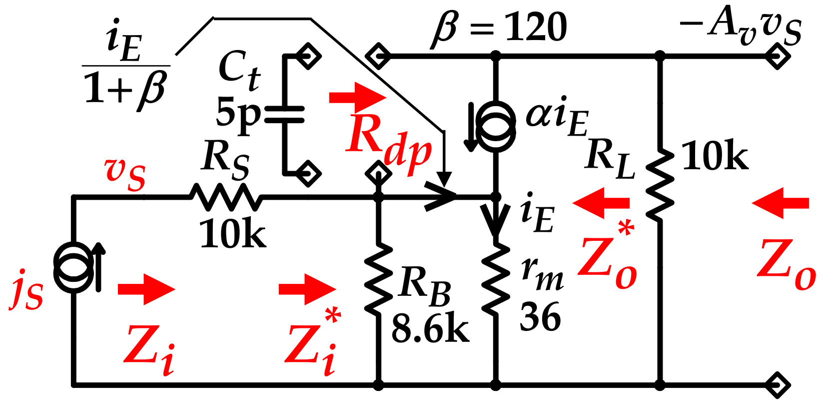
By GB trade-off: $R_{i\infty}^* = R_{im}^* \frac{R_{ni}^*}{R_{di}}$

$$= 2.9k \frac{39kHz}{3.2MHz} = 35\Omega \Rightarrow 31dB$$

By inspection: $R_{i\infty}^* = R_B \| r_m \| R_L = 35\Omega \Rightarrow 31dB$

Exercise 8.5

Use the EET to find Z_o and Z_o^* for the 1CE amplifier stage



Previous results:

$$A_v = A_{vm}\frac{1-s/\omega_z}{1+s/\omega_p} = 36dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S+R_B}\frac{\alpha R_L}{r_m+\frac{R_S\|R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_E/\alpha = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L} = 62$$

Outer output impedance Z_o :

Use the parallel EET with reference value $Z \equiv 1/sC_t$ infinite:

$$Z_o \equiv \frac{v_o}{j_S} = R_{om}\frac{1+sC_tR_{no}}{1+sC_tR_{do}}$$

$$R_{om} \equiv R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega$$

$R_{no} = R_{dp}$ **with output** v_o **nulled** $= R_{dp}$ **with input** j_S **shorted**

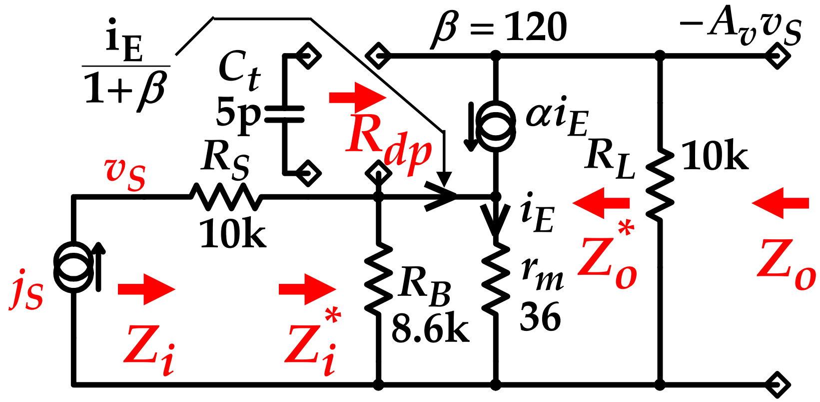
$$= R_S\|R_B\|(1+\beta)r_m = 2.2k$$

$R_{do} = R_{dp}$ **with input** j_S **open** $= R_d$ **for the voltage gain** A

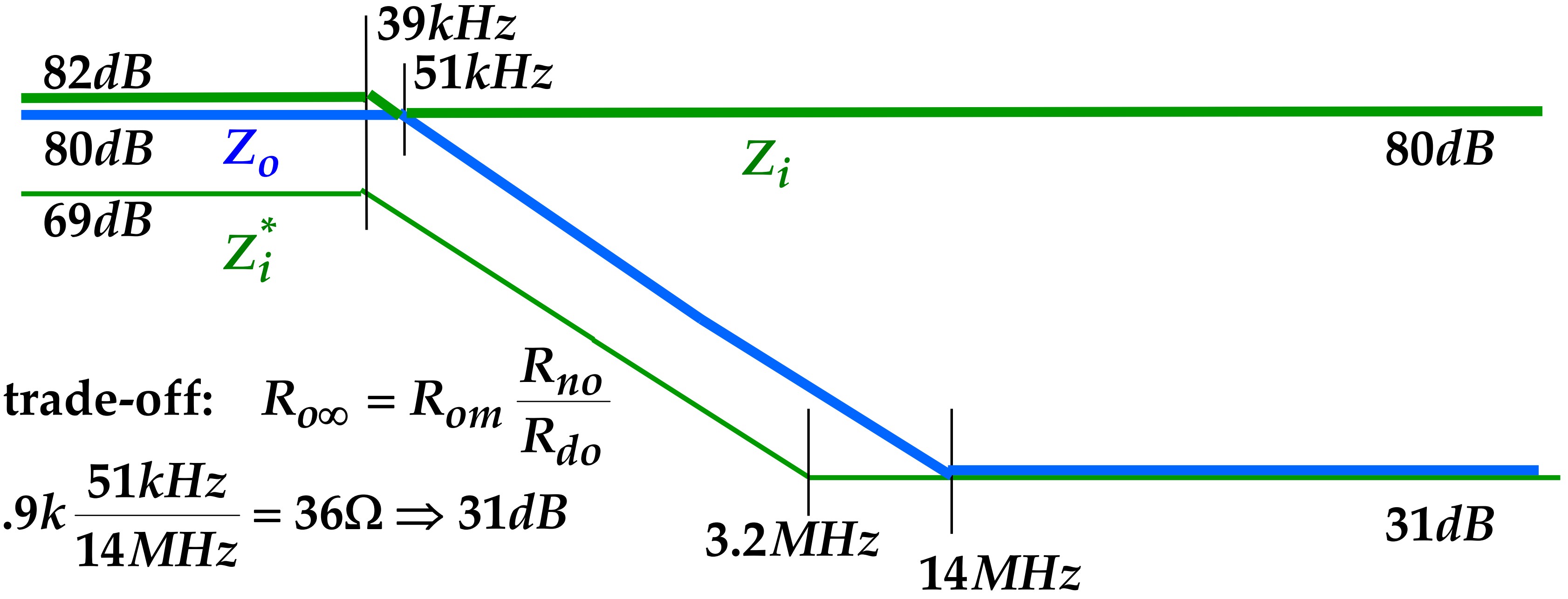
$$= 620k$$

Exercise 8.5: - Solution

Use the EET to find Z_o and Z_o^* for the 1CE amplifier stage



$$Z_o = 80dB \frac{1 + \frac{s/2\pi}{14MHz}}{1 + \frac{s/2\pi}{51kHz}}$$



Check:

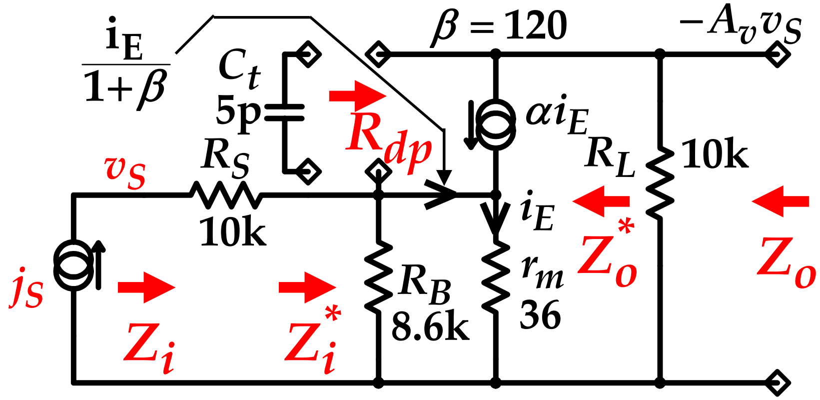
By GB trade-off: $R_{o\infty} = R_{om}\frac{R_{no}}{R_{do}}$

$= 2.9k\frac{51kHz}{14MHz} = 36\Omega \Rightarrow 31dB$

By inspection: $R_{o\infty} = R_S \| R_B \| r_m \| R_L = 36\Omega \Rightarrow 31dB$

Exercise 8.5: - Solution

Use the EET to find Z_o and Z_o^* for the 1CE amplifier stage



$$R_{om} \equiv R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega$$

$$R_{no} = R_S \| R_B \| (1+\beta) r_m = 2.2k$$

$$R_{do} \equiv \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} R_L = 620k$$

Inner output impedance Z_o^* :

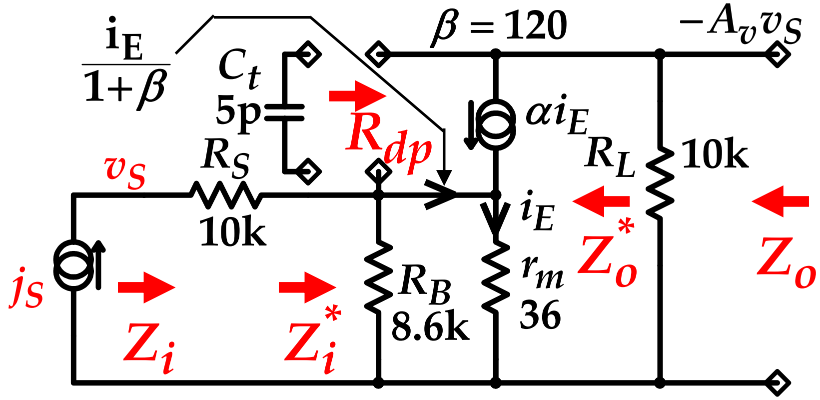
$$Z_o^* = Z_o\Big|_{R_L\to\infty} = R_{om} \frac{1+sC_tR_{no}}{1+sC_tR_{do}}\Bigg|_{R_L\to\infty} = R_{om}^* \frac{1+sC_tR_{no}}{1+sC_tR_{do}^*}$$

$$R_{om}^* = R_{om}\Big|_{R_L\to\infty} = R_L\Big|_{R_L\to\infty} = \infty$$

$$R_{do}^* = R_{do}\Big|_{R_L\to\infty} = \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} R_L\Bigg|_{R_L\to\infty} = \infty$$

Exercise 8.5: - Solution

Use the EET to find Z_o and Z_o^* for the 1CE amplifier stage



$$R_{om} \equiv R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega$$

$$R_{no} = R_S \| R_B \| (1+\beta) r_m = 2.2k$$

$$R_{do} \equiv \frac{R_S \| R_B \| (1+\beta) r_m}{R_S \| R_B \| r_m \| R_L} R_L = 620k$$

Inner output impedance Z_o^* :

Because R_{om}^* and R_{do}^* both are infinite, change the Z_o reference value from R_{om} to $R_{o\infty}$. Then:

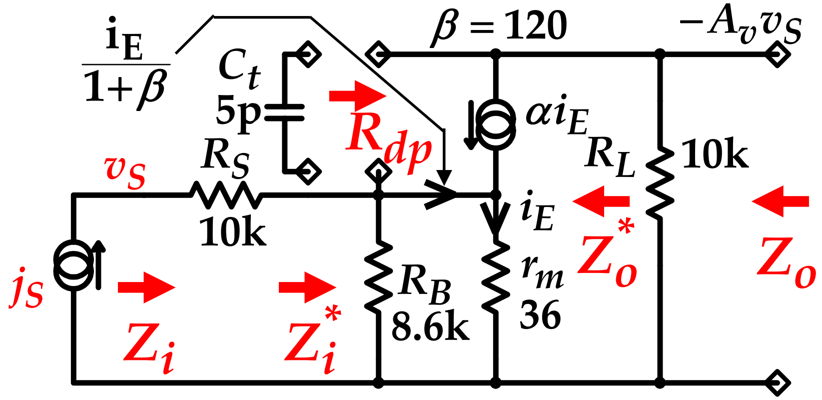
$$Z_o^* = Z_o\Big|_{R_L\to\infty} = R_{om} \frac{1 + sC_t R_{no}}{1 + sC_t R_{do}}\Bigg|_{R_L\to\infty} = R_{o\infty} \frac{1 + 1/sC_t R_{no}}{1 + 1/sC_t R_{do}}\Bigg|_{R_L\to\infty}$$

$$= R_{o\infty}^* \left(1 + 1/sC_t R_{no}\right)$$

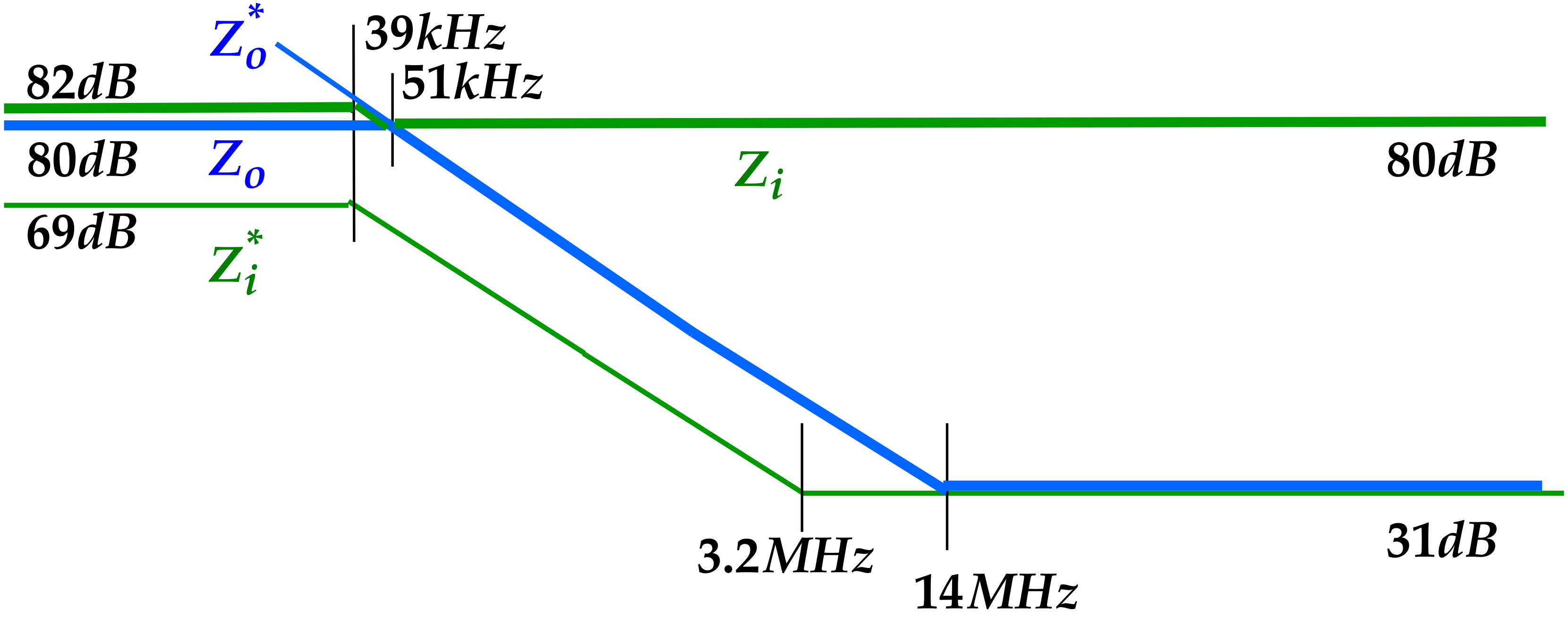
$$R_{o\infty}^* = R_{om} \frac{R_{no}}{R_{do}}\Bigg|_{R_L\to\infty} = R_S \| R_B \| r_m \| R_L \Big|_{R_L\to\infty} = R_S \| R_B \| r_m = 36\Omega \Rightarrow 31dB$$

Exercise 8.5: - Solution

Use the EET to find Z_o and Z_o^* for the 1CE amplifier stage



$$Z_o^* = 31dB\left(1+\frac{14MHz}{s/2\pi}\right)$$



9. THE DT AND THE CT:

The Dissection Theorem and the Chain Theorem

How to find the gain of a multistage amplifier as the product of separately calculated low entropy factors

Null Double Injection (ndi)

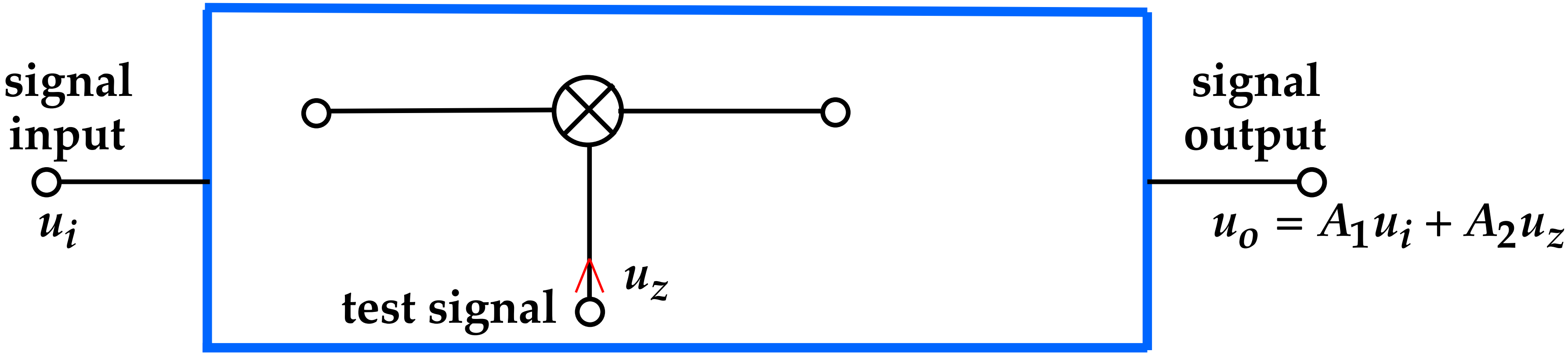
Usually, a transfer function (TF) is calculated as a response to a single independent excitation.

However, large analysis benefits accrue when certain constraints are imposed on several excitations present simultaneously.

For any linear system model:

The input is an independent signal, the output is a proportional dependent signal.

Consider a second input, an injected "test signal" u_z :



signal input

u_i

test signal

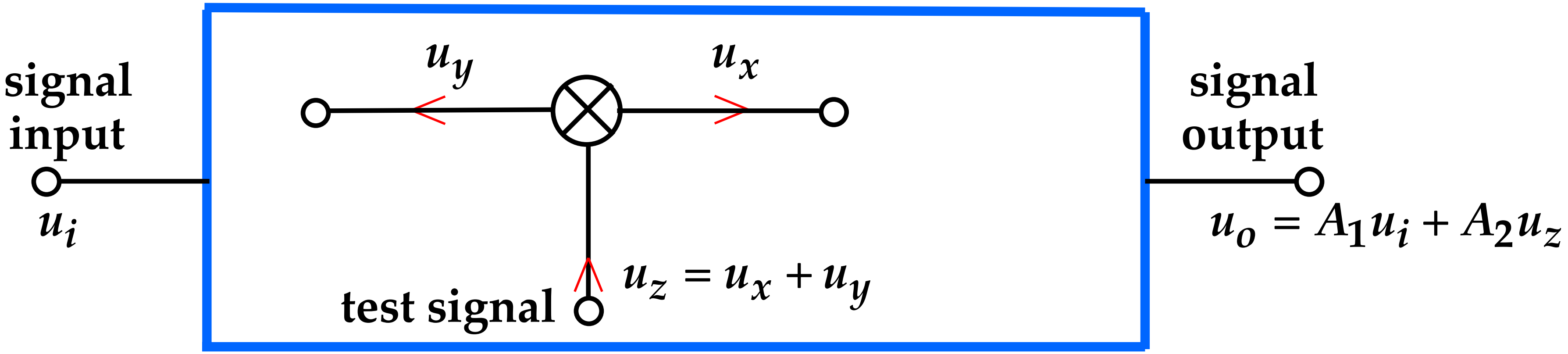
u_z

signal output

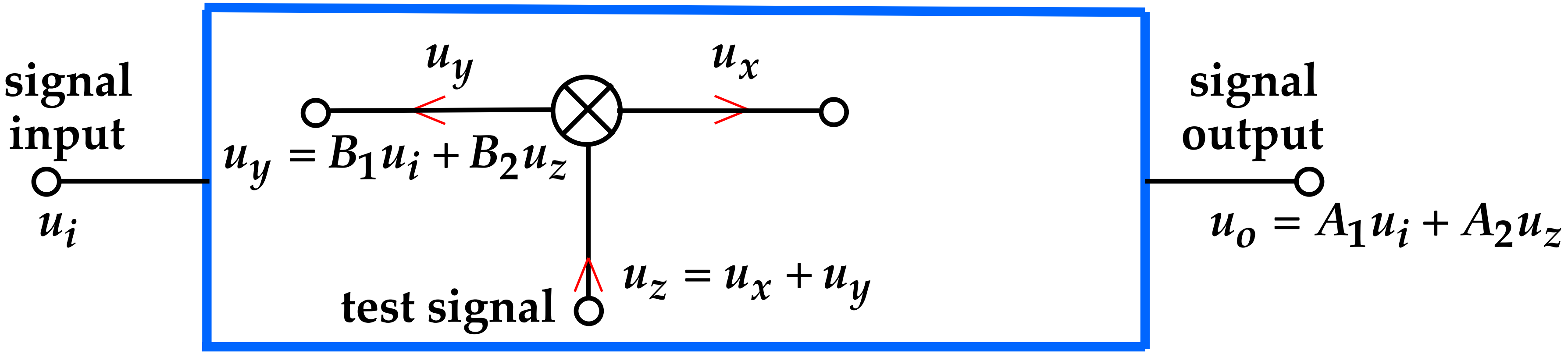
$u_o = A_1 u_i + A_2 u_z$

Since the model is linear, the output is now a linear sum of the values it would have with each input alone.

There are now two more dependent signals, u_x and u_y, where $u_x + u_y = u_z$:

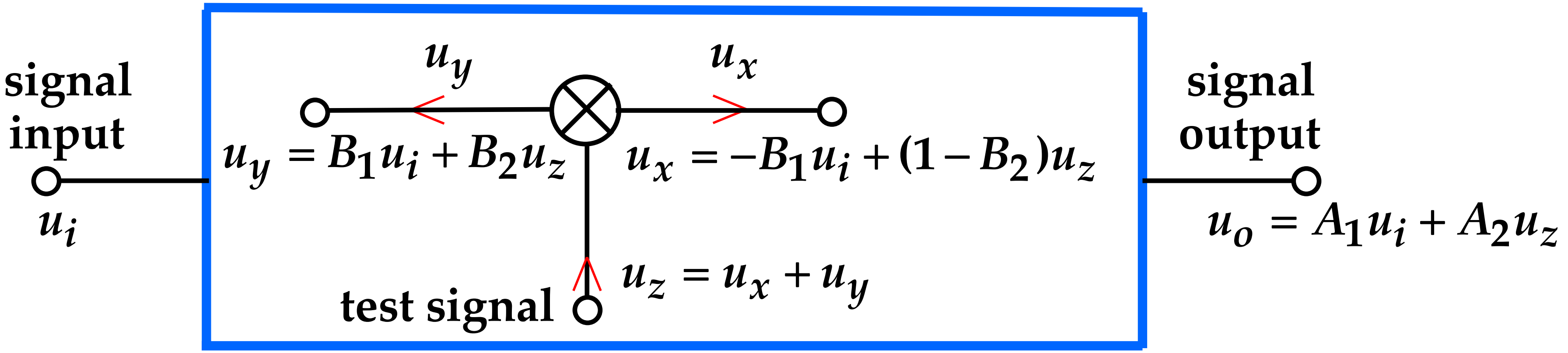


There are now two more dependent signals, u_x and u_y, where $u_x + u_y = u_z$:



The dependent signal u_y is also a linear sum of the values it would have with each input alone.

There are now two more dependent signals, u_x and u_y, where $u_x + u_y = u_z$:

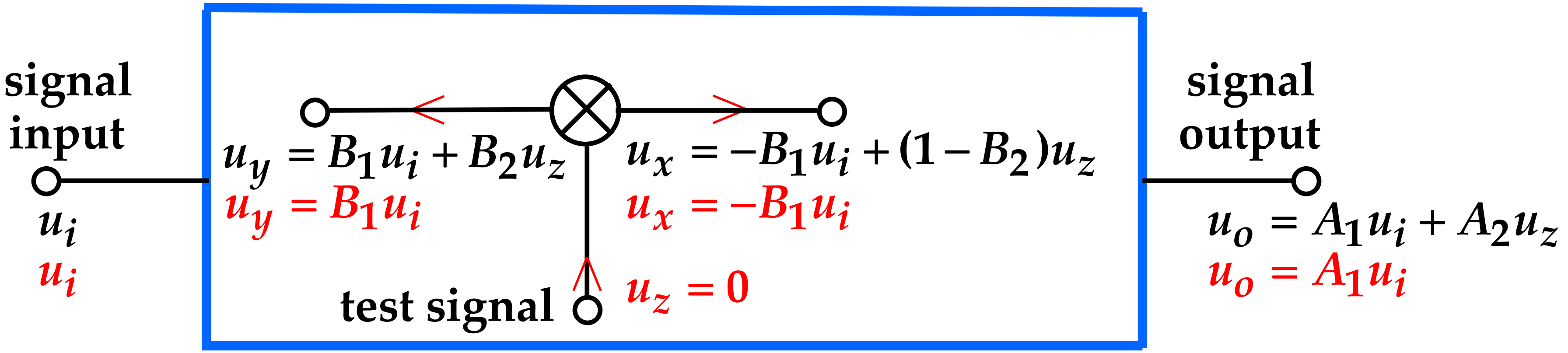


The dependent signal u_y is also a linear sum of the values it would have with each input alone.

By virtue of $u_z = u_x + u_y$, the independent signal u_x can also be expressed in terms of B_1 and B_2.

Several transfer functions (TFs) can be defined:

Special case 1: $u_z = 0$



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$u_y = B_1 u_i$

test signal

$u_x = -B_1 u_i + (1 - B_2) u_z$

$u_x = -B_1 u_i$

$u_z = 0$

signal output

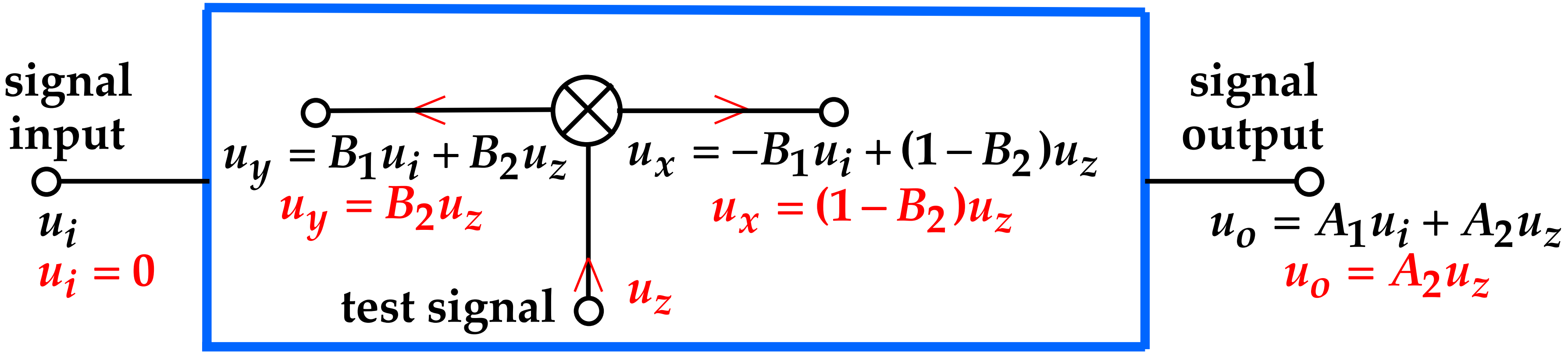
$u_o = A_1 u_i + A_2 u_z$

$u_o = A_1 u_i$

$$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0} = A_1$$

Several transfer functions (TFs) can be defined:

Special case 2: $u_i = 0$



signal input

u_i

$u_i = 0$

$u_y = B_1 u_i + B_2 u_z$

$u_y = B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$u_x = (1 - B_2) u_z$

test signal

u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$u_o = A_2 u_z$

$$H \equiv \left. \frac{u_o}{u_i} \right|_{u_z = 0} = A_1$$

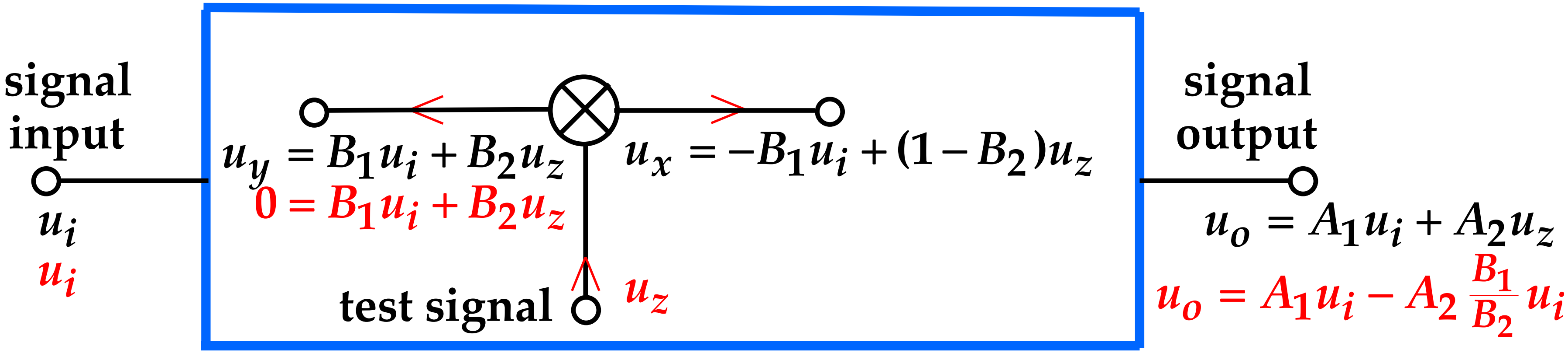
$$T \equiv \left. \frac{u_y}{u_x} \right|_{u_i = 0} = \frac{B_2}{1 - B_2}$$

These are *single injection* (si) TFs

Several transfer functions (TFs) can be defined:

Special case 3: $u_y = 0$

The two independent signals u_i and u_z can be mutually adjusted to null u_y



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$0 = B_1 u_i + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

test signal u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$u_o = A_1 u_i - A_2 \frac{B_1}{B_2} u_i$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}$$

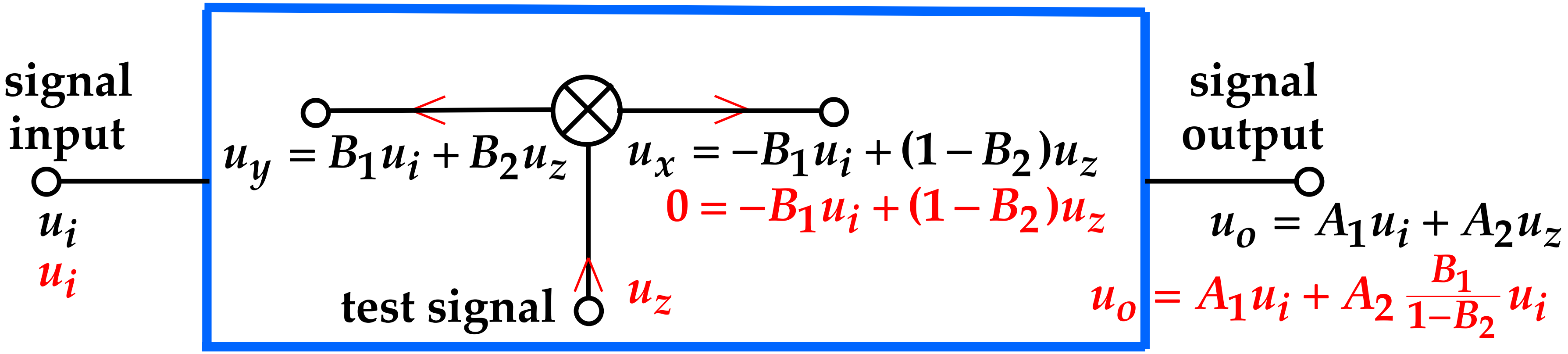
component of u_o from u_i

component of u_o from u_z
adjusted to null u_y

Several transfer functions (TFs) can be defined:

Special case 4: $u_x = 0$

The two independent signals u_i and u_z can be mutually adjusted to null u_x



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$0 = -B_1 u_i + (1 - B_2) u_z$

test signal u_z

u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$u_o = A_1 u_i + A_2 \frac{B_1}{1-B_2} u_i$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} = A_1 + A_2 \frac{B_1}{1 - B_2}$$

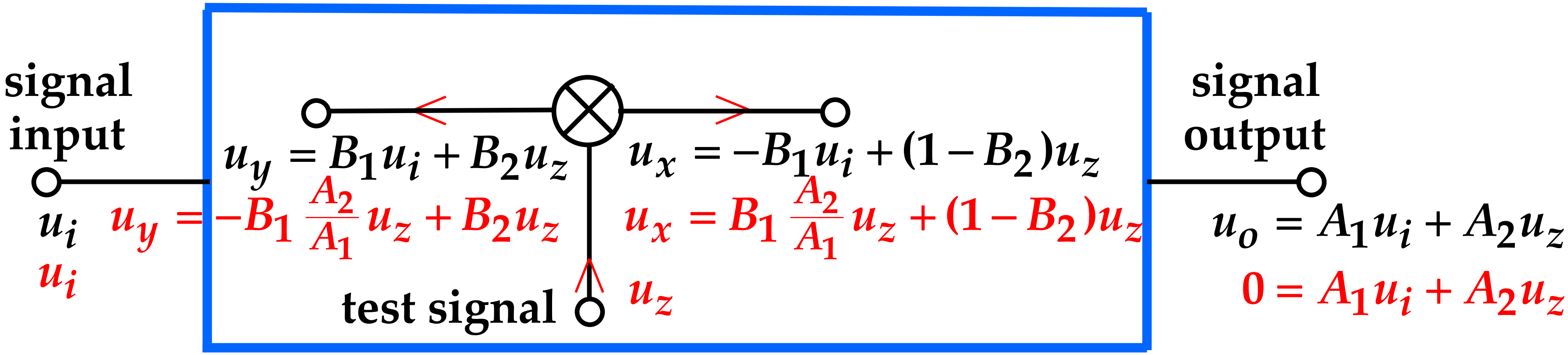
component of u_o from u_i

component of u_o from u_z
adjusted to null u_x

Several transfer functions (TFs) can be defined:

Special case 5: $u_o = 0$

The two independent signals u_i and u_z can be mutually adjusted to null u_o



signal input

u_i

u_i

$u_y = B_1 u_i + B_2 u_z$

$u_y = -B_1 \frac{A_2}{A_1} u_z + B_2 u_z$

$u_x = -B_1 u_i + (1 - B_2) u_z$

$u_x = B_1 \frac{A_2}{A_1} u_z + (1 - B_2) u_z$

test signal

u_z

signal output

$u_o = A_1 u_i + A_2 u_z$

$0 = A_1 u_i + A_2 u_z$

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2 \frac{B_1}{B_2}$$

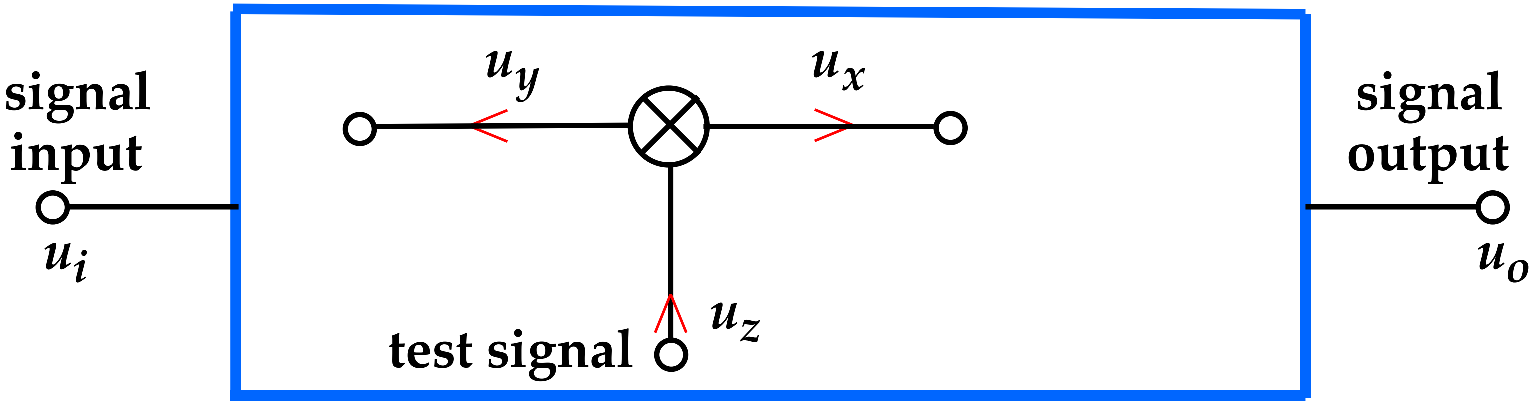
$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} = A_1 + A_2 \frac{B_1}{1 - B_2}$$

$$T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0} = \frac{A_1 B_2 - A_2 B_1}{A_1 - (A_1 B_2 - A_2 B_1)}$$

These are *null double injection* (ndi) TFs

Assembled results, so far:

First level TF: $$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0} = A_1$$ **(si)**



Second level TFs:

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} = A_1 - A_2\frac{B_1}{B_2}$$ **(ndi)**

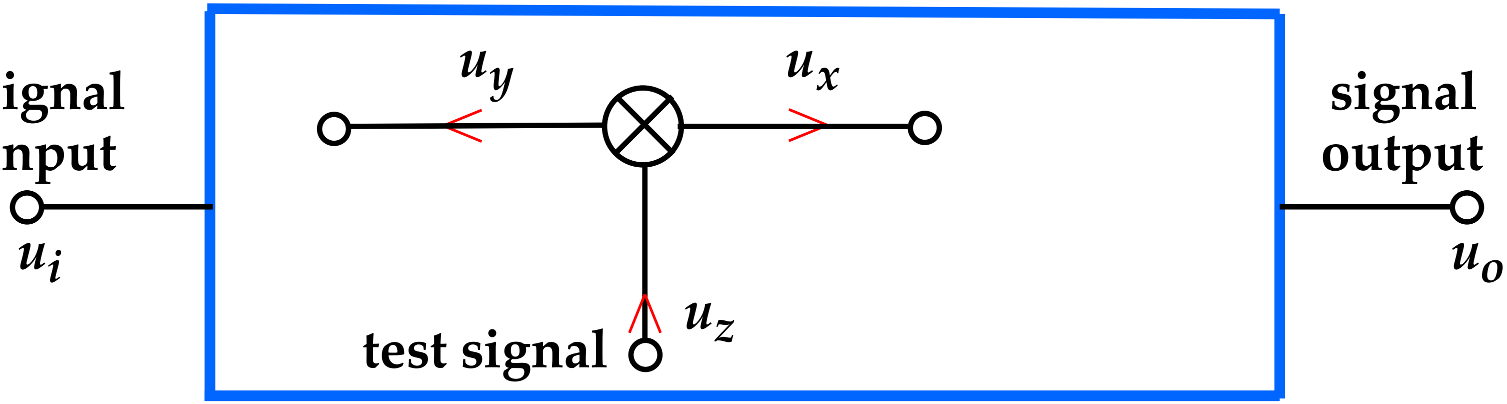
$$T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0} = \frac{A_1B_2 - A_2B_1}{A_1 - \left(A_1B_2 - A_2B_1\right)}$$ **(ndi)**

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} = A_1 + A_2\frac{B_1}{1-B_2}$$ **(ndi)**

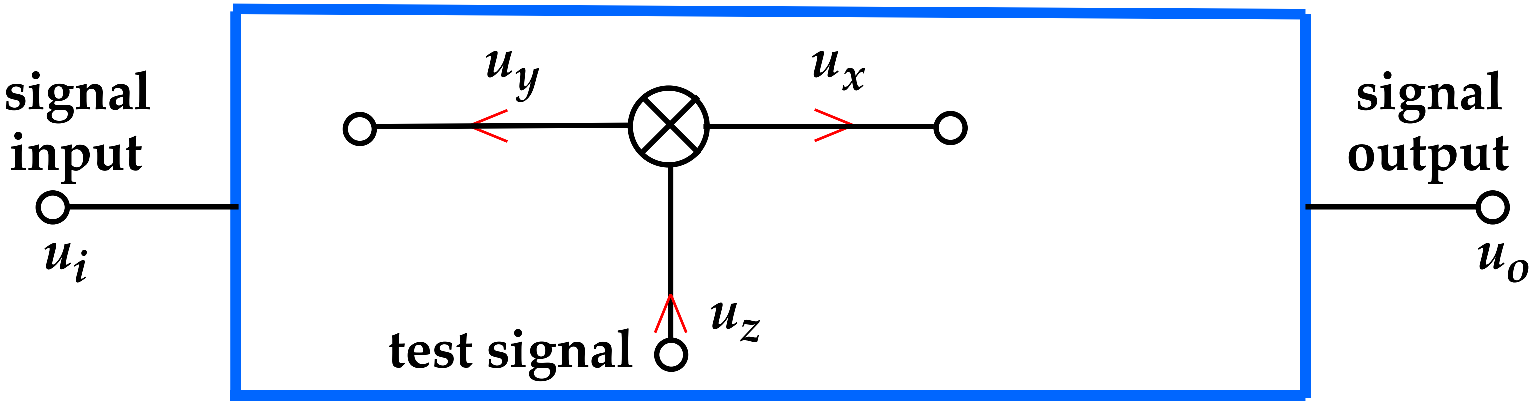
$$T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0} = \frac{B_2}{1-B_2}$$ **(si)**

Note that A_2 and B_1 occur only as a product A_2B_1.

The benefit to be gained from these definitions is that there are useful relations between these several TFs that do not involve the *A*'s and *B*'s.



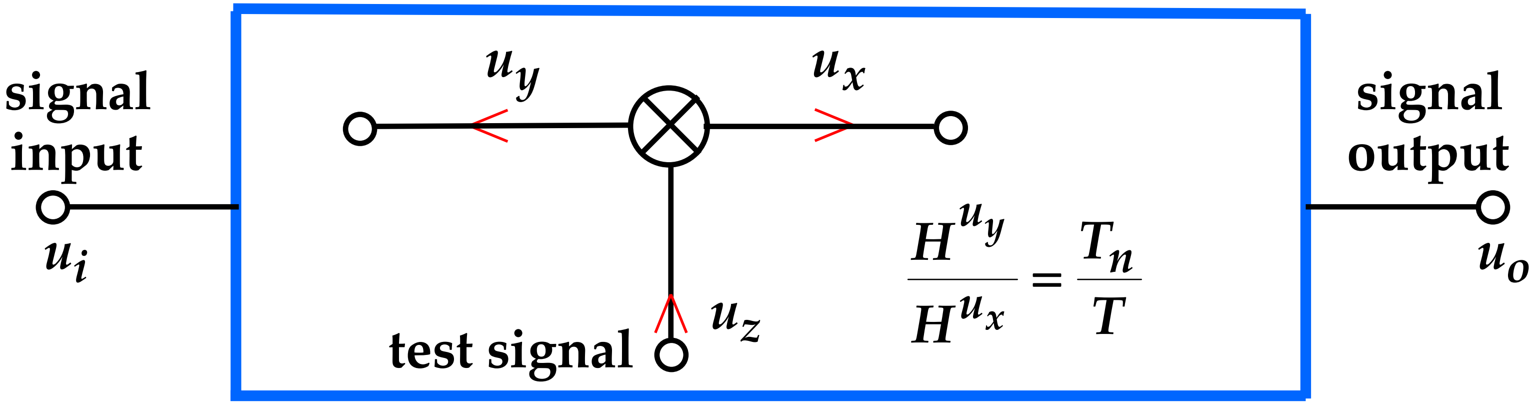
The benefit to be gained from these definitions is that there are useful relations between these several TFs that do not involve the A's and B's.



The 4 second level TFs are defined in terms of the 4 original parameters A_1, A_2, B_1, B_2. Since A_2 and B_1 occur only as a product A_2B_1, there are actually only 3 parameters and there must be a relation between the 4 second level TFs, which is

Redundancy Relation: $$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

The benefit to be gained from these definitions is that there are useful relations between these several TFs that do not involve the A's and B's.



$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

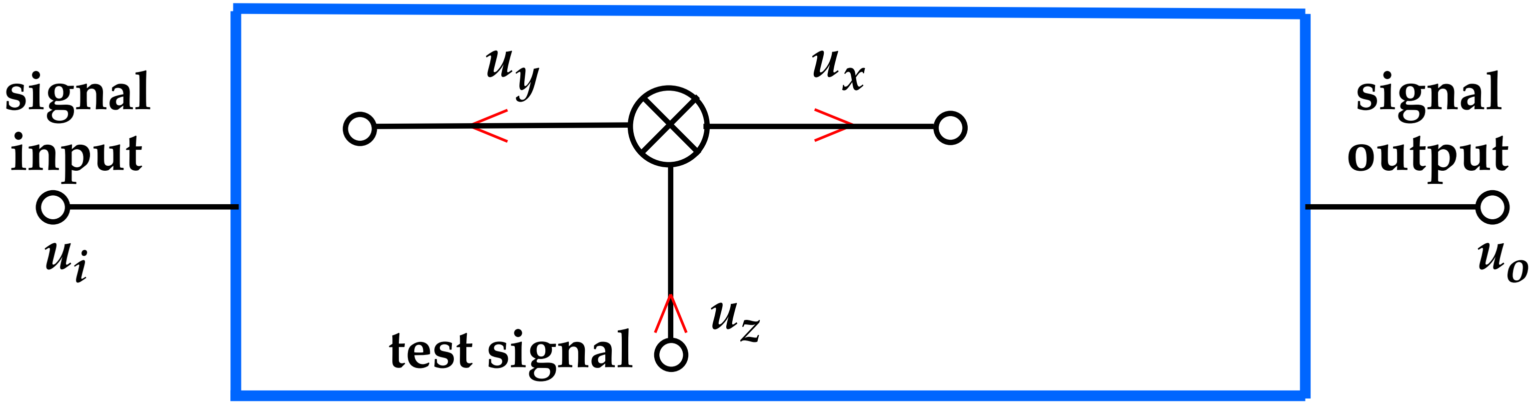
A consequence of the Redundancy Relation is that the first level TF H can be expressed in terms of any three of the four second level TFs H^{u_y}, H^{u_x}, T_n, T.

Two useful versions are:

$$H = H^{u_y}\frac{1+\dfrac{1}{T_n}}{1+\dfrac{1}{T}} \qquad H = H^{u_y}\frac{T}{1+T} + H^{u_x}\frac{1}{1+T}$$

These two versions, and the redundancy relation, can easily be verified by substitution of the definitions. After this, the A's and B's are no longer required, and will not appear again.

Dissection Theorem (DT)



$$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0}$$

$$H = \overset{\text{ndi}}{H^{u_y}}\,\frac{1+\overset{\text{ndi}}{\frac{1}{T_n}}}{1+\underset{\text{si}}{\frac{1}{T}}} = \underbrace{H^{u_y}\frac{T}{1+T} + \overset{\text{ndi}}{H^{u_x}}\frac{1}{1+T}}_{\text{second level TFs}}$$

first level TF

Notation:
Superscript signal is signal being nulled

$$H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} \qquad T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} \qquad T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0}$$

Redundancy Relation:

$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

These results constitute the Dissection Theorem (DT), so named because it shows that a first level TF can be "dissected" into three second level TFs established in terms of an injected test signal.

The DT is completely general, and applies to any TF of a linear system model.

For example, *H* could be a voltage gain, current gain, or an input or output impedance.

There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

Why are ndi calculations *always* simpler and easier than si calculations?

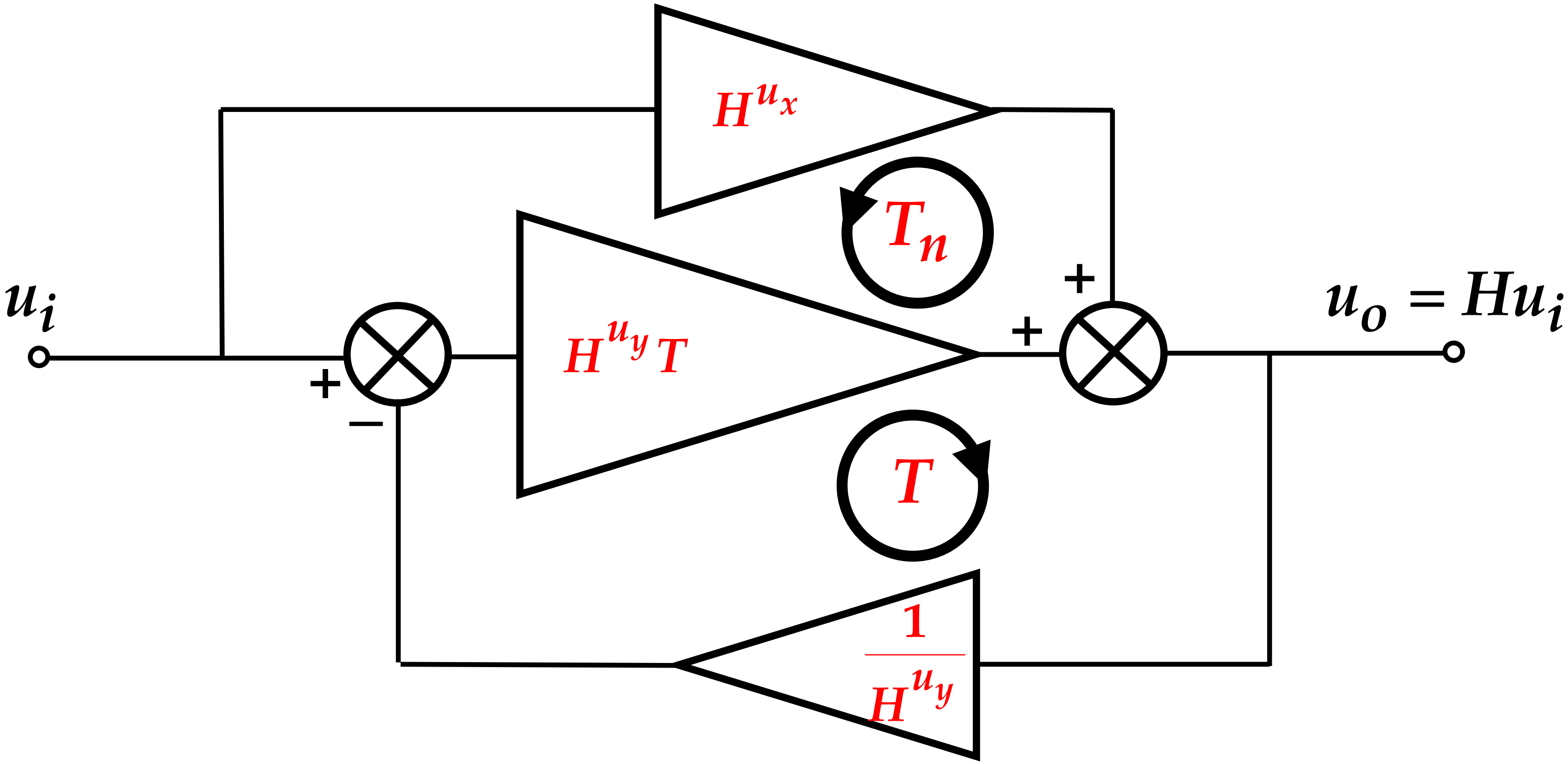
There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

Why are ndi calculations *always* simpler and easier than si calculations?

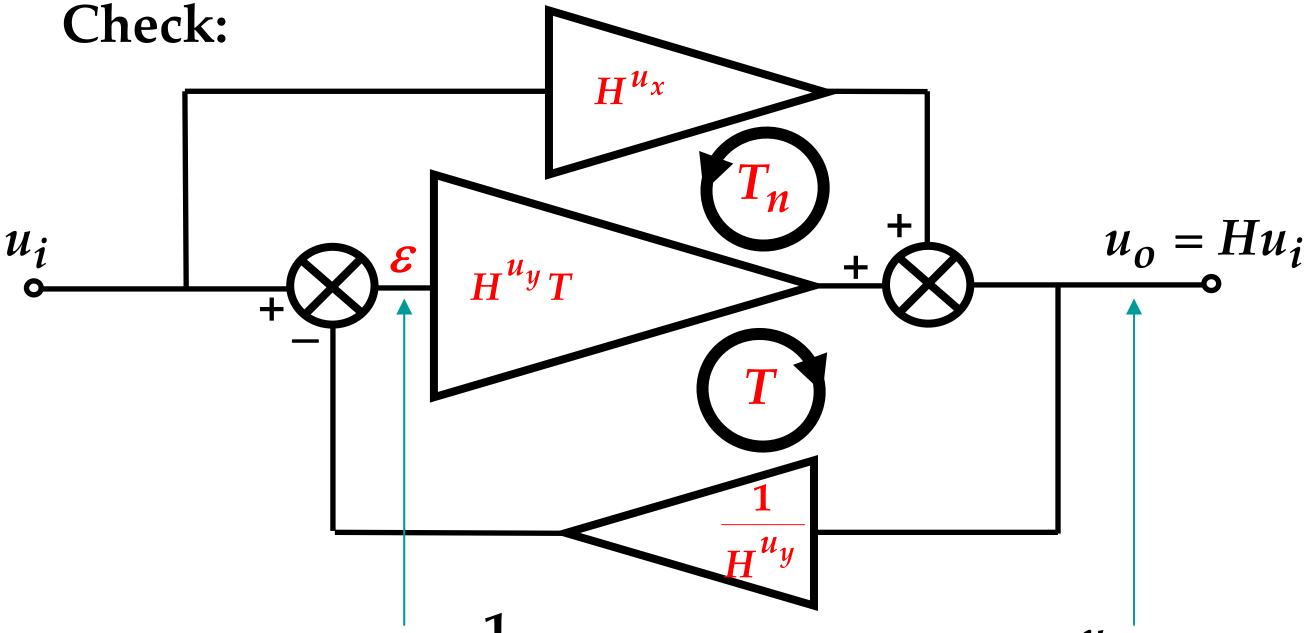
Because any element that supports a null signal does not contribute to the result, and because if one signal is nulled, often other signals are automatically nulled as well, and therefore several elements may be absent from the result.

The Dissection Theorem can be represented by the block diagram



Important: The individual blocks do not necessarily represent identifiable parts of the actual circuit!

Check:



$$T_n = H^{u_y} T \frac{1}{H^{u_x}}$$

$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

$$\varepsilon = u_i - \frac{1}{H^{u_y}} u_o$$

$$u_o = H^{u_y} T \varepsilon + H^{u_x} u_i$$

$$u_o = H^{u_y} T \left(u_i - \frac{1}{H^{u_y}} u_o \right) + H^{u_x} u_i$$

$$(1+T) u_o = \left(H^{u_y} T + H^{u_x} \right) u_i$$

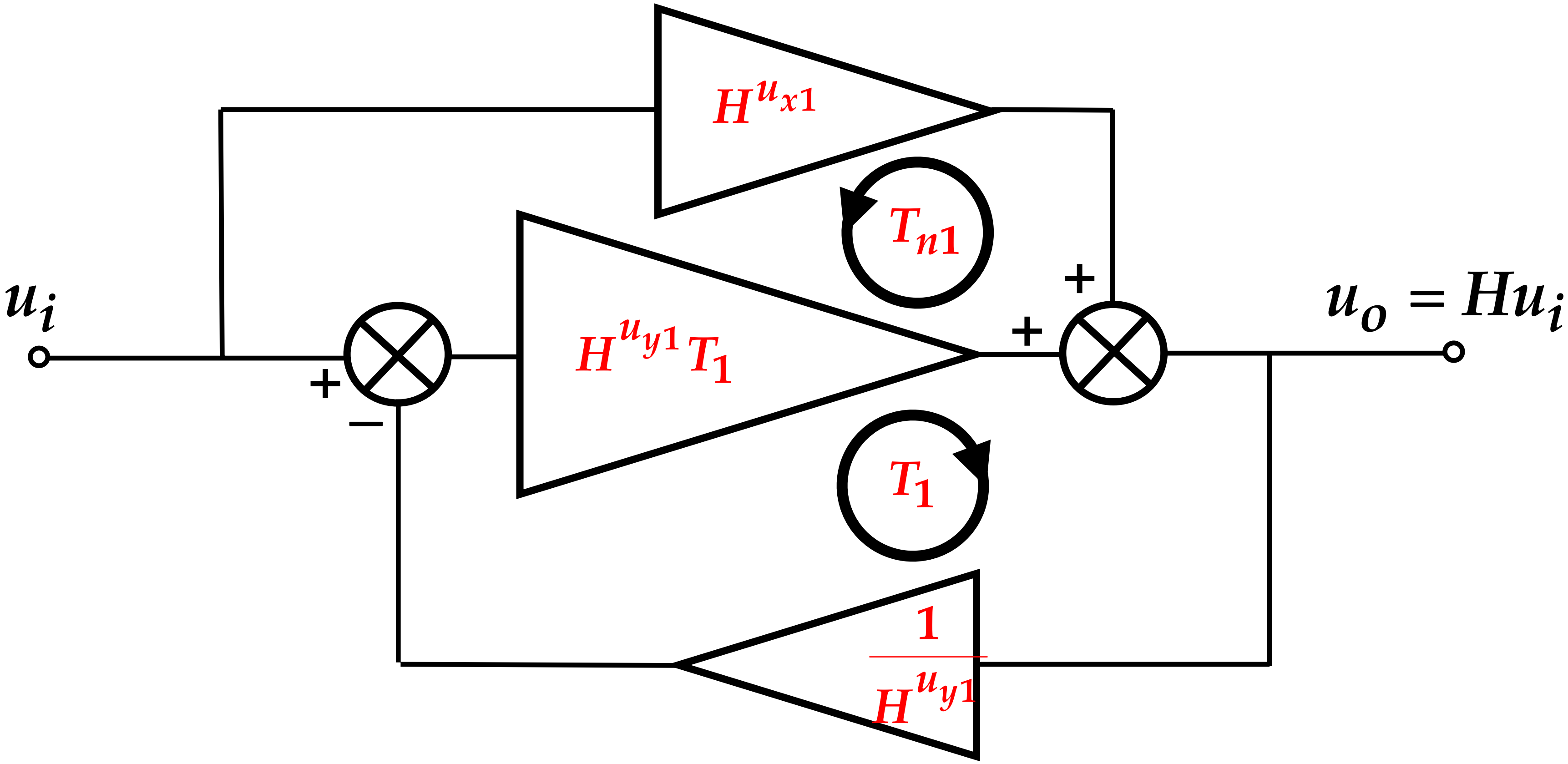
$$H = H^{u_y} \frac{T}{1+T} + H^{u_x} \frac{1}{1+T}$$

So far, nothing has been said about *where* in the system model the test signal is injected.

Different test signal injection points define different sets of second level TFs. Nevertheless, when a mutually consistent set is substituted into the DT, *the same H* results:

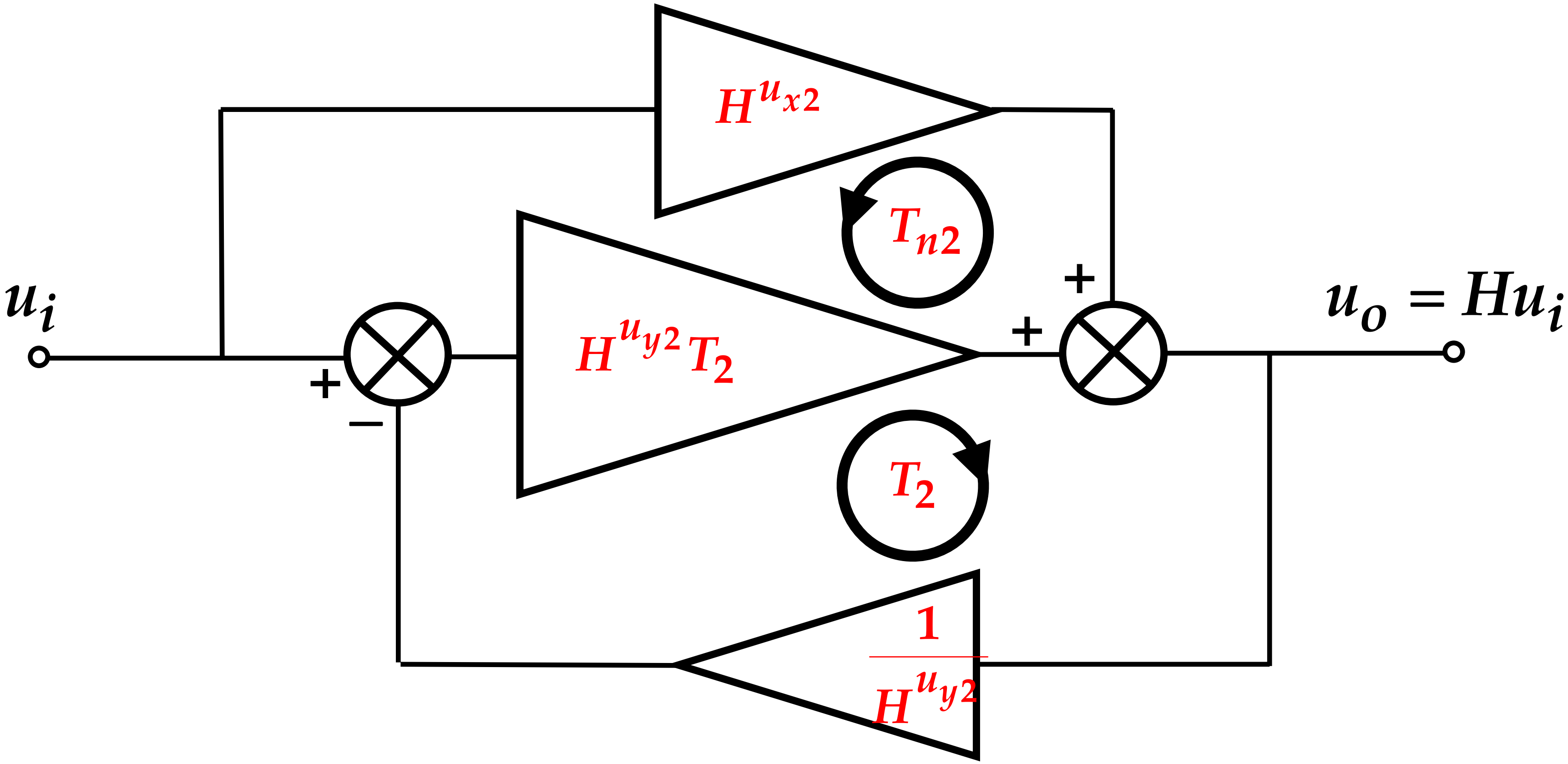
$$H = H^{u_{y1}} \frac{1+\frac{1}{T_{n1}}}{1+\frac{1}{T_1}} = H^{u_{y2}} \frac{1+\frac{1}{T_{n2}}}{1+\frac{1}{T_2}} = \ldots$$

This means that the blocks in the block diagram have different values for different test signal injection points:



Important: The individual blocks do not necessarily represent identifiable parts of the actual circuit!

This means that the blocks in the block diagram have different values for different test signal injection points:



Important: **The individual blocks do not necessarily represent identifiable parts of the actual circuit!**

Not only does the DT implement the Design & Conquer objective, but the DT is itself a Low Entropy Expression, and *much greater* benefits accrue if the second level TFs have useful physical interpretations.

Thus, the second level TFs themselves contain the useful design-oriented information and you may never need to actually substitute them into the theorem.

For example, if $T, T_n >> 1$, $H \approx H^{u_y}$

How to determine the physical interpretations of the second level TFs?

What kind of signal (voltage or current) is injected, and where it is injected, defines an "injection configuration."

Therefore, ***the key decision in applying the DT is choosing a test signal injection point so that at least one of the second level TFs has the physical interpretation you want it to have.***

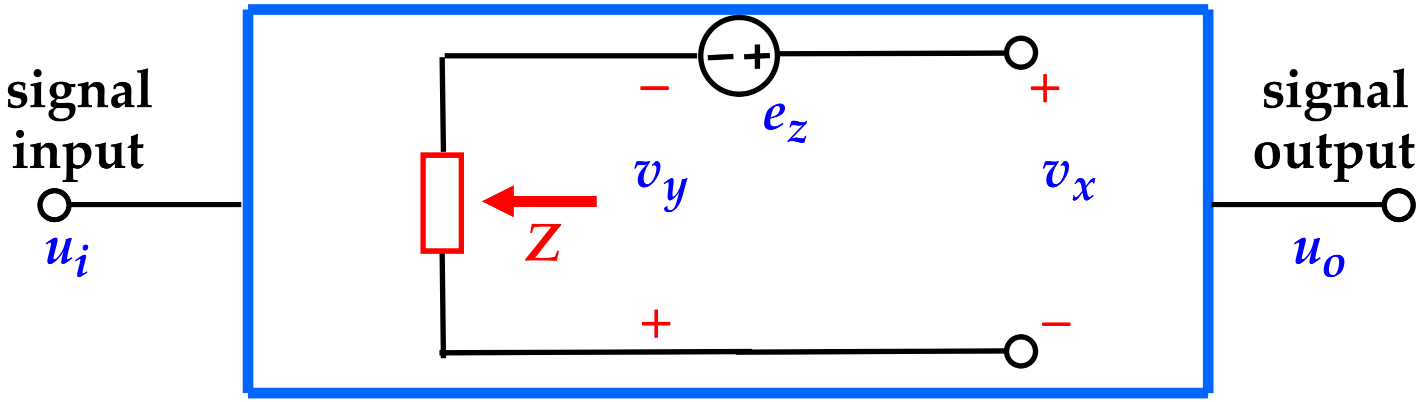
Specific injection configurations for the DT lead to the:

- **Extra Element Theorem (EET)**
- **Chain Theorem (CT)**
- **General Feedback Theorem (GFT)**

As usual, dual forms of the theorem emerge depending upon whether the injected signal u_z is a voltage or a current.

The Extra Element Theorem

Inject a test voltage e_z in series with an element Z such that v_y appears across Z:



The DT is:

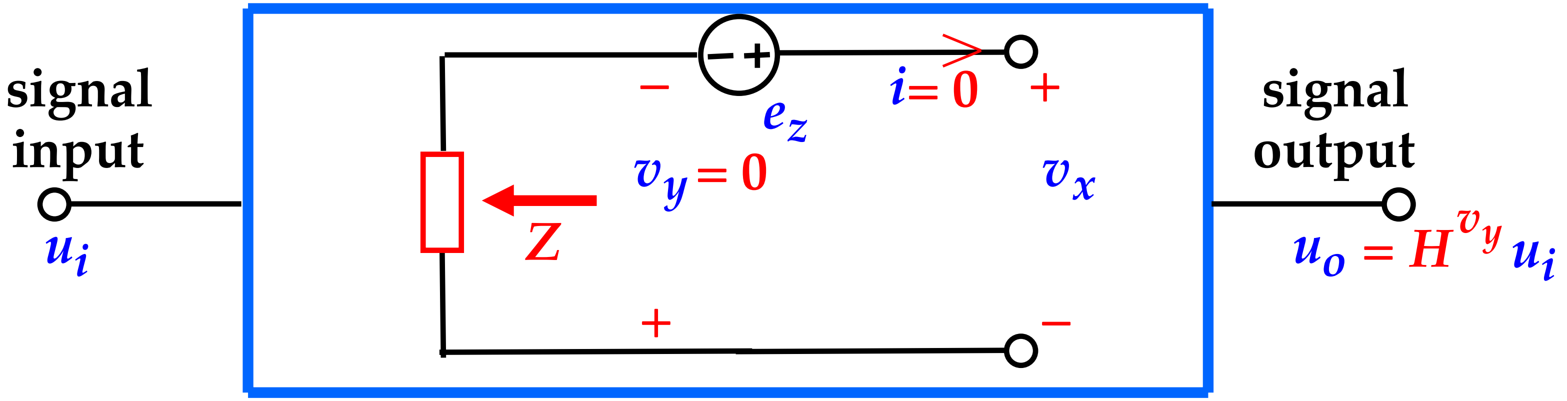
$$H = H^{v_y} \frac{1 + \frac{1}{T_{nv}}}{1 + \frac{1}{T_v}}$$

where:

$$H^{v_y} \equiv \left.\frac{u_o}{u_i}\right|_{v_y=0} \qquad T_v \equiv \left.\frac{v_y}{v_x}\right|_{u_i=0} \qquad T_{nv} \equiv \left.\frac{v_y}{v_x}\right|_{u_o=0}$$

The Extra Element Theorem

To find H^{v_y}, assume that e_z and u_i have been mutually adjusted to null v_y :



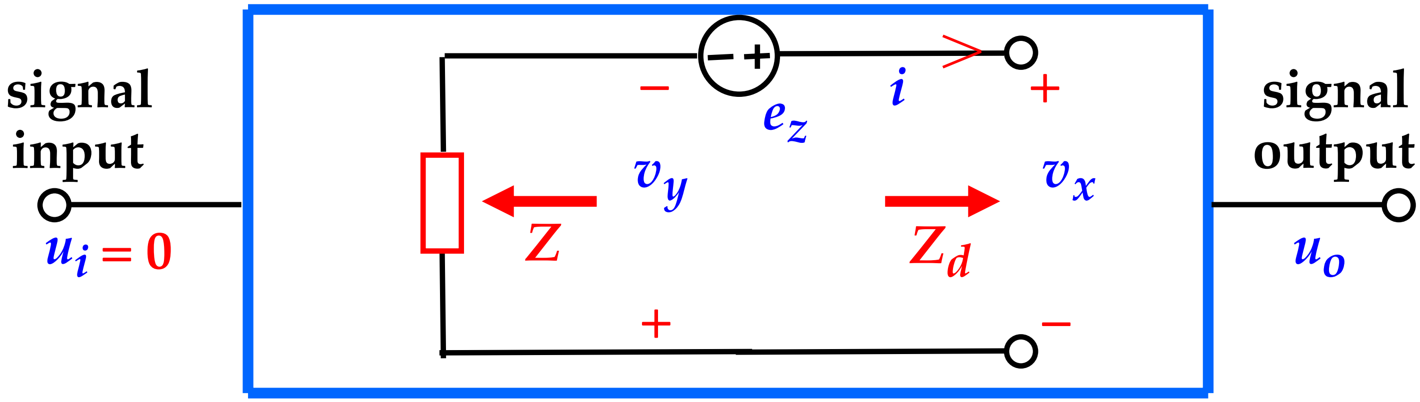
If $v_y = 0$, there is no current through Z, and so the current i into the test port is also zero, which is the condition that would exist if there were no injected test signal and Z were open. Therefore,

$$H\big|_{Z=\infty} = H^{v_y} \equiv \left.\frac{u_o}{u_i}\right|_{v_y=0}$$

where $H\big|_{Z=\infty}$ is the first level TF H when $Z = \infty$.

The Extra Element Theorem

To find T_v, set $u_i = 0$:



The si driving point impedance Z_d looking into the test port is

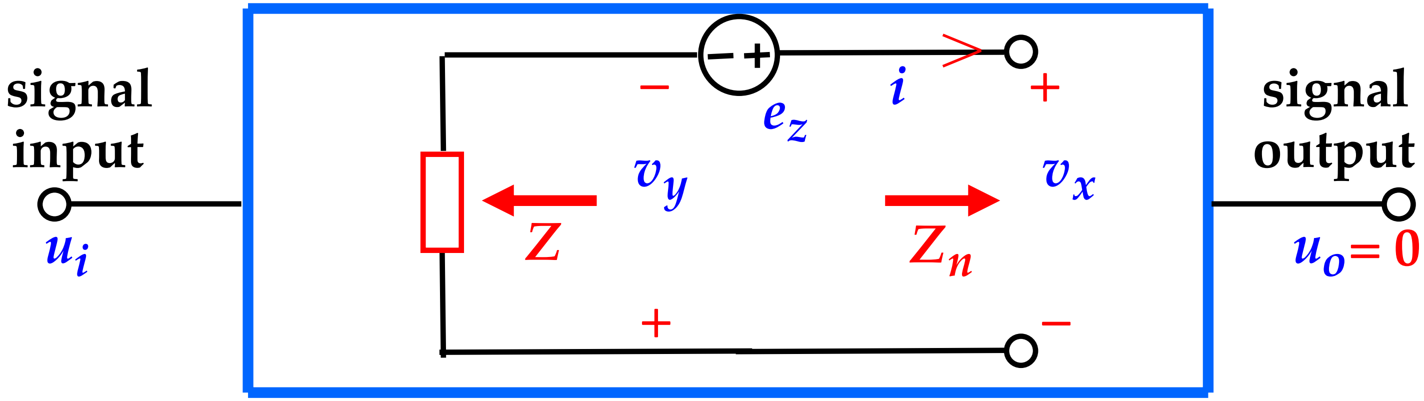
$$Z_d \equiv \left. \frac{v_x}{i} \right|_{u_i=0}$$

Since Z and Z_d are in series with the same current i,

$$T_v = \left. \frac{v_y}{v_x} \right|_{u_i=0} = \frac{Z}{Z_d}$$

The Extra Element Theorem

To find T_{nv}, assume that e_z and u_i have been mutually adjusted to null u_o :



The ndi driving point impedance Z_n looking into the test port is

$$Z_n \equiv \left.\frac{v_x}{i}\right|_{u_o=0}$$

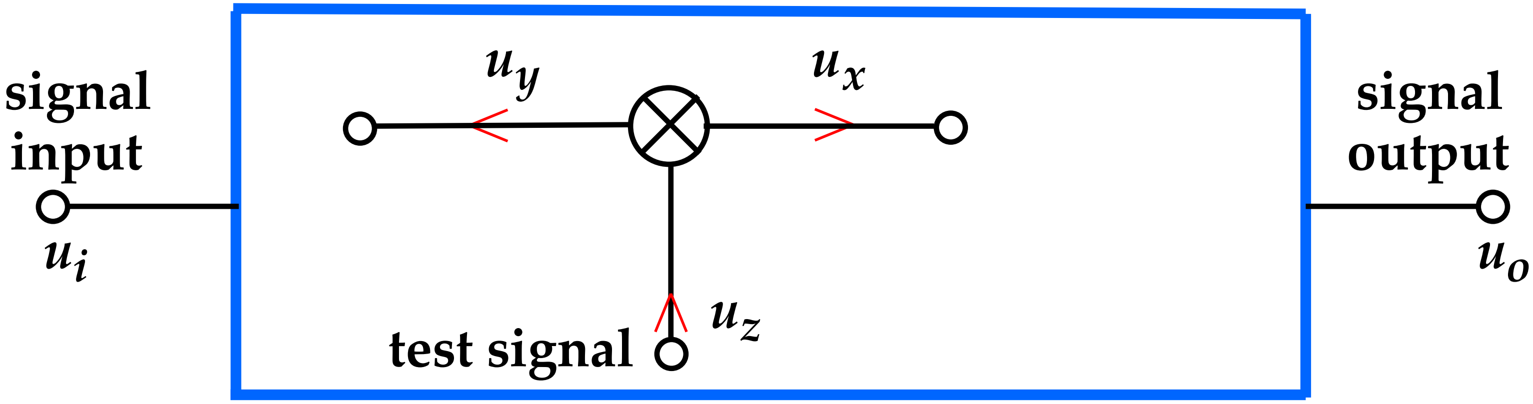
Since Z and Z_n are in series with the same current i,

$$T_{nv} = \left.\frac{v_y}{v_x}\right|_{u_o=0} = \frac{Z}{Z_n}$$

With the second level TFs replaced by the new definitions, the DT morphs into the Extra Element Theorem (EET):

$$H = H\big|_{Z=\infty}\frac{1+\frac{Z_n}{Z}}{1+\frac{Z_d}{Z}}$$

Dissection Theorem (DT)



$$H \equiv \left.\frac{u_o}{u_i}\right|_{u_z=0}$$

$$H = H^{\overset{\text{ndi}}{u_y}}\,\frac{1+\frac{1}{\overset{\text{ndi}}{T_n}}}{1+\frac{1}{\underset{\text{si}}{T}}} \underbrace{= H^{u_y}\frac{T}{1+T} + H^{\overset{\text{ndi}}{u_x}}\frac{1}{1+T}}_{\text{second level TFs}}$$

first level TF

Notation:
Superscript signal is signal being nulled

Redundancy Relation:

$$\frac{H^{u_y}}{H^{u_x}} = \frac{T_n}{T}$$

$$\overbrace{H^{u_y} \equiv \left.\frac{u_o}{u_i}\right|_{u_y=0} \qquad T_n \equiv \left.\frac{u_y}{u_x}\right|_{u_o=0}}^{}$$

$$H^{u_x} \equiv \left.\frac{u_o}{u_i}\right|_{u_x=0} \qquad T \equiv \left.\frac{u_y}{u_x}\right|_{u_i=0}$$

There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

Why are ndi calculations *always* simpler and easier than si calculations?

There are many reasons why the Dissection Theorem is useful.

The *minimum* benefit of the DT is that it embodies the "Divide and Conquer" approach, because one complicated calculation is replaced by three calculations, two of which are ndi calculations and are therefore *simpler* and *easier* than si calculations.

Why are ndi calculations *always* simpler and easier than si calculations?

Because any element that supports a null signal does not contribute to the result, and because if one signal is nulled, often other signals are automatically nulled as well, and therefore several elements may be absent from the result.

Not only does the DT implement the Design & Conquer objective, but the DT is itself a Low Entropy Expression, and *much greater* benefits accrue if the second level TFs have useful physical interpretations.

Thus, the second level TFs themselves contain the useful design-oriented information and you may never need to actually substitute them into the theorem.

For example, if $T, T_n >> 1$, $H \approx H^{u_y}$

How to determine the physical interpretations of the second level TFs?

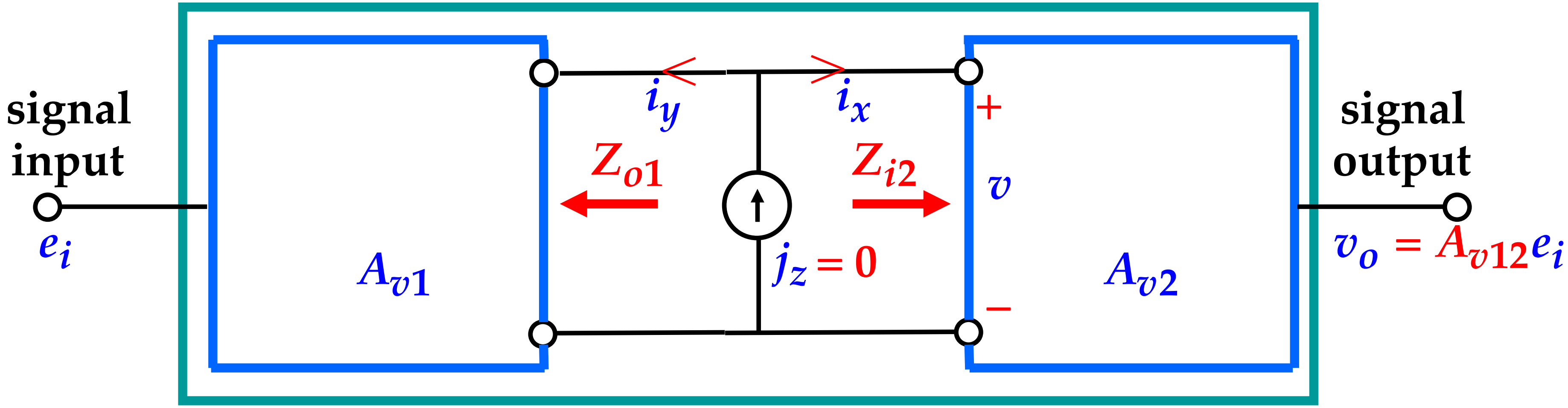
What kind of signal (voltage or current) is injected, and where it is injected, defines an "injection configuration."

Therefore, ***the key decision in applying the DT is choosing a test signal injection point so that at least one of the second level TFs has the physical interpretation you want it to have.***

Another special case of the DT leads to the Chain Theorem (CT).

The test signal injection configuration is such that the entire signal from the input flows to the output (no bypass paths).

The Chain Theorem (CT)



The gain $A_{v12} \equiv \dfrac{v_o}{e_i}$ is given by the DT:

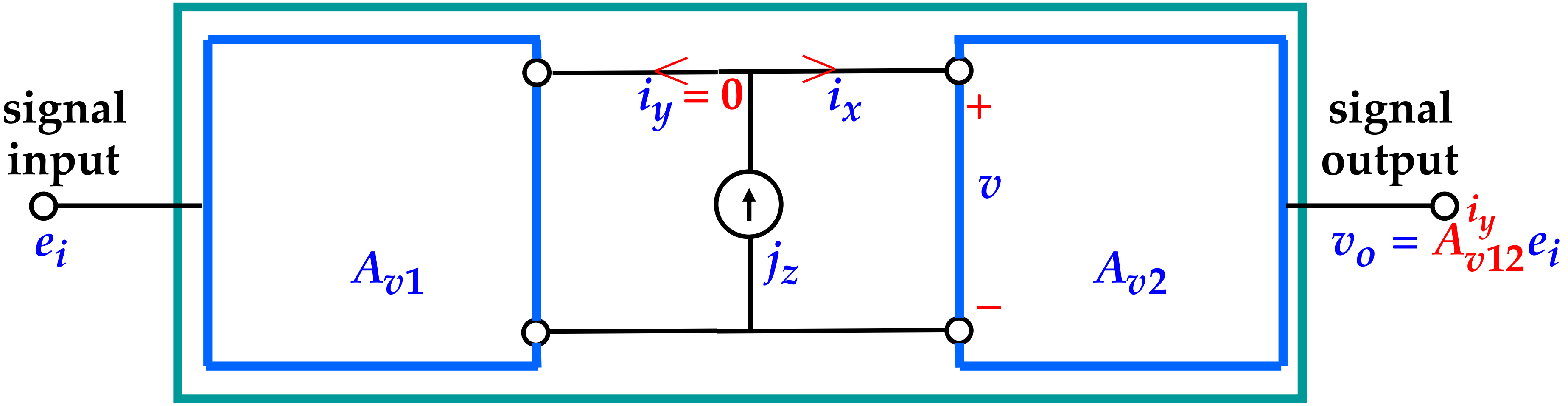
$$A_{v12} = A_{v12}^{i_y} \frac{1 + \dfrac{1}{T_{ni}}}{1 + \dfrac{1}{T_i}}$$

The TF $T_{ni} \equiv i_y / i_x \big|_{v_o = 0}$ is an ndi calculation with the output v_o nulled.

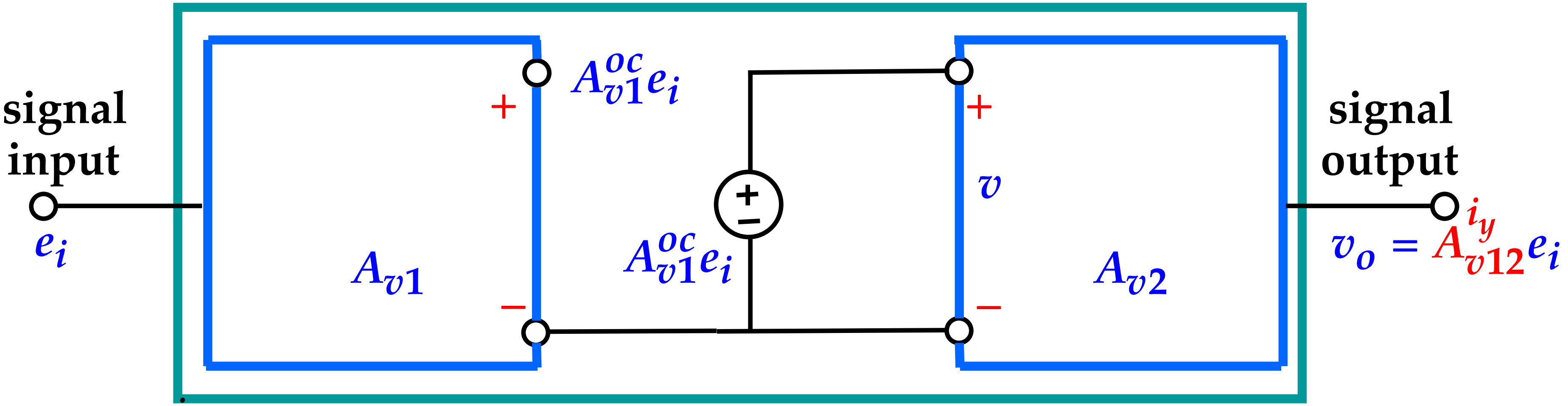
If v_o is nulled, so is i_x, so $T_{ni} = \infty$.

This implies that T_{ni} is infinite unless the signal can bypass the injection point.

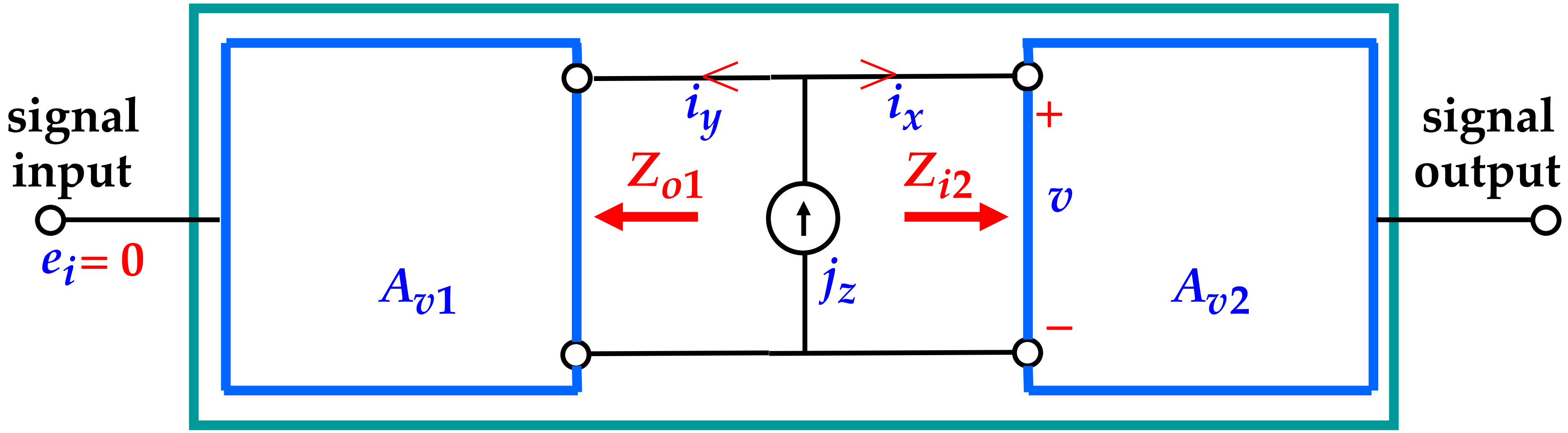
The Chain Theorem (CT)



Nulled i_y means that the A_{v1} box is unloaded, so the input voltage to the A_{v2} box is the open-circuit (oc) output voltage of the A_{v1} box.



Thus, $A_{v12}^{i_y} = A_{v1}^{oc} A_{v2}$ is the *voltage-buffered gain* of the two stages.



Also
$$T_i \equiv \left.\frac{i_y}{i_x}\right|_{e_i=0} = \left.\frac{v/Z_{o1}}{v/Z_{i2}}\right|_{e_i=0} = \frac{Z_{i2}}{Z_{o1}},$$

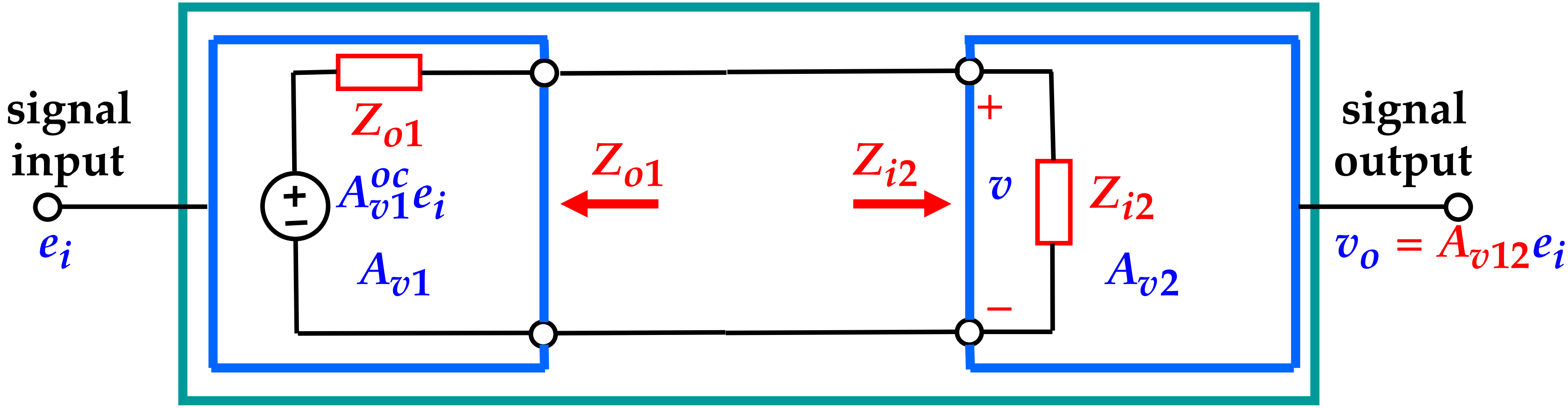
so the DT becomes
$$A_{v12} = A_{v1}^{oc} A_{v2} \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$

This can be interpreted as

$$\begin{bmatrix} \text{gain} \\ \text{of the two stages} \end{bmatrix} = \begin{bmatrix} \text{voltage buffered gain} \\ \text{of the two stages} \end{bmatrix} \text{X} \begin{bmatrix} \text{voltage loading factor} \\ \text{between the two stages} \end{bmatrix}$$

The Chain Theorem (CT)

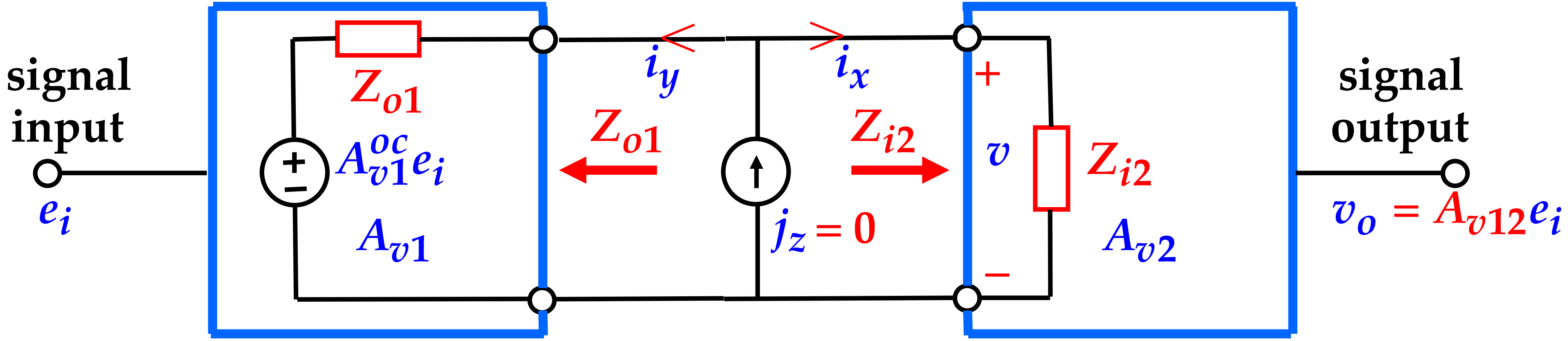
This is exactly the result that would be obtained directly from the model:



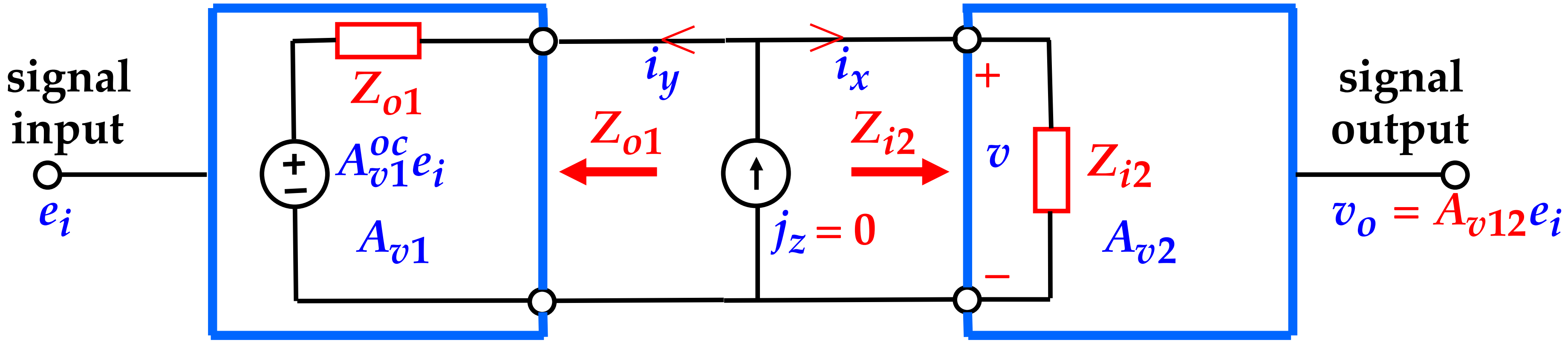
$$A_{v12} = A_{v1}^{oc} A_{v2} \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$

The Chain Theorem (CT)

A useful application of the DT with $T_{ni} = \infty$ is to assemble the properties of a 2-stage amplifier from the properties of each separate stage.

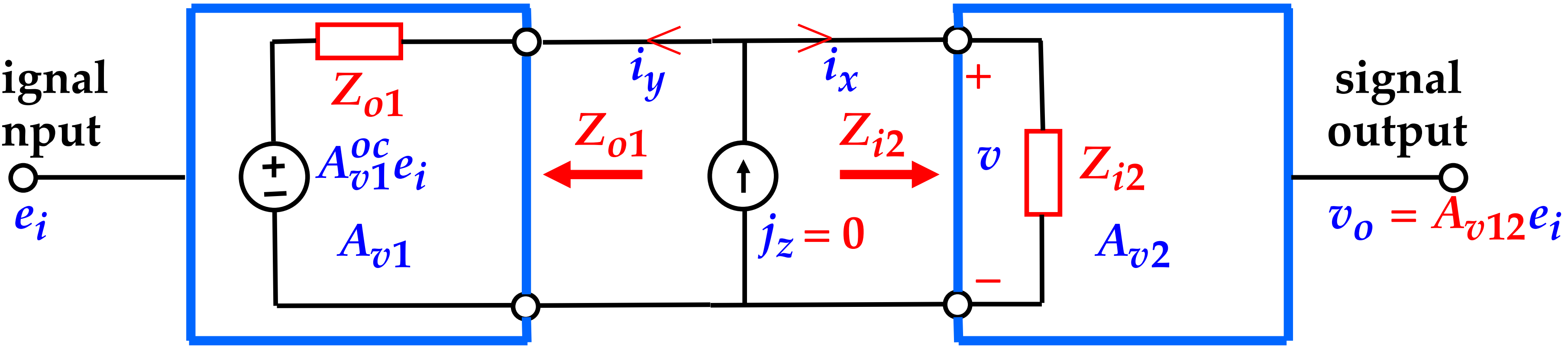


The Chain Theorem (CT)



This "Divide and Conquer" approach avoids analysis of both stages simultaneously.

The Chain Theorem (CT)



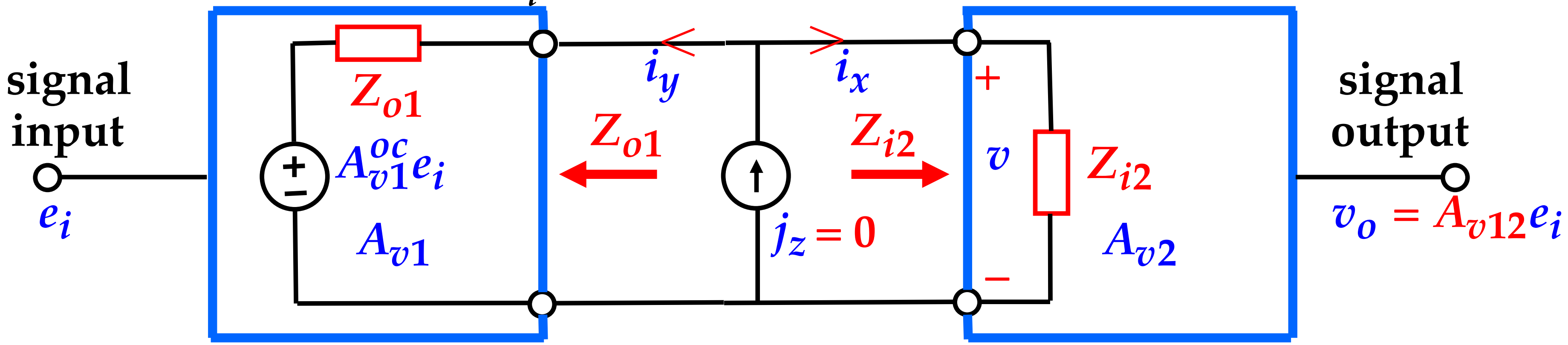
$$A_{v12} = A_{v1}^{oc}A_{v2}\frac{1}{1+\frac{1}{T_i}} = A_{v1}^{oc}A_{v2}D_i$$

$$T_i \equiv \frac{Z_{i2}}{Z_{o1}} \qquad D_i \equiv \frac{1}{1+\frac{1}{T_i}} = \frac{T_i}{1+T_i} = \frac{Z_{i2}}{Z_{i2}+Z_{o1}}$$

where $A_{v1}^{oc}A_{v2}$ is the "voltage buffered" gain that would occur if there were a buffer between the two stages, and D_i is a "discrepancy factor" that accounts for the interaction between the two stages which results from the loading of the first stage by the input of the second stage.

The Chain Theorem (CT)

$$A_{v12} = A_{v1}^{oc} A_{v2} \frac{1}{1+\frac{1}{T_i}} = A_{v1}^{oc} A_{v2} D_i$$



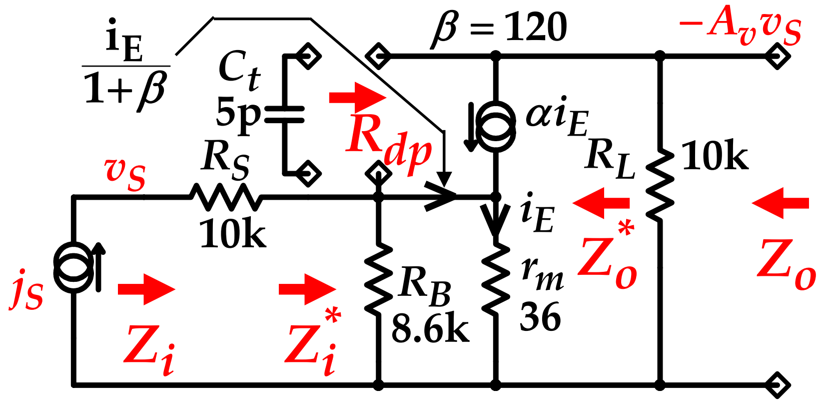
Since all TFs will be in factored pole-zero form, the only place where additional approximation may be needed resides inside the D_i, where the sum of two TFs is required.

"Doing the algebra on the graph" can be conducted in two ways:

$1+T_i$ can be found as the sum of the TFs 1 and T_i, dominated by the larger;

D_i can be found from $\frac{1}{D_i} = 1 + \frac{1}{T_i} = \frac{1}{1} + \frac{1}{T_i}$ as the reciprocal sum of 1 and T_i, dominated by the smaller.

Let each stage be the 1CE stage previously treated.



$$A_v = A_{vm}\frac{1-s/\omega_z}{1+s/\omega_p} = 36dB\frac{1-\dfrac{s/2\pi}{880MHz}}{1+\dfrac{s/2\pi}{51kHz}}$$

$$A_{vm} \equiv \frac{R_B}{R_S+R_B}\frac{\alpha R_L}{r_m+\dfrac{R_S\|R_B}{1+\beta}} = 62 \Rightarrow 36dB$$

$$R_n = r_m = 36\Omega \qquad R_d = mR_L = 620k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L} = 62$$

$$Z_i = R_{im}\frac{1+sC_tR_{ni}}{1+sC_tR_{di}} = 82dB\frac{1+\dfrac{s/2\pi}{51kHz}}{1+\dfrac{s/2\pi}{39kHz}} \qquad Z_o = R_{om}\frac{1+sC_tR_{no}}{1+sC_tR_{do}}$$

$$R_{im} \equiv R_S + R_B\|(1+\beta)r_m = 13k \Rightarrow 82dB \text{ ref } 1\Omega \qquad R_{om} \equiv R_L = 10k \Rightarrow 80dB \text{ ref } 1\Omega$$

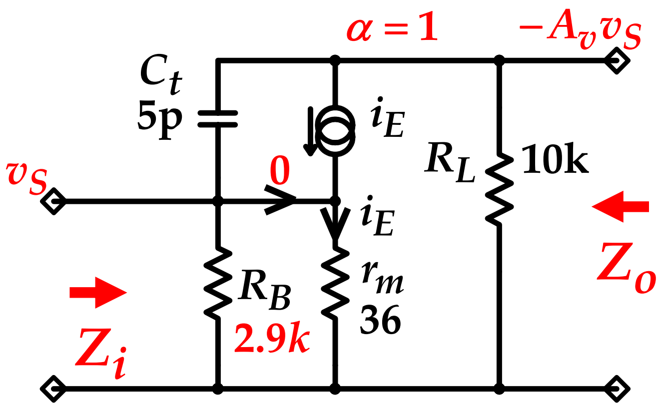
$$R_{ni} = mR_L \equiv \frac{R_S\|R_B\|(1+\beta)r_m}{R_S\|R_B\|r_m\|R_L}R_L = 620k \qquad R_{no} = R_S\|R_B\|(1+\beta)r_m = 2.2k$$

$$R_{di} = \frac{R_B\|(1+\beta)r_m}{R_B\|r_m\|R_L}R_L = 820k \qquad R_{do} = mR_L = 620k$$

However, to make the symbolic equations more compact, without loss of generality, let $R_S \rightarrow 0$ and $\alpha \rightarrow 1$ ($\beta \rightarrow \infty$).

To keep R_{im}^{*} the same, also let $R_B \| (1+\beta) r_m = 2.9k \rightarrow R_B$

The new 1CE stage is:



$$A_v = A_{vm}\frac{1 - s/\omega_z}{1 + s/\omega_p} = 49dB\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}}$$

$$A_{vm} = \frac{R_L}{r_m} = 280 \Rightarrow 49dB$$

$$R_n = r_m = 36\Omega \qquad R_d = R_L = 10k$$

$$\omega_z \equiv \frac{1}{C_t R_n} \qquad \omega_p \equiv \frac{1}{C_t R_d} \qquad m = 1$$

$$Z_i = R_{im}\frac{1 + sC_t R_L}{1 + sC_t R_L \frac{R_B}{R_B \| r_m \| R_L}} = 69dB\frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$$

$$Z_o = R_{om}\frac{1}{1 + sC_t R_L} = 80dB\frac{1}{1 + \frac{s/2\pi}{3.2MHz}}$$

$R_{im} = R_B = 2.9k \Rightarrow 69dB$ ref. 1Ω

$R_{om} = R_L = 10k \Rightarrow 80dB$ ref. 1Ω

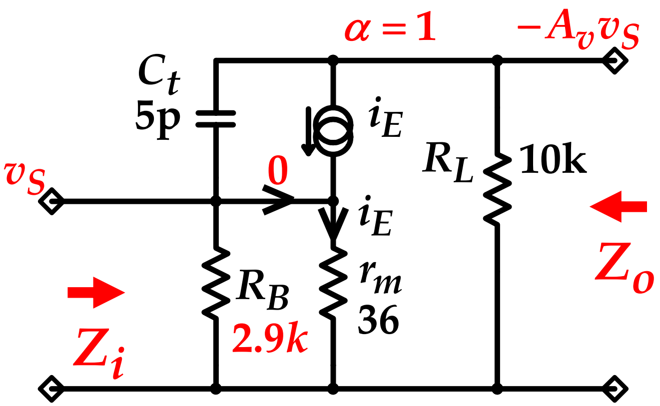
$R_{ni} = mR_L = 10k$

$R_{no} = 0$

$$R_{di} = \frac{R_B \| (1+\beta) r_m}{R_B \| r_m \| R_L} R_L = 820k$$

$R_{do} = mR_L = 10k$

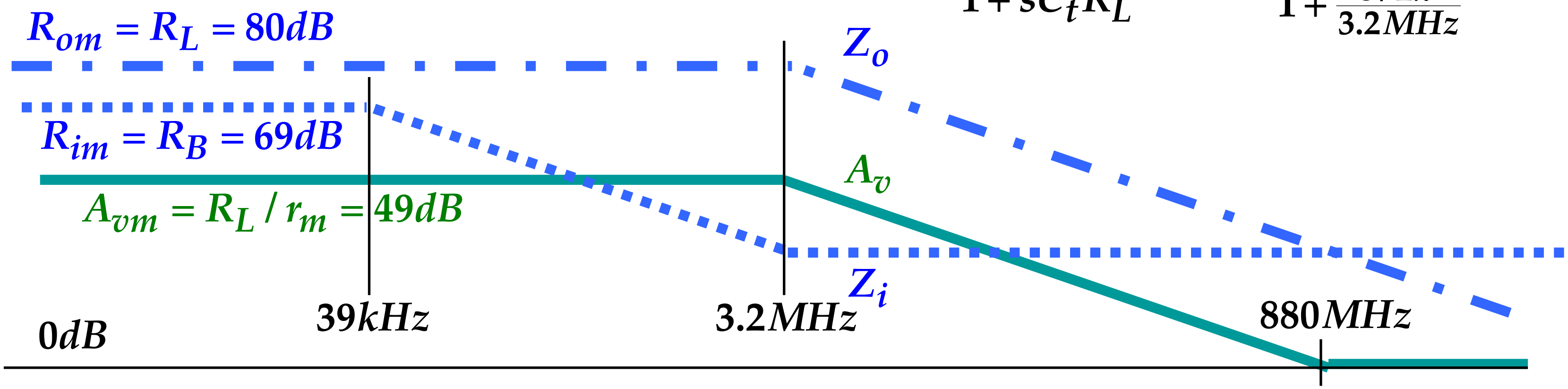
Note that letting $R_S \to 0$ reduces the "miller multiplier" m to 1.

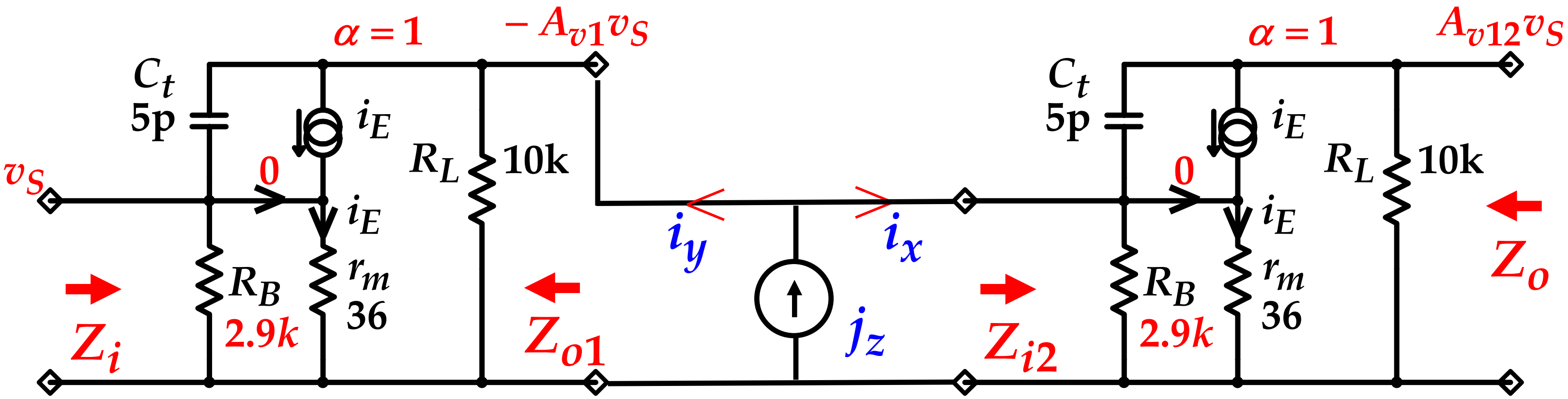


$$A_v = \frac{R_L}{r_m}\frac{1 - sC_t r_m}{1 + sC_t R_L} = 49dB\,\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}}$$

$$Z_i = R_B\frac{1 + sC_t R_L}{1 + sC_t R_L\frac{R_B}{R_B\|r_m\|R_L}} = 69dB\,\frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$$

$$Z_o = R_L\frac{1}{1 + sC_t R_L} = 80dB\,\frac{1}{1 + \frac{s/2\pi}{3.2MHz}}$$



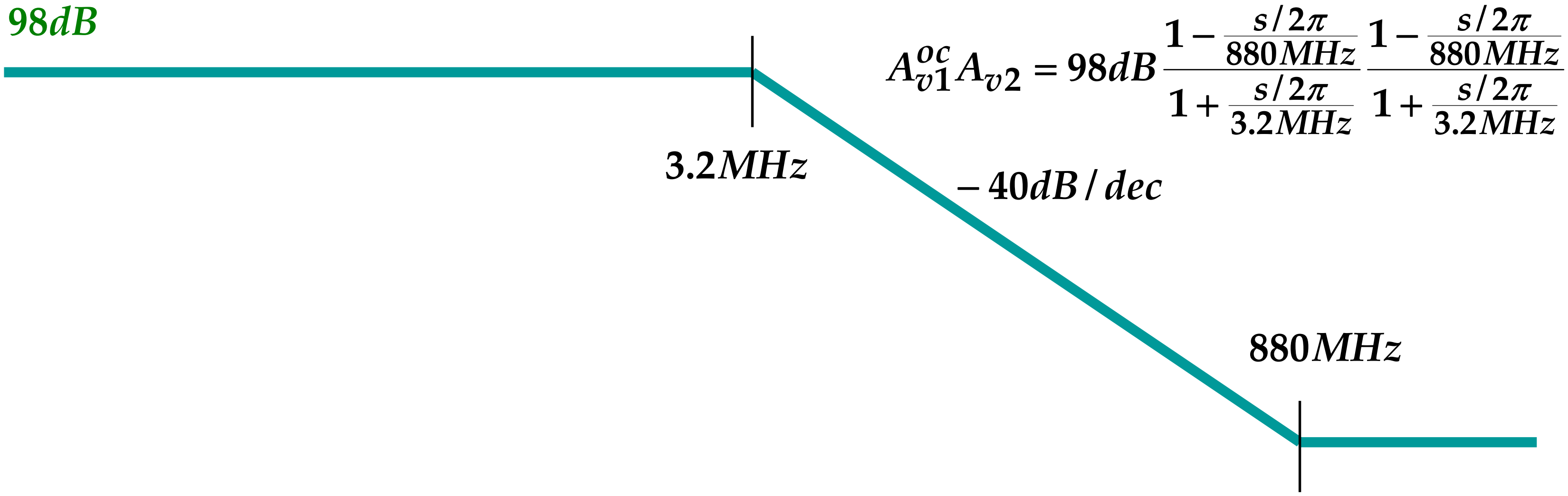


The DT gives

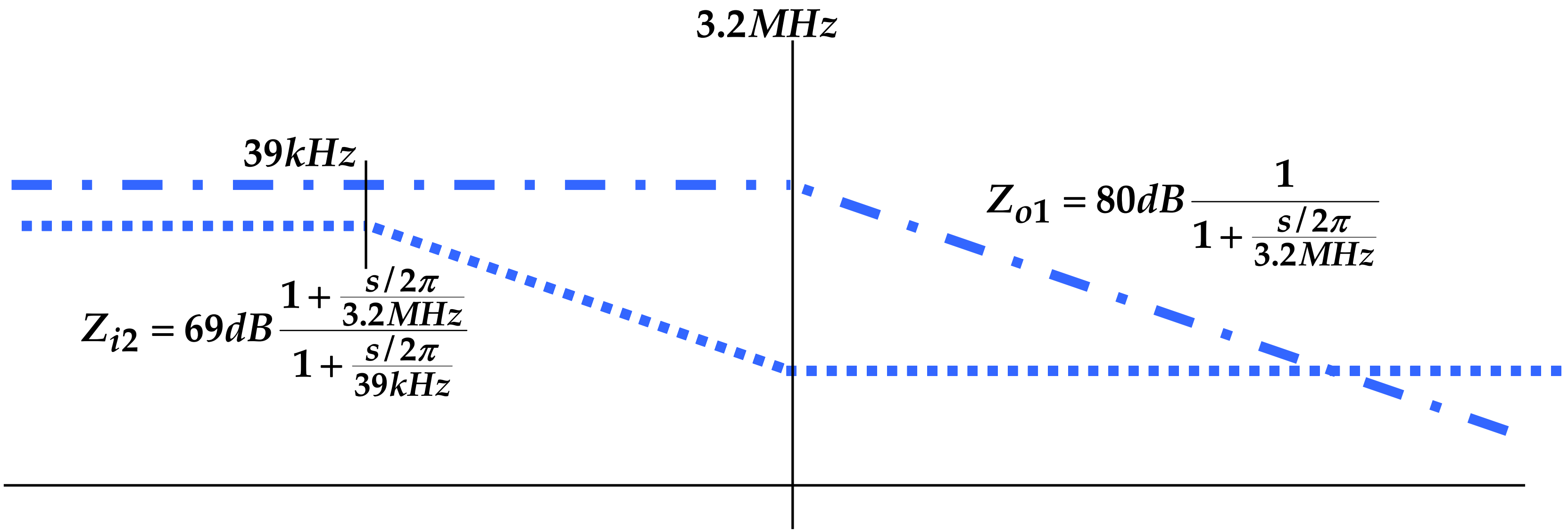
$$A_{v12} = A_{v1}^{oc} A_{v2} D_i \qquad (T_{ni} = \infty)$$

The buffered gain $A_{v1}^{oc} A_{v2}$ is the product of the two separate gains, where A_{v1} is already open-circuit:

$$A_{v1}^{oc} A_{v2} = 98dB \left(\frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{3.2MHz}} \right)^2$$



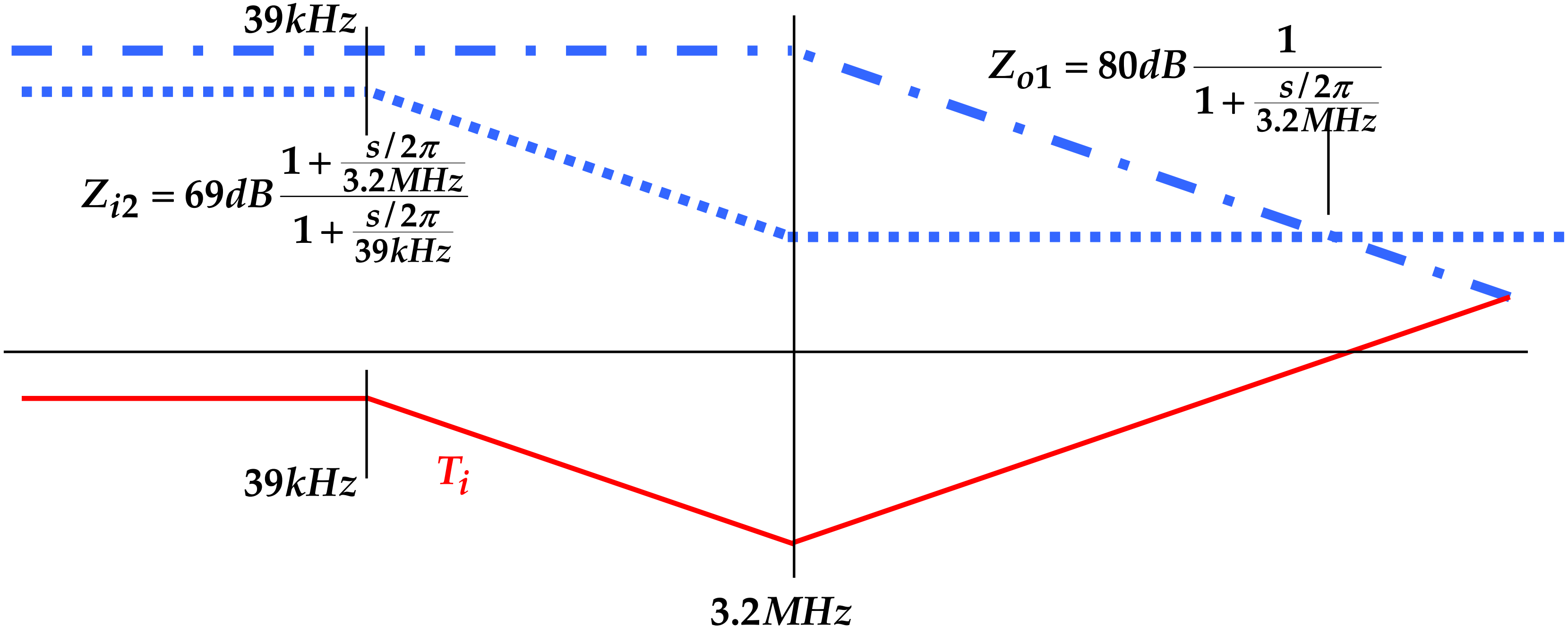
$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$



$$Z_{o1} = 80dB \frac{1}{1 + \frac{s/2\pi}{3.2MHz}}$$

$$Z_{i2} = 69dB \frac{1 + \frac{s/2\pi}{3.2MHz}}{1 + \frac{s/2\pi}{39kHz}}$$

$$T_i \equiv \frac{Z_{i2}}{Z_{o1}} = -11dB \frac{\left(1+\frac{s/2\pi}{3.2MHz}\right)^2}{1+\frac{s/2\pi}{39kHz}}$$

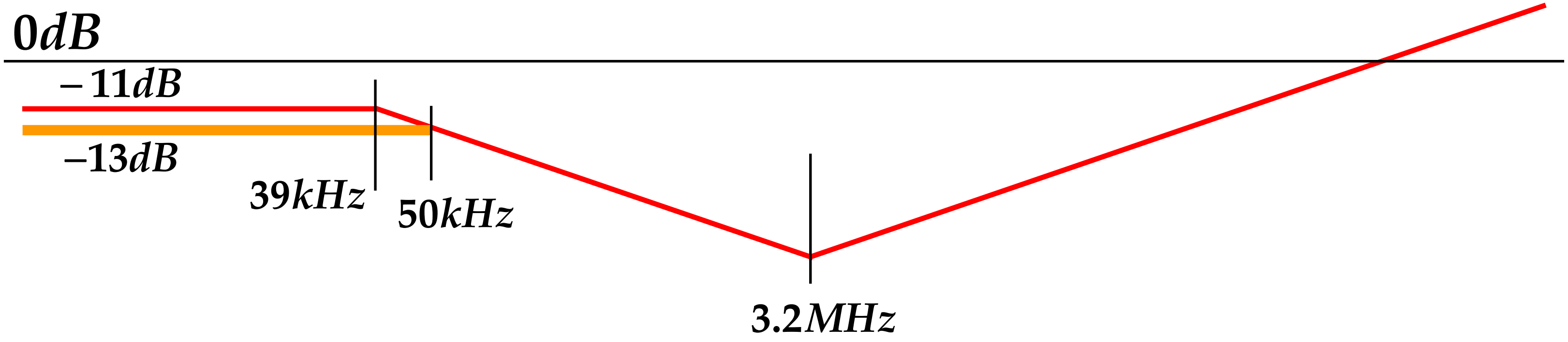


$$Z_{o1} = 80dB \frac{1}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Z_{i2} = 69dB \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{39kHz}}$$

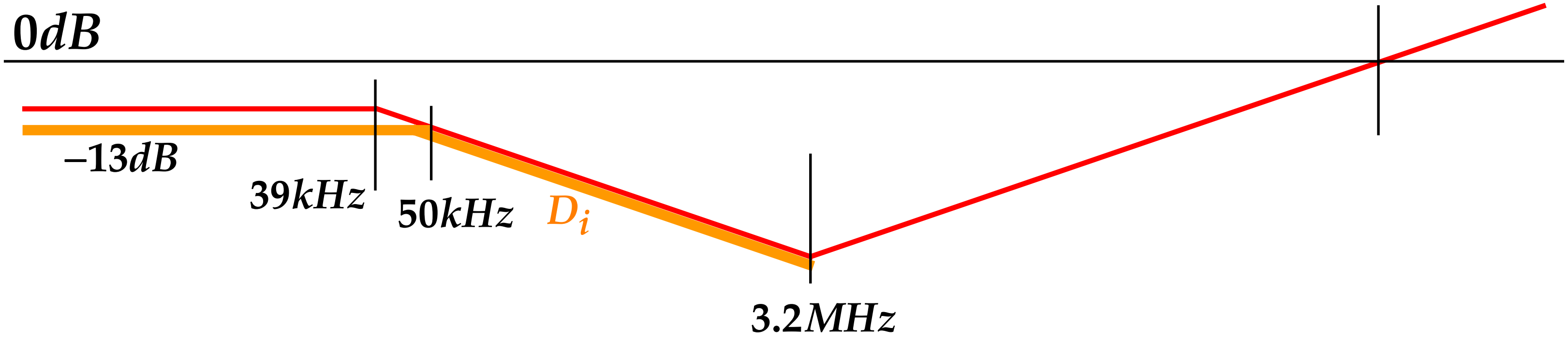
The discrepancy factor $D_i = \frac{1}{1+\frac{1}{T_i}}$ **or** $\frac{1}{D_i} = \frac{1}{1} + \frac{1}{T_i}$ **or** $D_i = 1 \| T_i$

is dominated by the smaller:



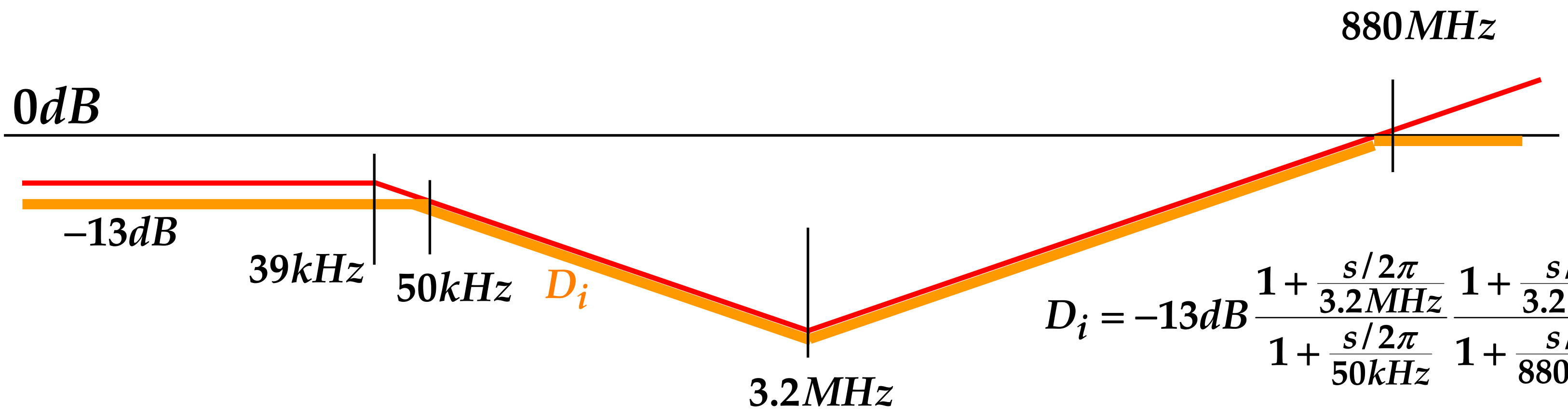
The discrepancy factor $D_i = \frac{1}{1+\frac{1}{T_i}}$ **or** $\frac{1}{D_i} = \frac{1}{1} + \frac{1}{T_i}$ **or** $D_i = 1 \| T_i$

is dominated by the smaller:



The discrepancy factor $D_i = \dfrac{1}{1+\frac{1}{T_i}}$ **or** $\dfrac{1}{D_i} = \dfrac{1}{1} + \dfrac{1}{T_i}$ **or** $D_i = 1 \| T_i$

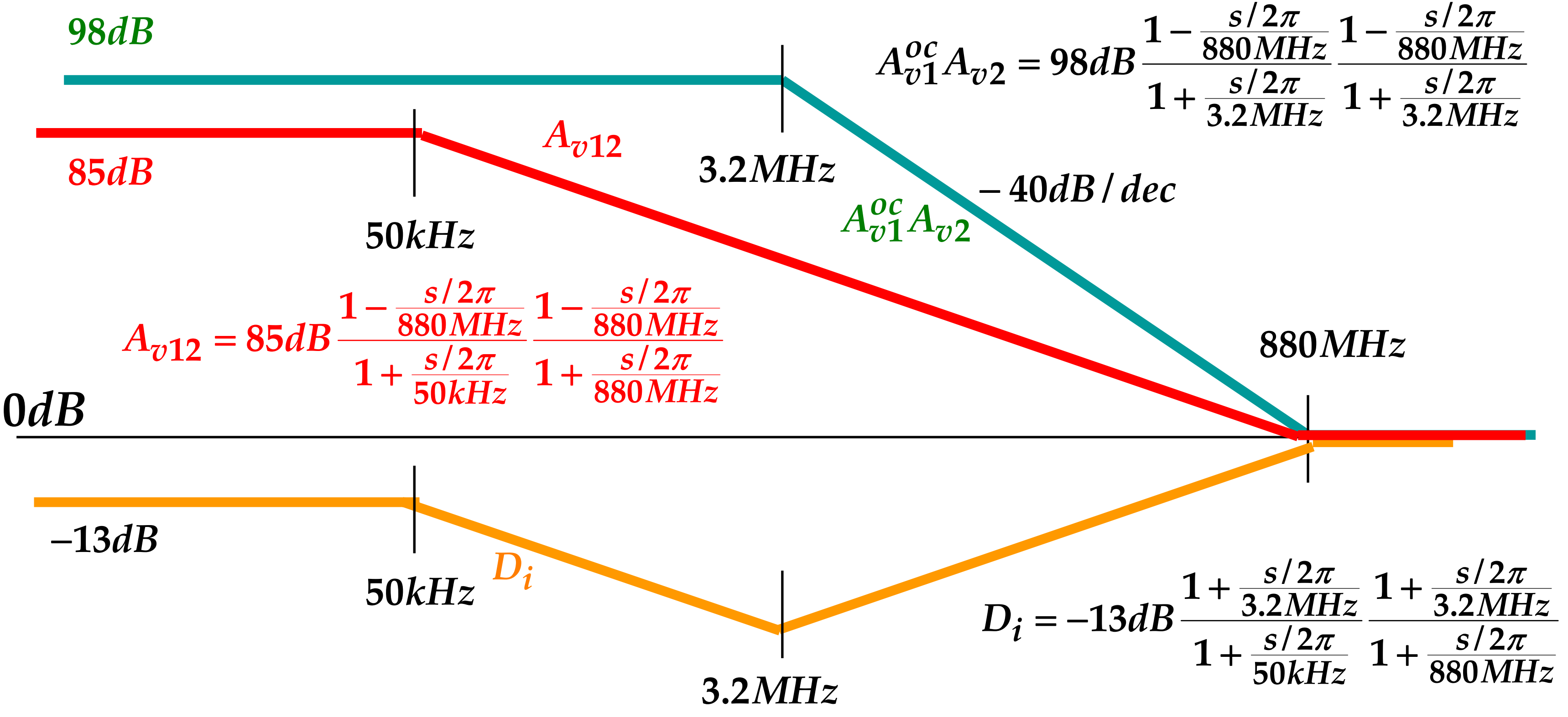
is dominated by the smaller:



$$D_i = -13dB \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{50kHz}} \frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{880MHz}}$$

All these graphical constructions can be conducted symbolically to give the result for D_i in low entropy factored pole-zero form.

Final step: assemble A_{v12} as the product of the buffered gain and the discrepancy factor:

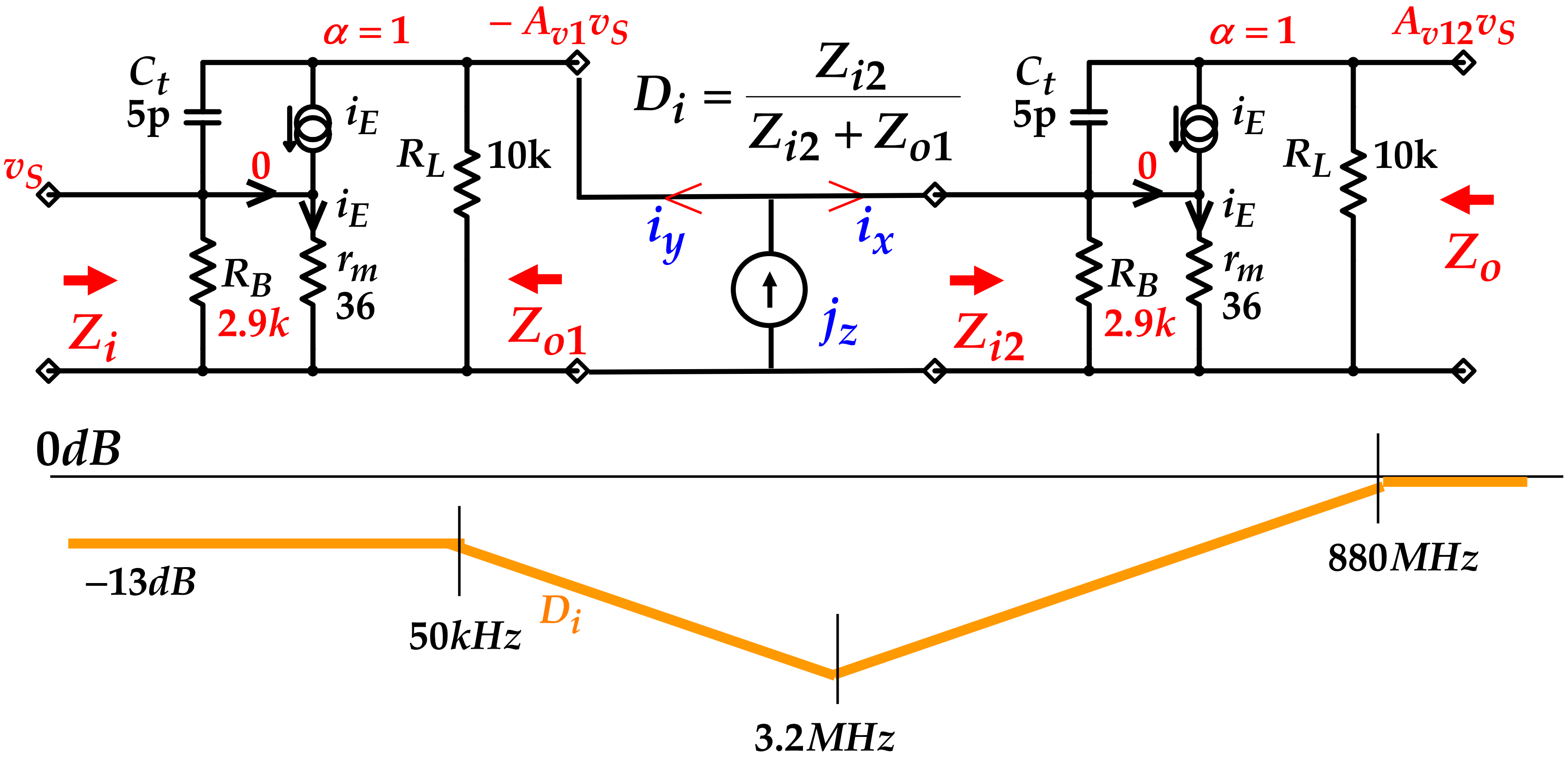


$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

$$A_{v12} = 85dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{50kHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{880MHz}}$$

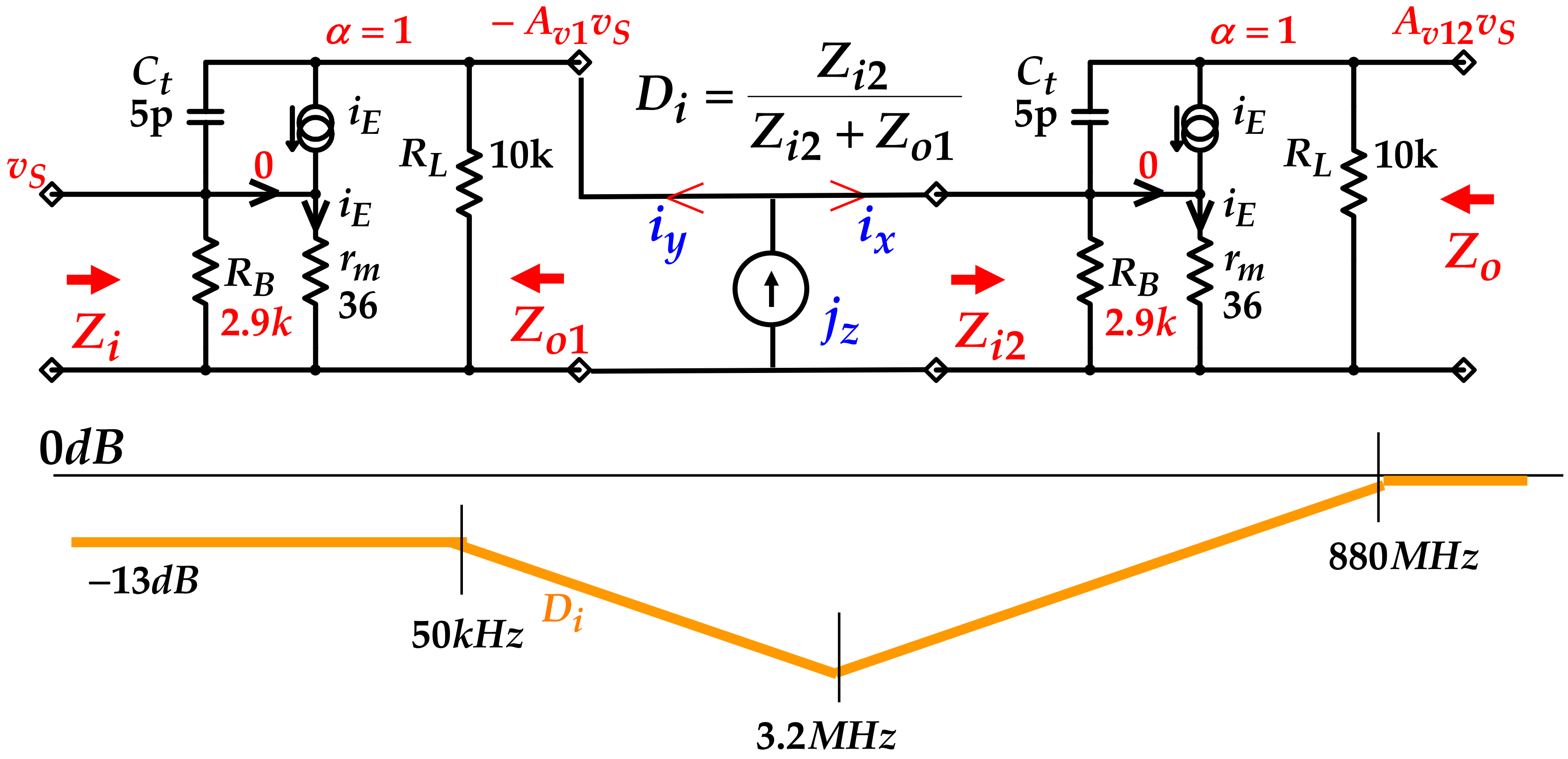
$$D_i = -13dB\,\frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{50kHz}}\,\frac{1+\frac{s/2\pi}{3.2MHz}}{1+\frac{s/2\pi}{880MHz}}$$

The fact that D_i is less than 1 over most of the frequency range indicates that the second stage imposes heavy loading upon the first stage:



$$D_i = \frac{Z_{i2}}{Z_{i2} + Z_{o1}}$$

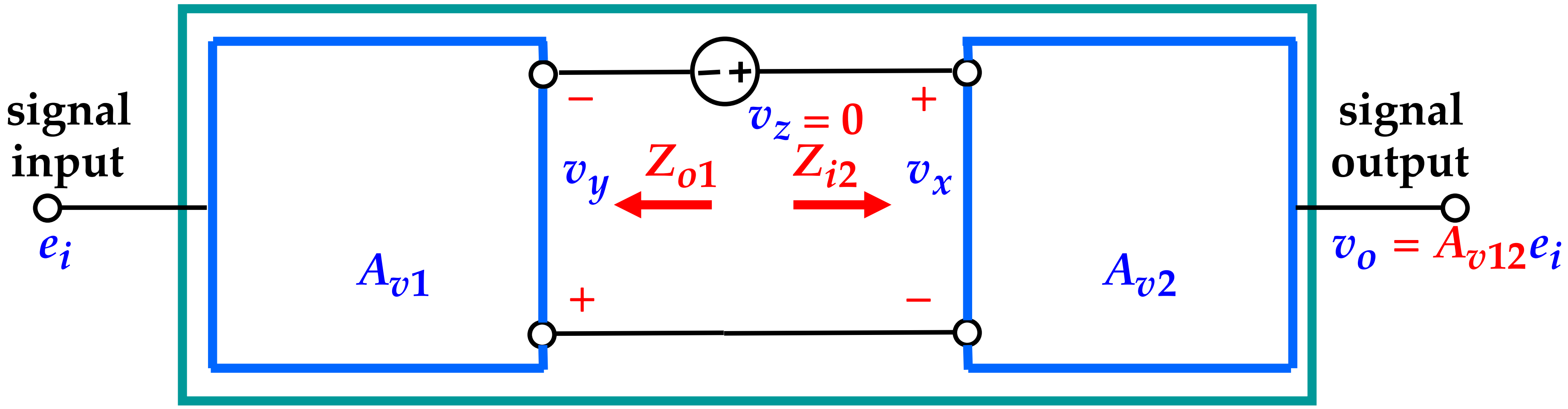
The fact that D_i is less than 1 over most of the frequency range indicates that the second stage imposes heavy loading upon the first stage:



$$D_i = \frac{Z_{i2}}{Z_{i2}+Z_{o1}}$$

This suggests that the first stage behaves more like a current source than a voltage source, and therefore that the analysis might be better undertaken using the dual form of the DT.

The Chain Theorem (CT)



The gain $A_{v12} \equiv \dfrac{v_o}{e_i}$ is given by the DT:

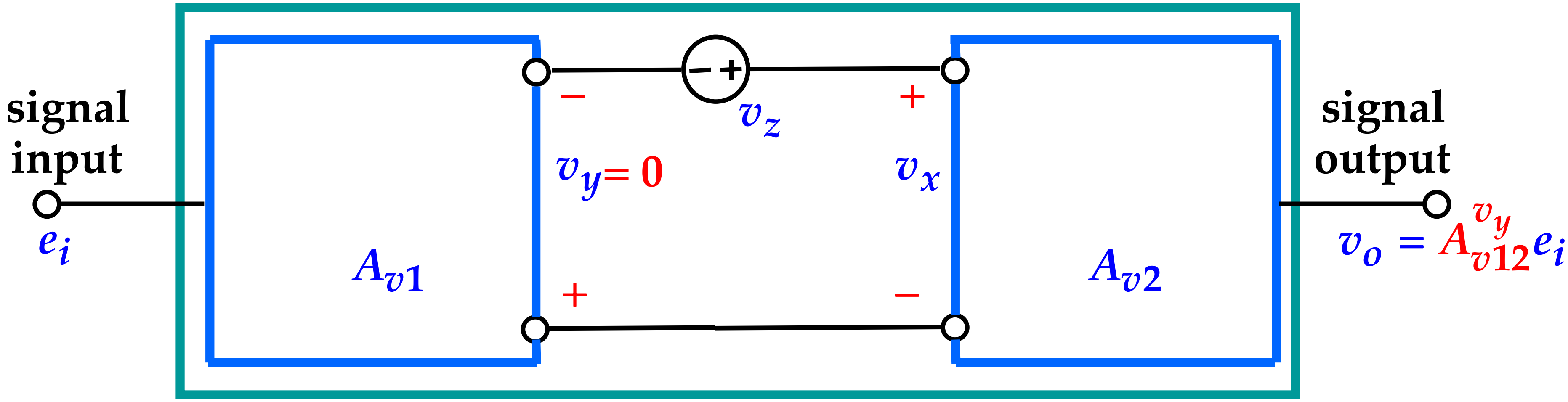
$$A_{v12} = A_{v12}^{v_y} \frac{1 + \dfrac{1}{T_{nv}}}{1 + \dfrac{1}{T_v}}$$

The TF $T_{nv} \equiv v_y / v_x \big|_{v_o = 0}$ is an ndi calculation with the output v_o nulled.

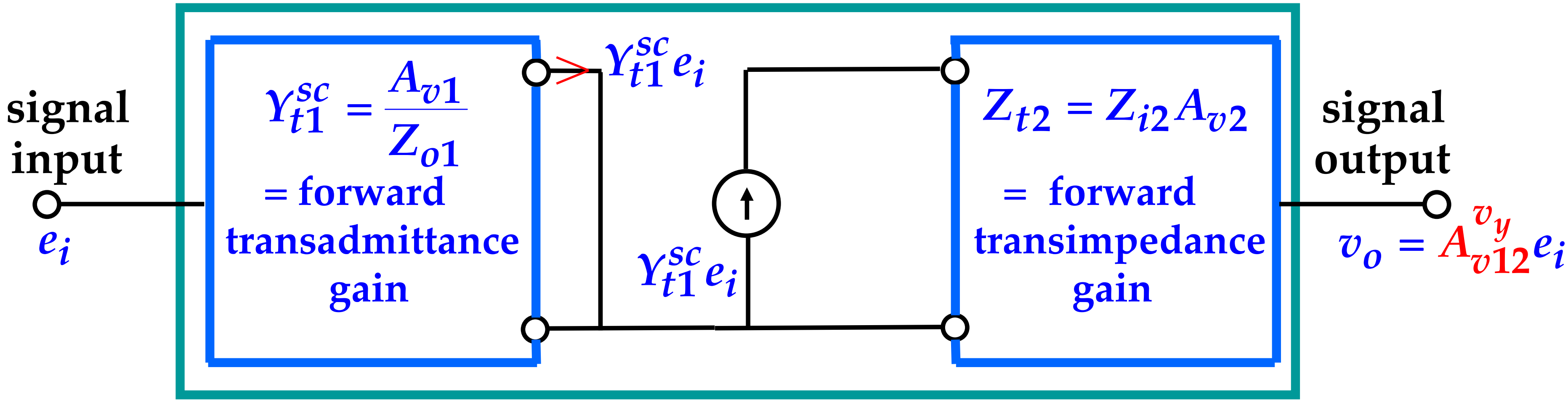
If v_o is nulled, so is v_x, so $T_{nv} = \infty$.

This implies that T_{nv} is infinite unless the signal can bypass the injection point.

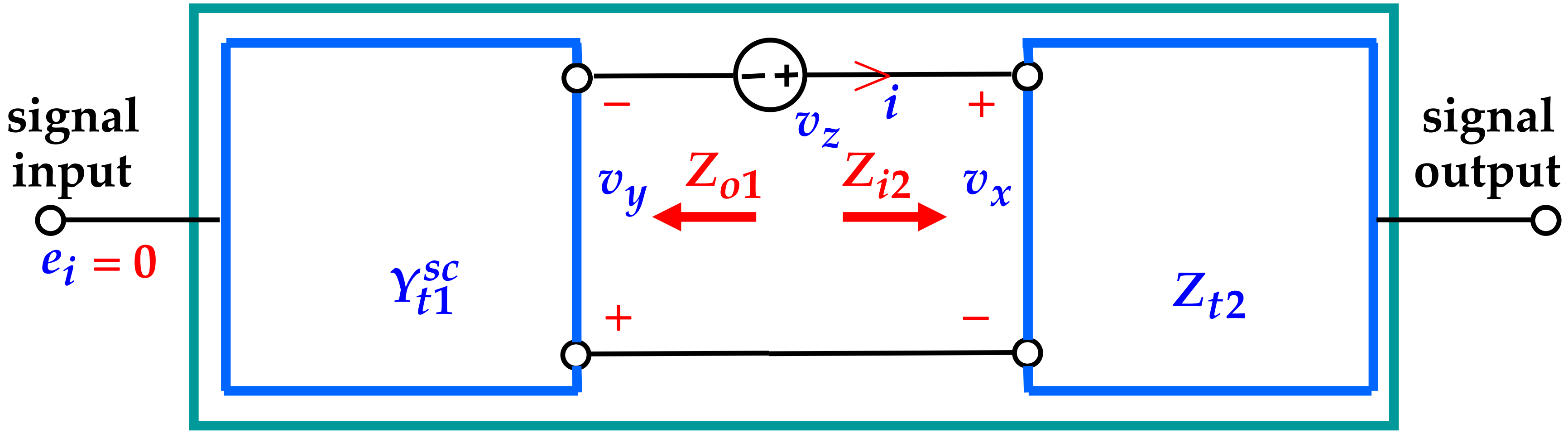
The Chain Theorem (CT)



Nulled v_y means that the A_{v1} box is shorted, so the input current to the A_{v2} box is the short-circuit (sc) output current of the A_{v1} box.



Thus, $A_{v12}^{v_y} = Y_{t1}^{sc} Z_{t2}$ is the *current - buffered gain* of the two stages.



Also

$$T_v \equiv \left.\frac{v_y}{v_x}\right|_{e_i=0} = \left.\frac{iZ_{o1}}{iZ_{i2}}\right|_{e_i=0} = \frac{Z_{o1}}{Z_{i2}},$$

so the DT becomes

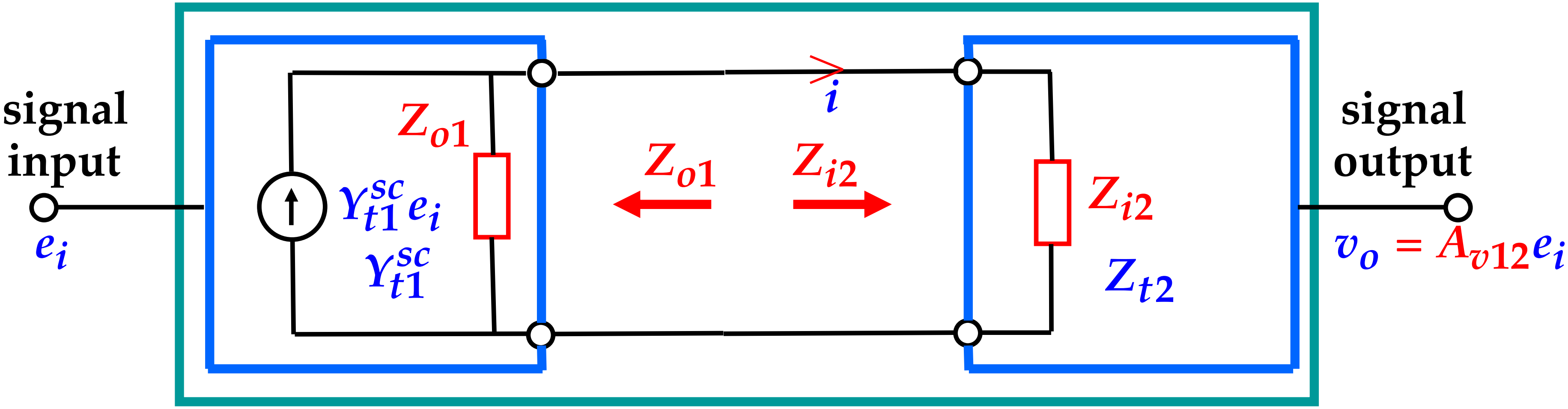
$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

This can be interpreted as

$$\left[\begin{array}{c}\text{gain}\\ \text{of the two stages}\end{array}\right] = \left[\begin{array}{c}\text{current buffered gain}\\ \text{of the two stages}\end{array}\right] \text{X} \left[\begin{array}{c}\text{current loading factor}\\ \text{between the two stages}\end{array}\right]$$

The Chain Theorem (CT)

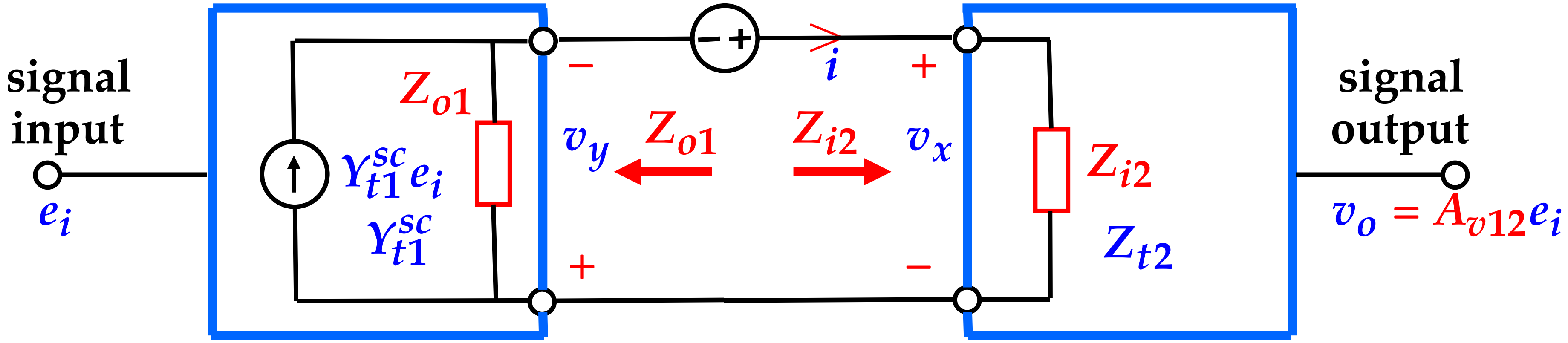
This is exactly the result that would be obtained directly from the model:



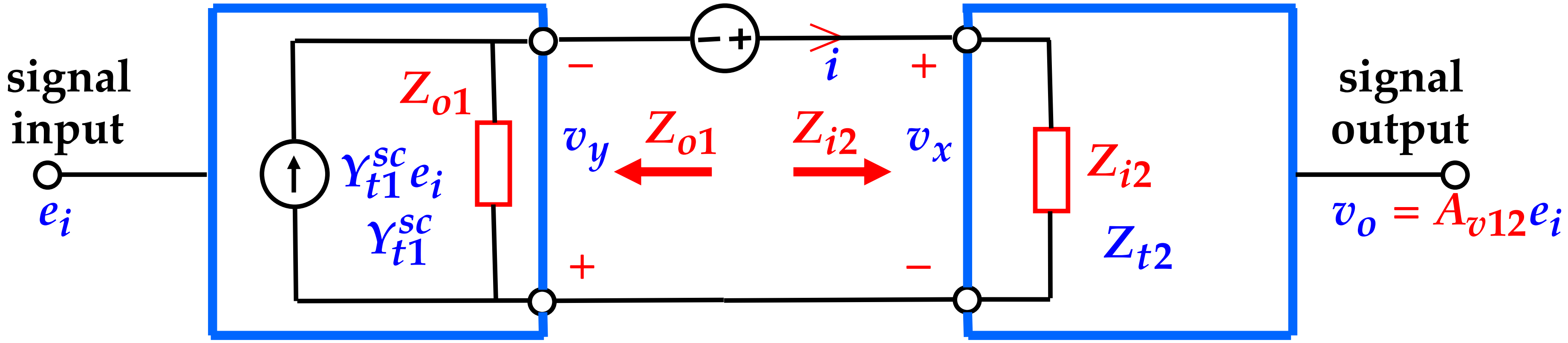
$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

The Chain Theorem (CT)

A useful application of the DT with $T_{nv} = \infty$ is to assemble the properties of a 2-stage amplifier from the properties of each separate stage.

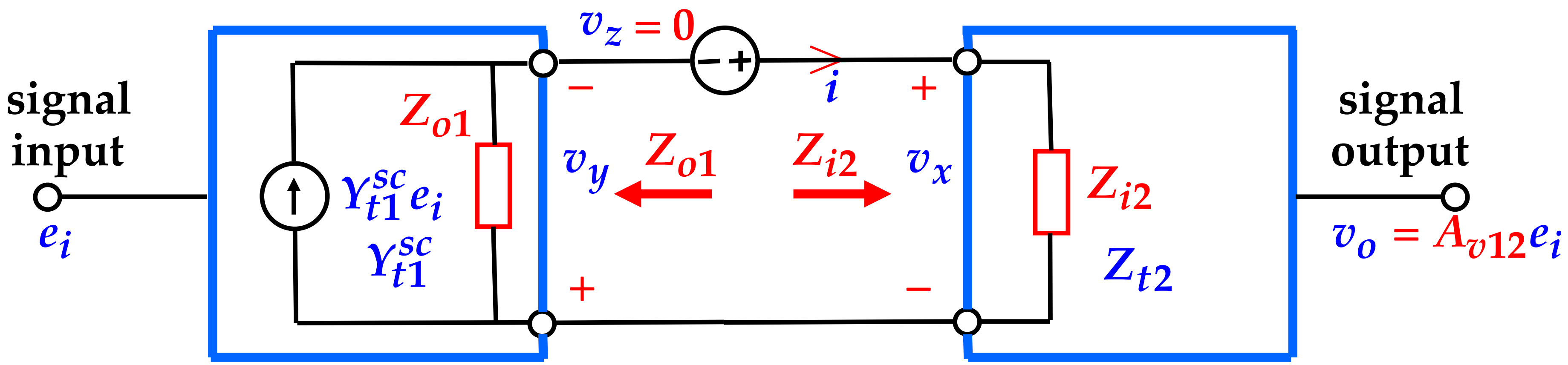


The Chain Theorem (CT)



This "Divide and Conquer" approach avoids analysis of both stages simultaneously.

The Chain Theorem (CT)



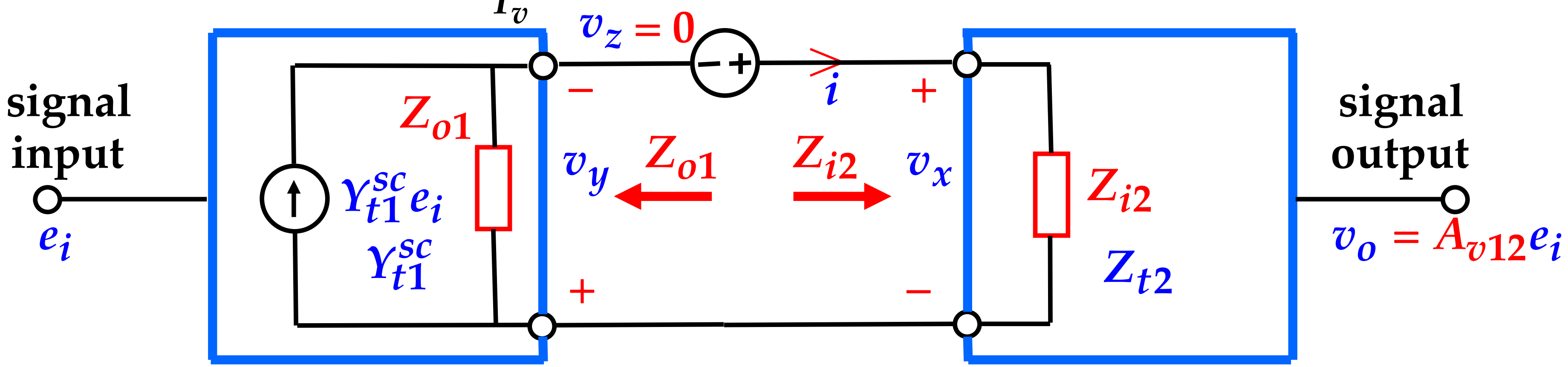
$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{1}{1+\frac{1}{T_v}} = Y_{t1}^{sc} Z_{t2} D_v$$

$$T_v \equiv \frac{Z_{o1}}{Z_{i2}} \qquad D_v \equiv \frac{1}{1+\frac{1}{T_v}} = \frac{T_v}{1+T_v} = \frac{Z_{o1}}{Z_{i2}+Z_{o1}}$$

where $Y_{t1}^{sc} Z_{t2}$ is the "current buffered" gain that would occur if there were a buffer between the two stages, and D_v is a "discrepancy factor" that accounts for the interaction between the two stages which results from the loading of the first stage by the input of the second stage.

The Chain Theorem (CT)

$$A_{v12} = Y_{t1}^{sc} Z_{t2} \frac{1}{1+\frac{1}{T_v}} = Y_{t1}^{sc} Z_{t2} D_v$$

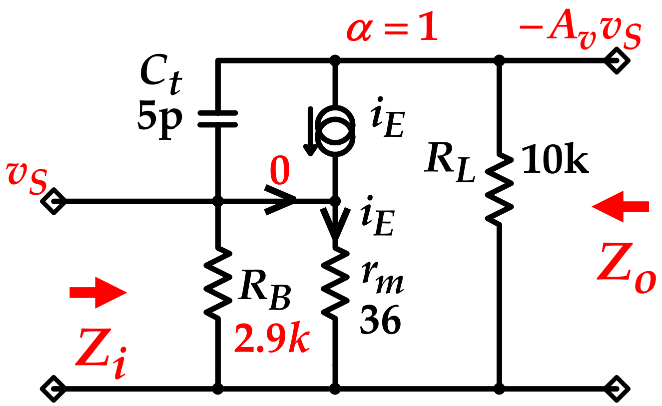


Since all TFs will be in factored pole-zero form, the only place where additional approximation may be needed resides inside the D_v, where the sum of two TFs is required.

"Doing the algebra on the graph" can be conducted in two ways:

$1+T_v$ can be found as the sum of the TFs 1 and T_v, dominated by the larger;

D_v can be found from $\frac{1}{D_v} = 1 + \frac{1}{T_v} = \frac{1}{1} + \frac{1}{T_v}$ as the reciprocal sum of 1 and T_v dominated by the smaller.

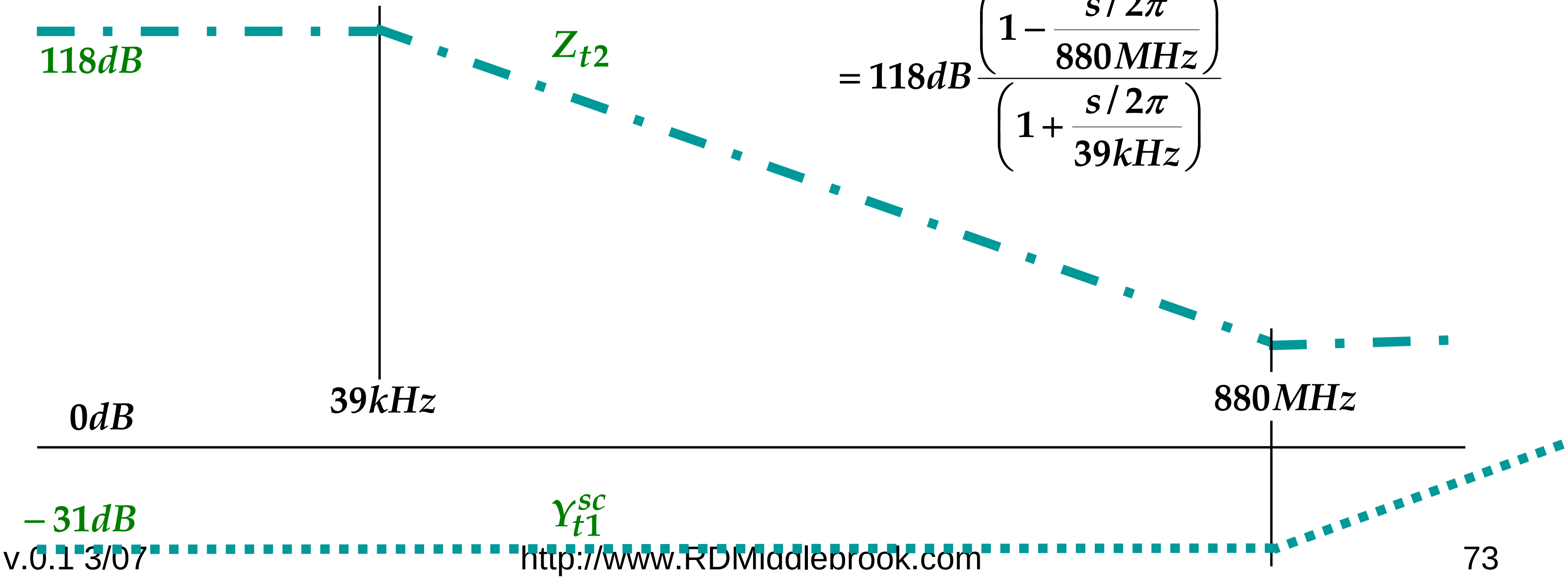


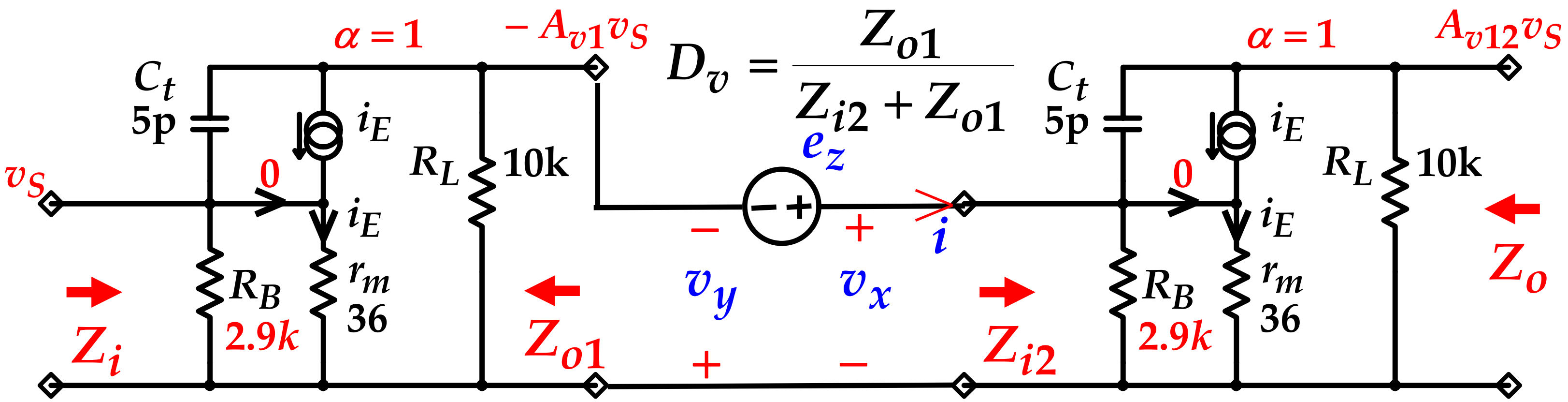
$$Y_t^{sc} = \frac{A_v}{Z_o} = \frac{1}{r_m}(1 - sC_t r_m)$$

$$= -31dB\left(1 - \frac{s/2\pi}{880MHz}\right)$$

$$Z_t = Z_i A_v = \frac{R_B R_L}{r_m}\frac{(1 - sC_t r_m)}{\left(1 + sC_t R_L \frac{R_B}{R_B\|r_m\|R_L}\right)}$$

$$= 118dB\frac{\left(1 - \frac{s/2\pi}{880MHz}\right)}{\left(1 + \frac{s/2\pi}{39kHz}\right)}$$





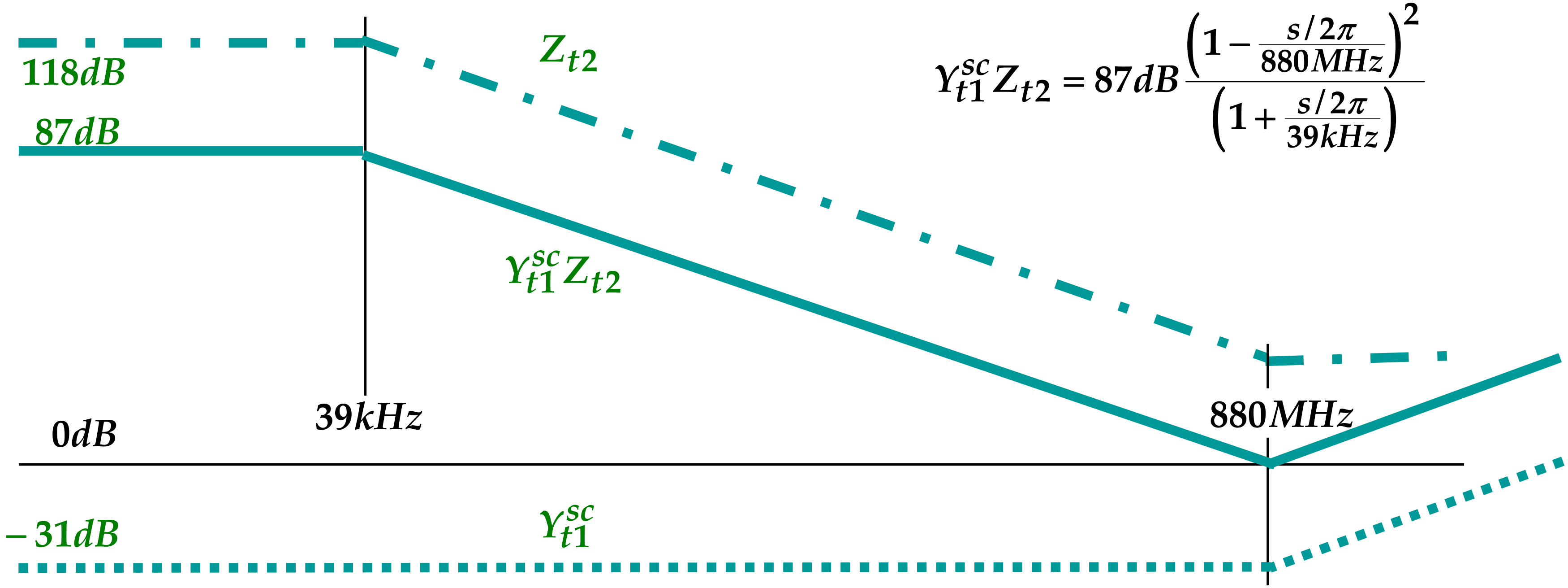
$$D_v = \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

The DT gives

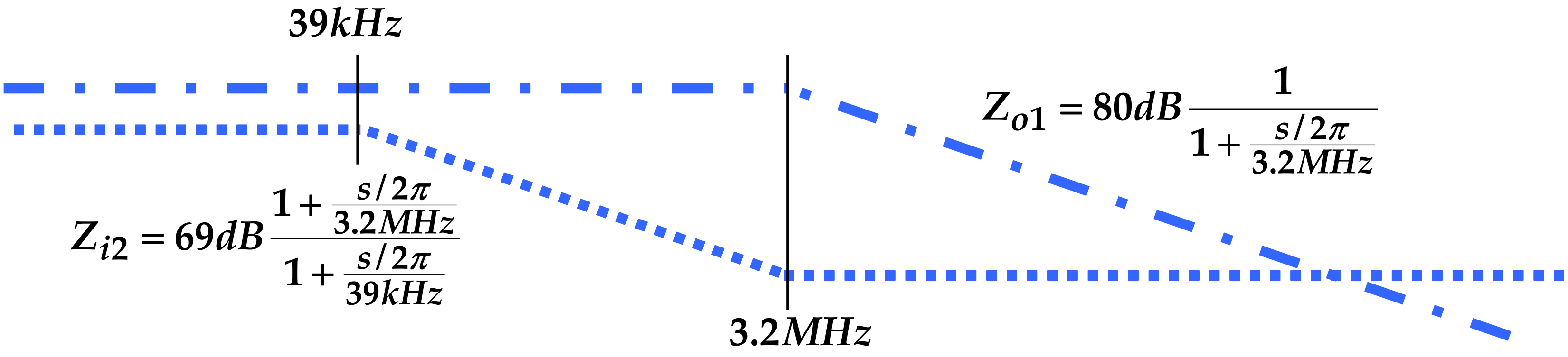
$$A_{v12} = Y_{t1}^{sc} Z_{t2} D_v \qquad (T_{nv} = \infty)$$

The current buffered gain $Y_{t1}^{sc} Z_{t2}$ is the product of the two separate gains:

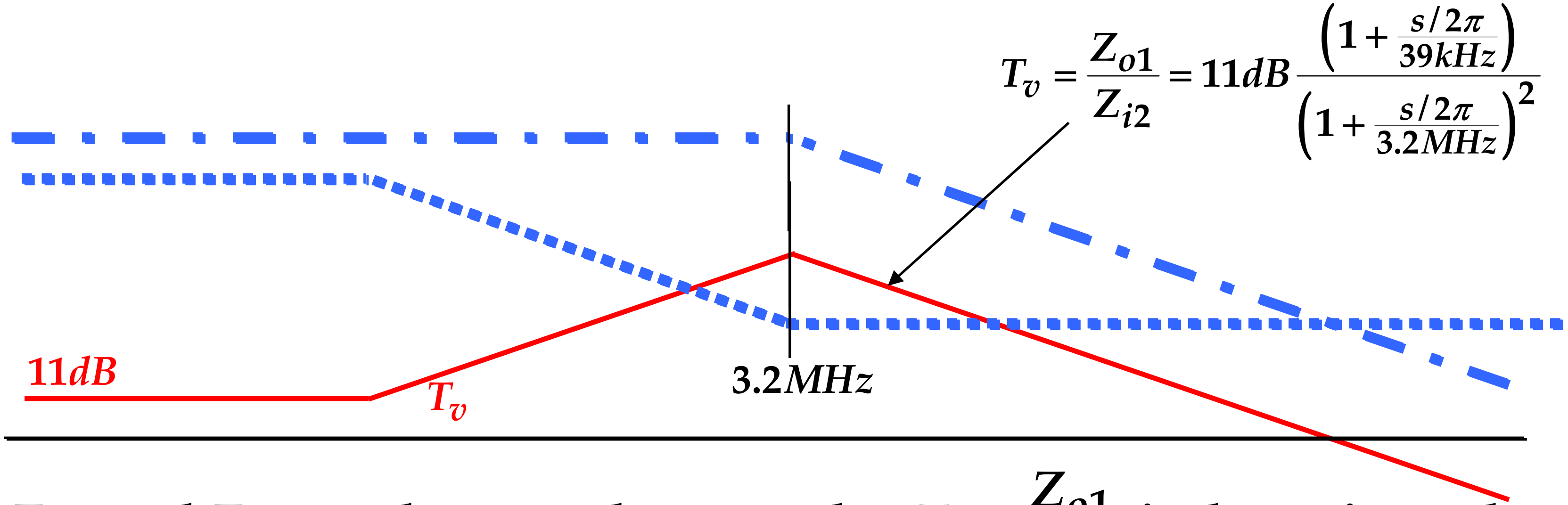
$$Y_{t1}^{sc} Z_{t2} = 87dB \frac{\left(1 - \frac{s/2\pi}{880MHz}\right)^2}{\left(1 + \frac{s/2\pi}{39kHz}\right)}$$



$$Y_{t1}^{sc}Z_{t2} = 87dB\frac{\left(1-\frac{s/2\pi}{880MHz}\right)^2}{\left(1+\frac{s/2\pi}{39kHz}\right)}$$



Z_{o1} and Z_{i2} are the same, but note that $T_v = \dfrac{Z_{o1}}{Z_{i2}}$ is the reciprocal of $T_i = \dfrac{Z_{i2}}{Z_{o1}}$:

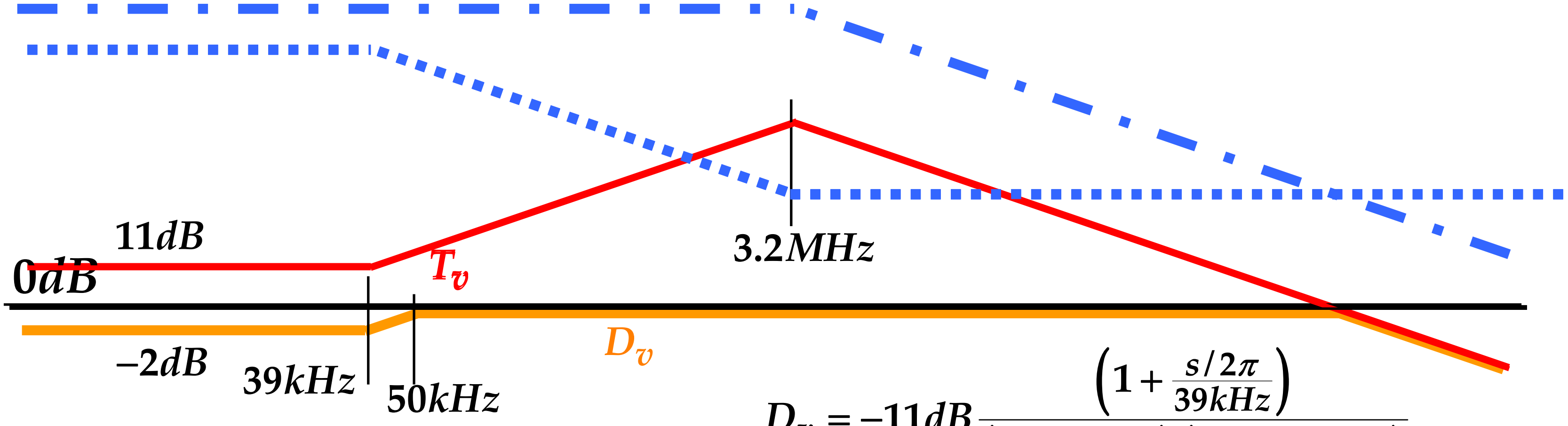


$$T_v = \frac{Z_{o1}}{Z_{i2}} = 11dB \frac{\left(1+\frac{s/2\pi}{39kHz}\right)}{\left(1+\frac{s/2\pi}{3.2MHz}\right)^2}$$

Z_{o1} and Z_{i2} are the same, but note that $T_v = \frac{Z_{o1}}{Z_{i2}}$ is the reciprocal of $T_i = \frac{Z_{i2}}{Z_{o1}}$:

The discrepancy factor $D_v = \frac{1}{1+\frac{1}{T_v}}$ **or** $\frac{1}{D_v} = \frac{1}{1} + \frac{1}{T_v}$ **or** $D_v = 1 \| T_v$

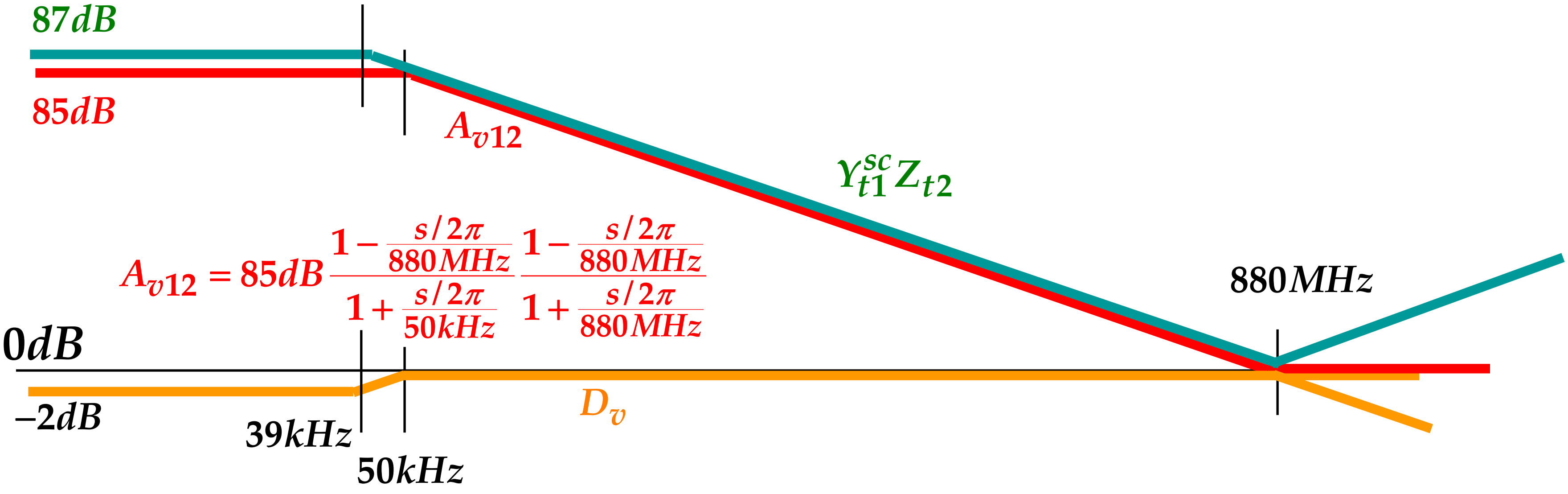
is dominated by the smaller:



$$D_v = -11dB \frac{\left(1+\frac{s/2\pi}{39kHz}\right)}{\left(1+\frac{s/2\pi}{50kHz}\right)\left(1+\frac{s/2\pi}{880MHz}\right)}$$

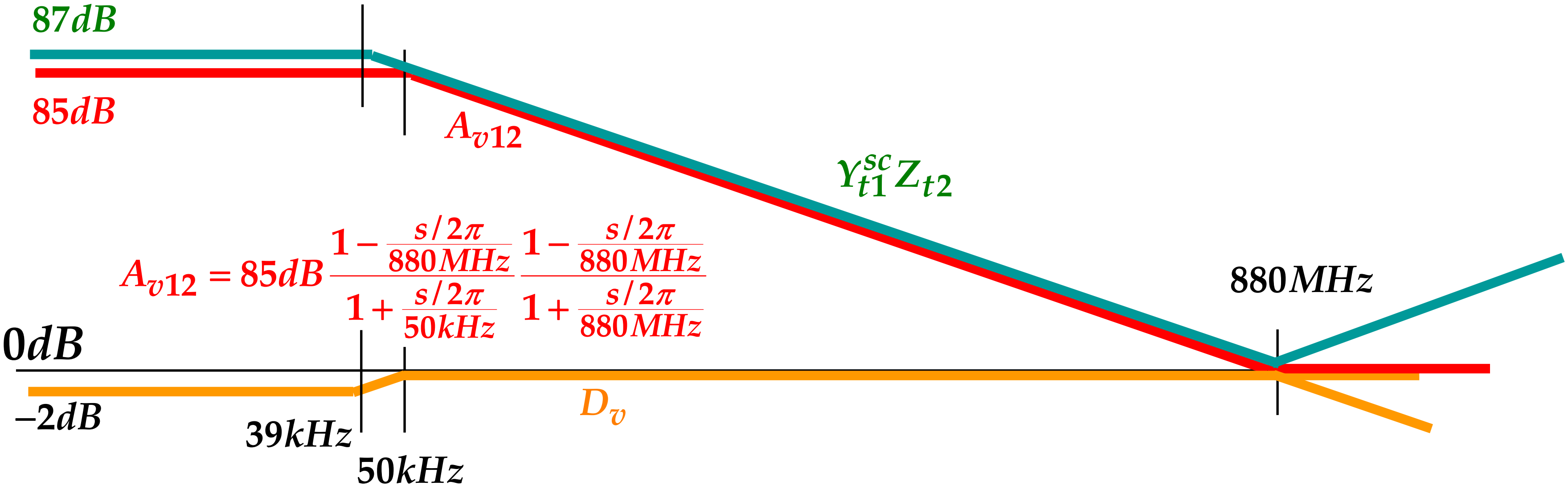
All these graphical constructions can be conducted symbolically to give the result for D_v **in low entropy factored pole-zero form.**

Final step: assemble A_{v12} as the product of the buffered gain and the discrepancy factor:



$$A_{v12} = 85dB \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{50kHz}} \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{880MHz}}$$

Final step: assemble A_{v12} as the product of the buffered gain and the discrepancy factor:



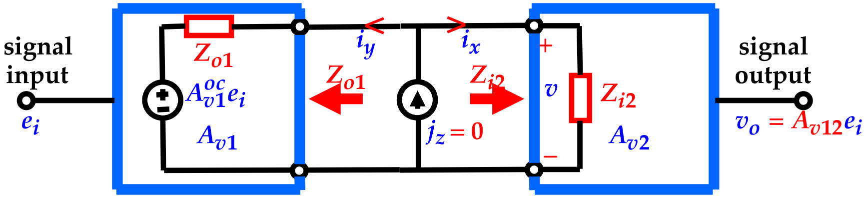
$$A_{v12} = 85dB \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{50kHz}} \frac{1 - \frac{s/2\pi}{880MHz}}{1 + \frac{s/2\pi}{880MHz}}$$

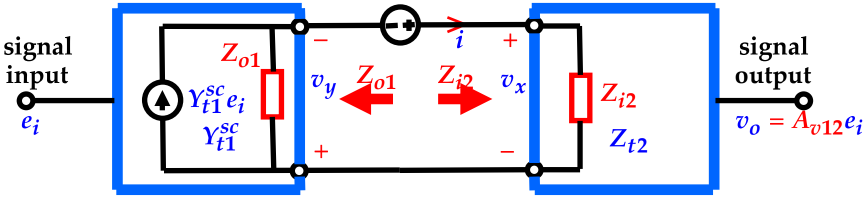
The fact that D_v is close to 1 over most of the frequency range confirms the expectation that the first stage behaves more like a current source than a voltage source.

Summary:

The DT allows assembly of the properties of a 2-stage amplifier from the properties of each separate stage.

This can be done by injection of either a test current j_z or a test voltage e_z at the interface:





$$A_{v12} = A_{v12}^{i_y} D_i$$

where $A_{v12}^{i_y} = A_{v1}^{oc} A_{v2}$

is the voltage buffered gain

and $D_i = \dfrac{Z_{i2}}{Z_{i2} + Z_{o1}}$

$$A_{v12} = A_{v12}^{v_y} D_v$$

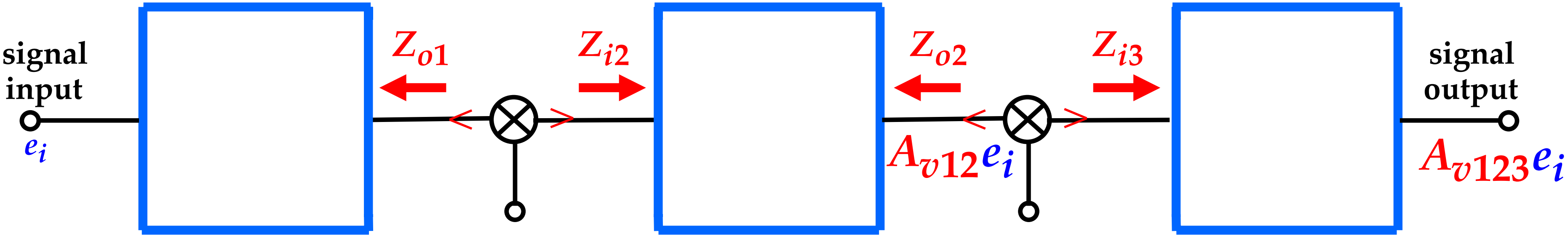
where $A_{v12}^{v_y} = Y_{t1}^{sc} Z_{t2}$

is the current buffered gain

and $D_v = \dfrac{Z_{o1}}{Z_{i2} + Z_{o1}}$

are the discrepancy factors representing the interface loading.

In principle, this procedure can be extended to the addition of extra stages:



In practice, this procedure becomes cumbersome because the discrepancy factor for the first interface changes when a second interface is added.

However, there is an alternative form for the gain of 2 stages that circumvents this problem.

The DT results already obtained are:

$$A_{v12} = A_{v12}^{i_y} D_i = A_{v12}^{i_y} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} \qquad A_{v12} = A_{v12}^{v_y} D_v = A_{v12}^{v_y} \frac{Z_{o1}}{Z_{i2} + Z_{o1}}$$

Rewrite:

$$\frac{1}{A_{v12}} \frac{Z_{i2}}{Z_{i2} + Z_{o1}} = \frac{1}{A_{v12}^{i_y}} \qquad \frac{1}{A_{v12}} \frac{Z_{o1}}{Z_{i2} + Z_{o1}} = \frac{1}{A_{v12}^{v_y}}$$

Add the two:

$$\frac{1}{A_{v12}} = \frac{1}{A_{v12}^{i_y}} + \frac{1}{A_{v12}^{v_y}}$$

$$\frac{1}{A_{v12}} = \frac{1}{A_{v12}^{i_y}} + \frac{1}{A_{v12}^{v_y}}$$

This simple and elegant result says that the interface discrepancy factors D_i and D_v are not needed, and the overall gain is a "parallel combination" of the two buffered gains:

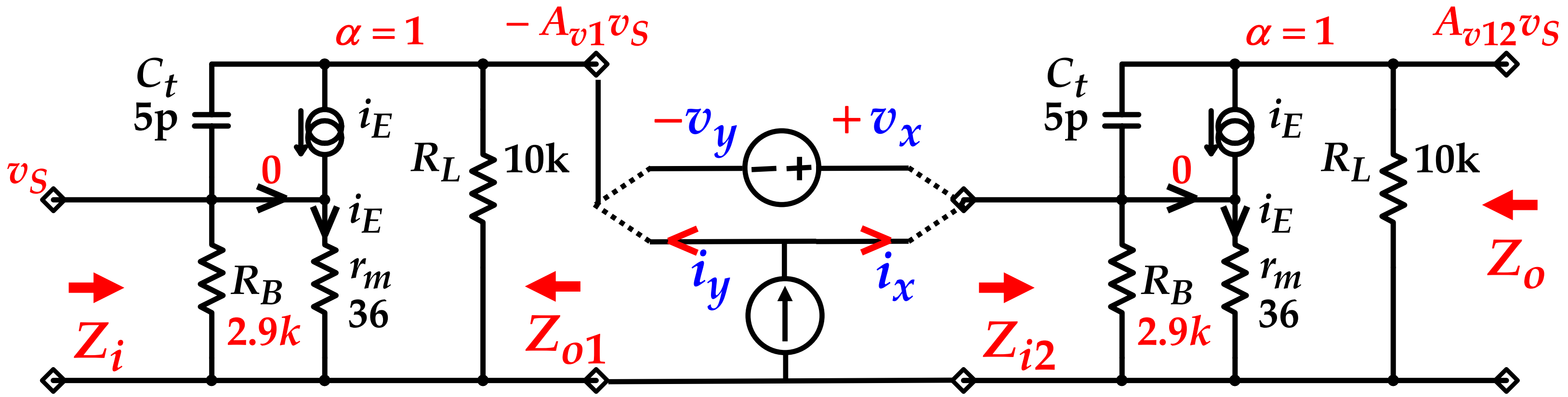
$$A_{v12} = A_{v12}^{i_y} \Big\| A_{v12}^{v_y}$$

where $A_{v12}^{i_y} = A_{v1}^{oc} A_{v2}$ = **voltage buffered gain of the 2 stages**

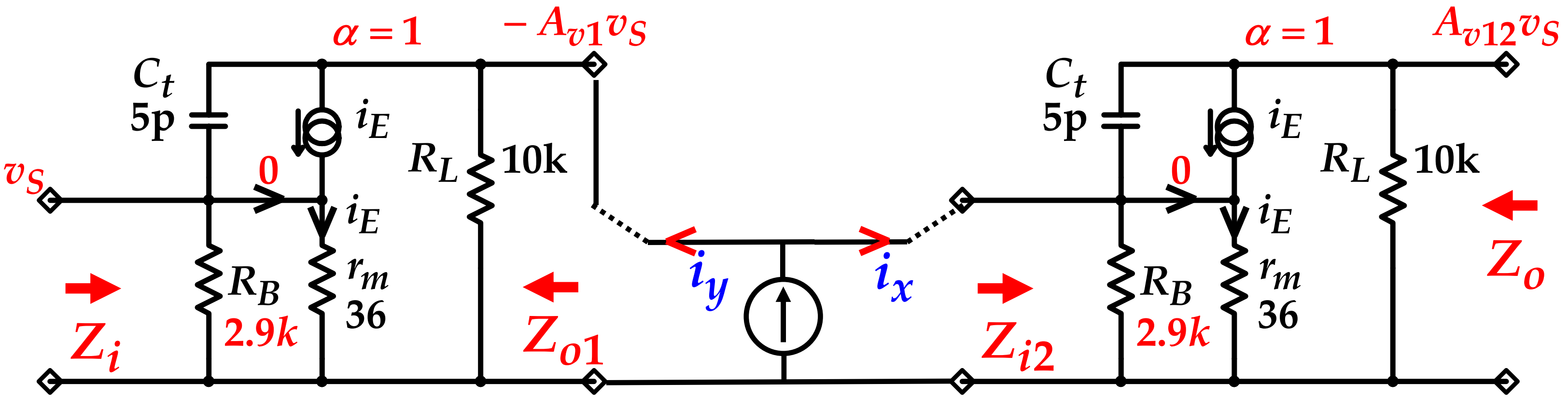
and $A_{v12}^{v_y} = Y_{t1}^{sc} Z_{t2}$ = **current buffered gain of the 2 stages**

This result is actually the Chain Theorem (CT), and A_{v1}^{oc}, A_{v2}, Y_{t1}^{sc}, Z_{t2} are the (reciprocals of the) chain parameters (c parameters).

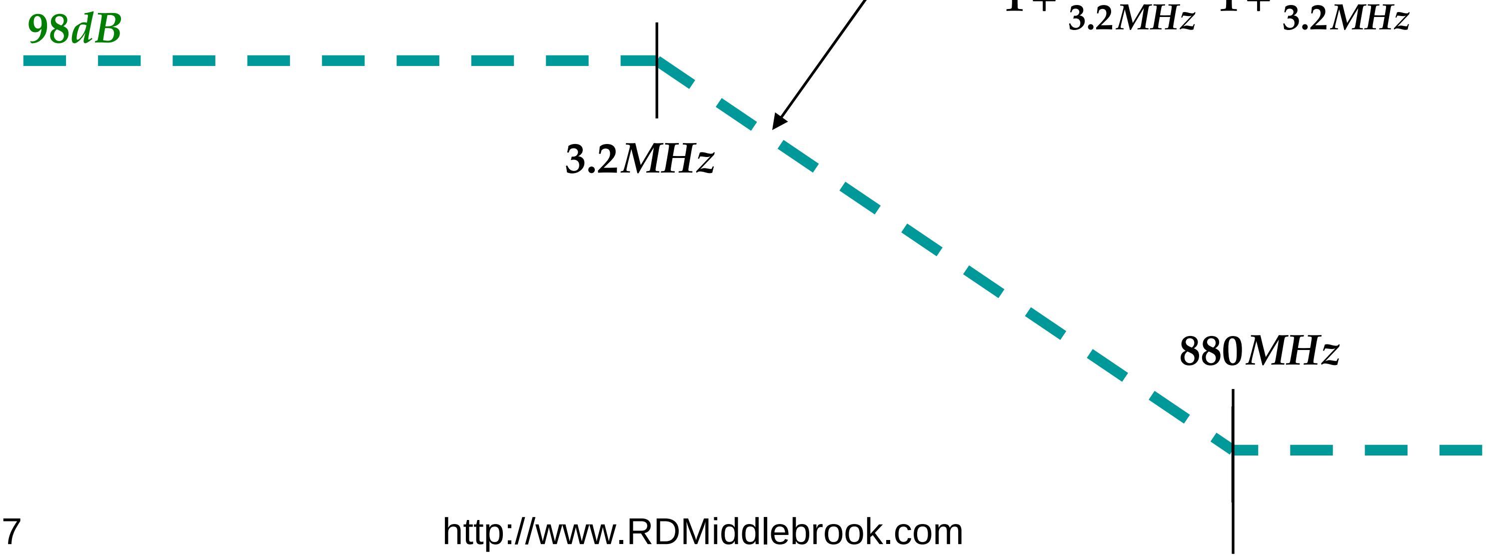
Rework the previous example:



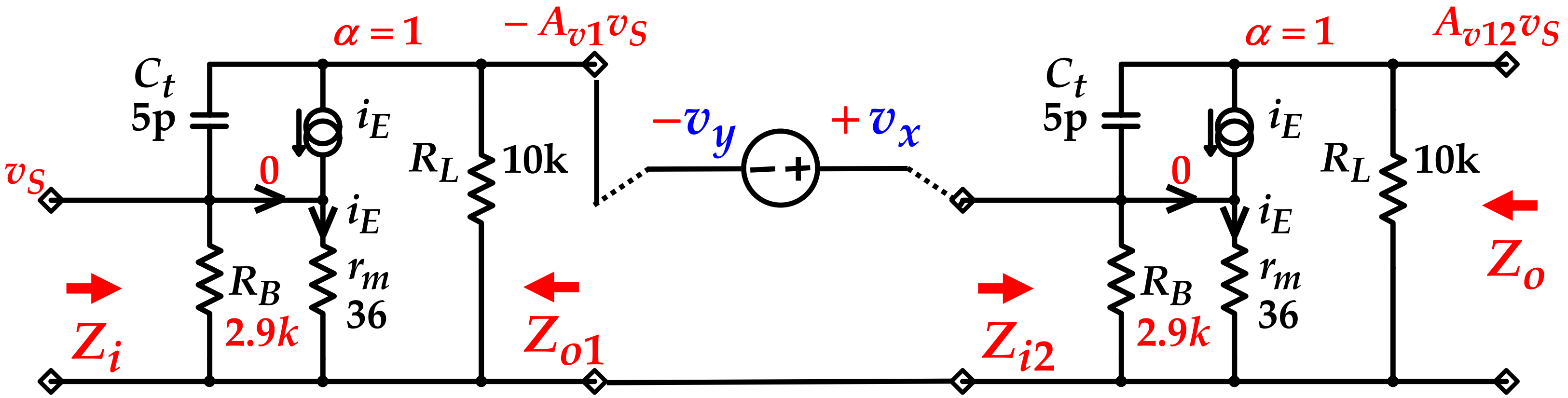
Rework the previous example:

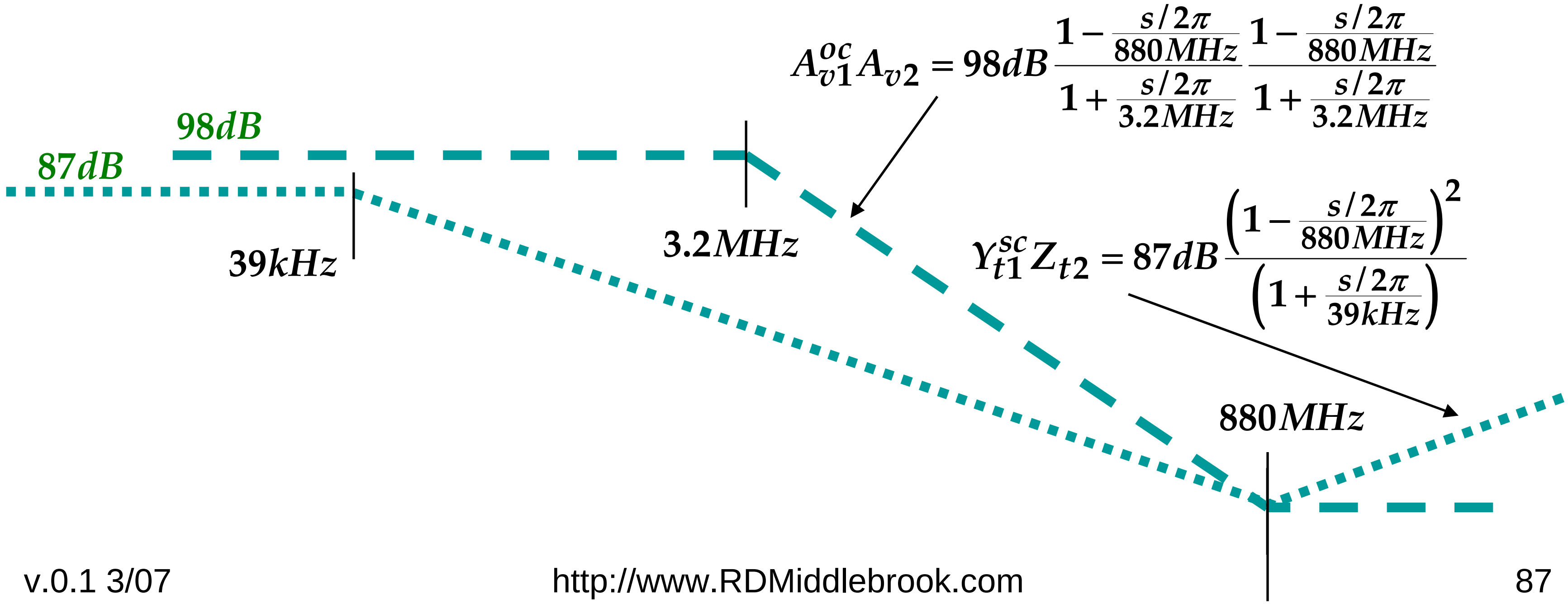


$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$



Rework the previous example:

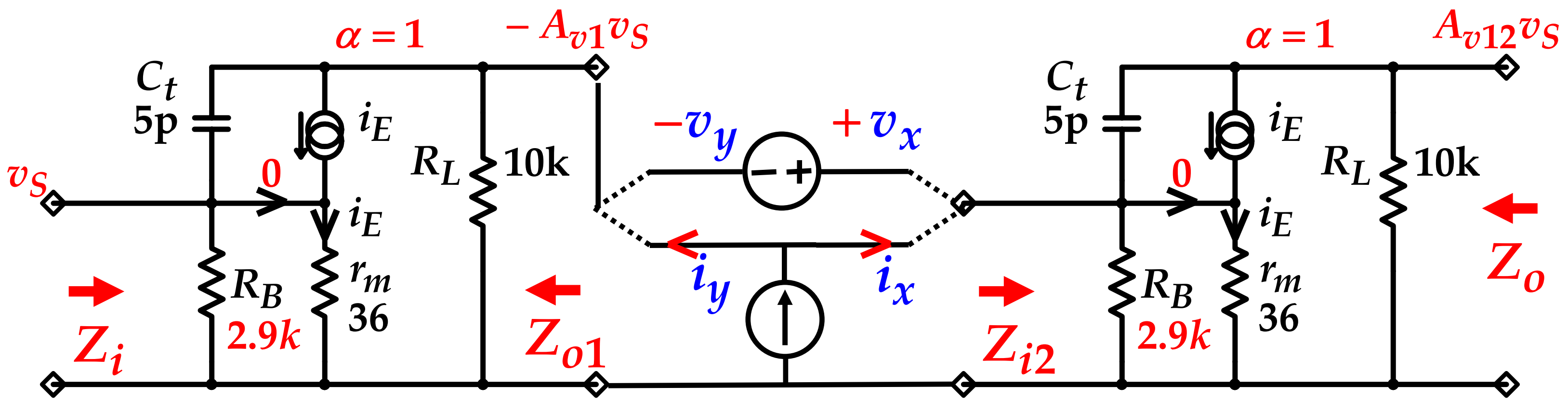


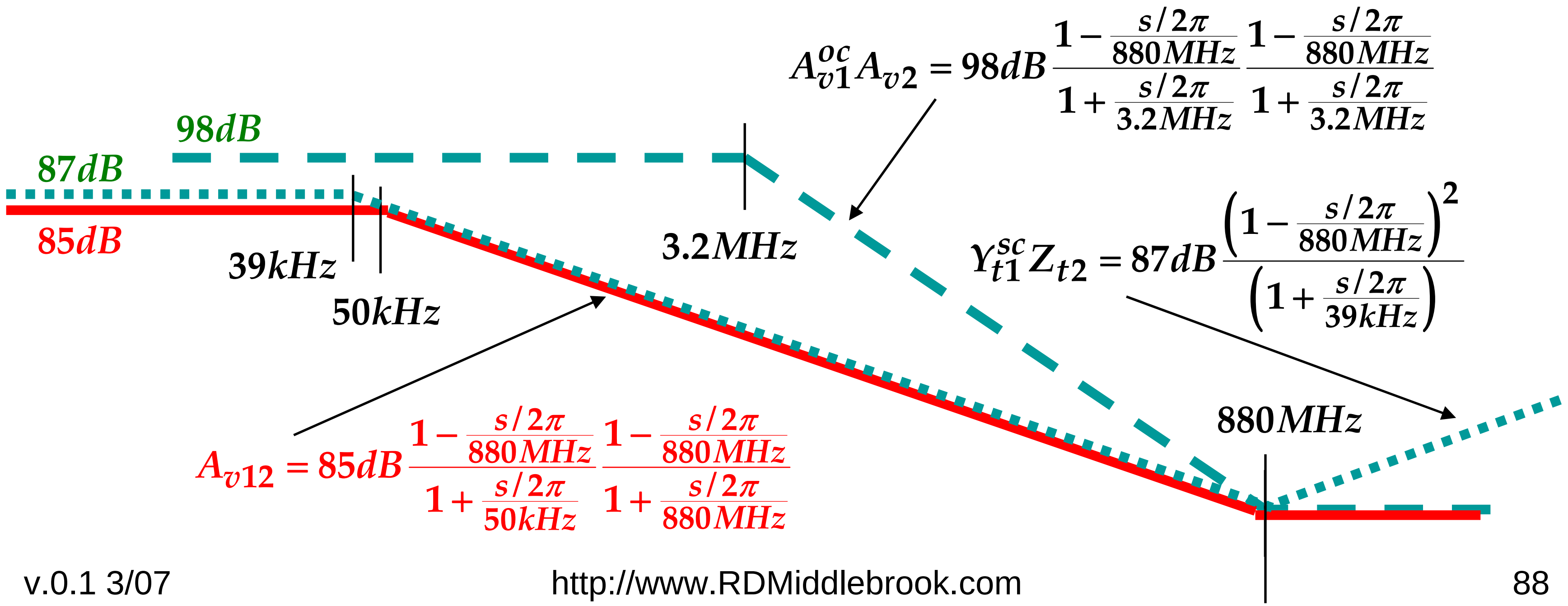


$$A_{v1}^{oc}A_{v2} = 98dB\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Y_{t1}^{sc}Z_{t2} = 87dB\frac{\left(1-\frac{s/2\pi}{880MHz}\right)^2}{\left(1+\frac{s/2\pi}{39kHz}\right)}$$

Rework the previous example:





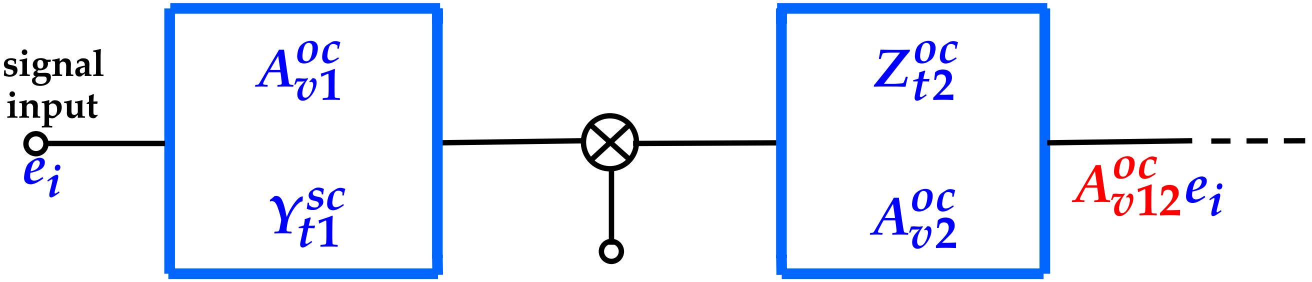
$$A_{v1}^{oc}A_{v2} = 98dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{3.2MHz}}$$

$$Y_{t1}^{sc}Z_{t2} = 87dB\,\frac{\left(1-\frac{s/2\pi}{880MHz}\right)^2}{\left(1+\frac{s/2\pi}{39kHz}\right)}$$

$$A_{v12} = 85dB\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{50kHz}}\,\frac{1-\frac{s/2\pi}{880MHz}}{1+\frac{s/2\pi}{880MHz}}$$

The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

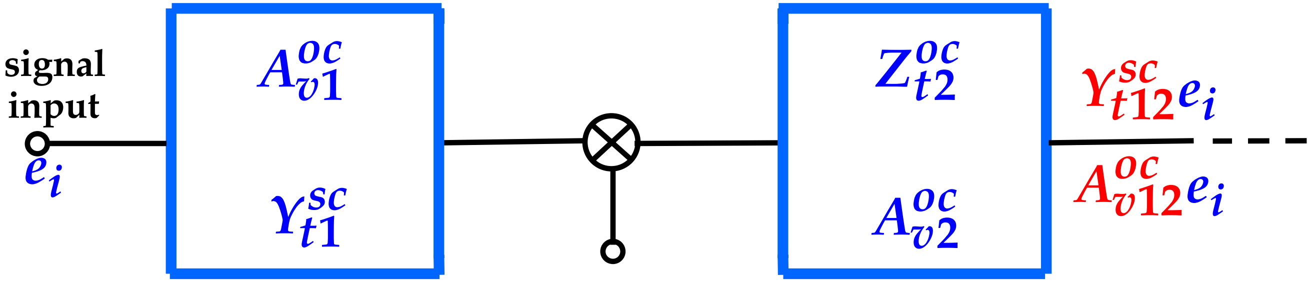
The procedure is:



Find A_{v1}^{oc} and Y_{t1}^{sc} of stage 1, and Z_{t2}^{oc} and A_{v2}^{oc} of stage 2.
Combine them by the CT to find A_{v12}^{oc}, as above,

The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

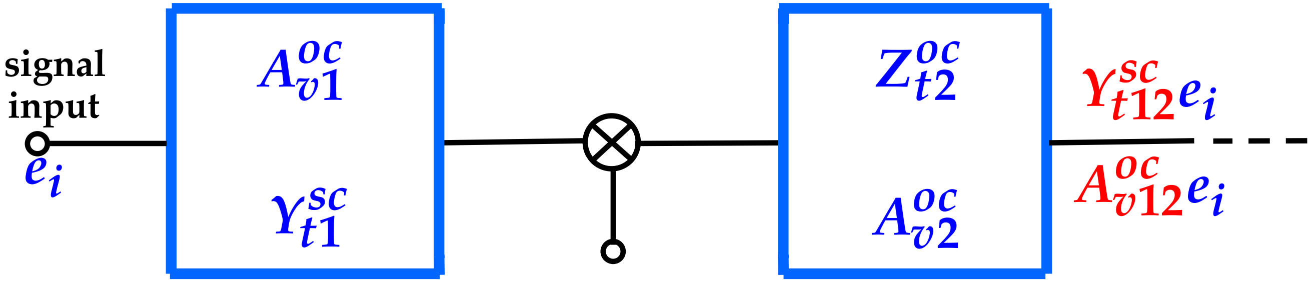
The procedure is:



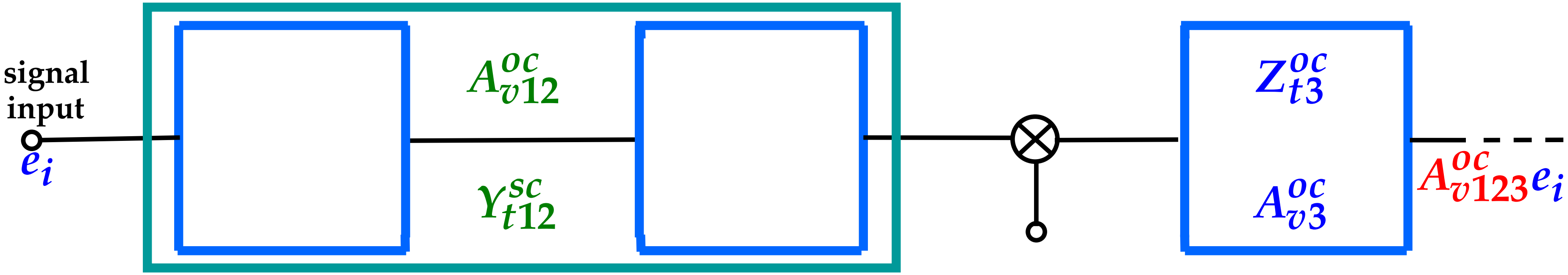
Find A_{v1}^{oc} and Y_{t1}^{sc} of stage 1, and Z_{t2}^{oc} and A_{v2}^{oc} of stage 2.
Combine them by the CT to find A_{v12}^{oc}, as above, and hence find Y_{t12}^{sc}.

The CT is the key to implementation of the "Divide and Conquer" approach to D-OA.

The procedure is:



Find A_{v1}^{oc} and Y_{t1}^{sc} of stage 1, and Z_{t2}^{oc} and A_{v2}^{oc} of stage 2.
Combine them by the CT to find A_{v12}^{oc}, as above, and hence find Y_{t12}^{sc}.



Find Z_{t3}^{oc} and A_{v3}^{oc} of stage 3.
Combine them by the CT to find A_{v123}^{oc}.

and so on...